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Generalized Reproducing Kernel Particle Method for Fragment-Impact and Fracture Modeling

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Generalized Reproducing Kernel Particle Method for Fragment-Impact and Fracture Modeling

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Structural Engineering with a specialization in Computational Science

by

Edouard Yreux

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Professor David Benson
Professor Michael Holst

2015
The dissertation of Edouard Yreux is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2015
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VITA

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ABSTRACT OF THE DISSERTATION

Generalized Reproducing Kernel Particle Method for Fragment-Impact and Fracture Modeling

by

Edouard Yreux

Doctor of Philosophy in Structural Engineering (with a specialization in Computational Science)

University of California, San Diego, 2015

Professor Jiun-Shyan Chen, Chair

Accurate modeling of fracture and material fragmentation remains a challenging problem in computational mechanics. Extrinsic enrichment of the approximation functions leads to an increase of degrees of freedom, while intrinsic enrichment near crack tips encounters compatibility issues. Furthermore, although material fragmentation can be effectively represented under a semi-Lagrangian meshfree framework in conjunction with the appropriate damage laws, approximation with high order bases remains difficult to construct in this class of problem due to the need of meeting the kernel
support coverage conditions in the fragmented areas. A new approach for constructing
the meshfree approximation functions is introduced in this dissertation to enhance the
flexibility in their construction and to address the above mentioned difficulties in fracture
and fragment-impact modeling of materials and solids.

First, the construction of meshfree approximation functions with arbitrary order
of completeness is revisited and extended with a more general approach. Approximation
properties of the proposed family of approximation functions are examined and their
unique features are utilized to achieve oscillation diminishing properties and $p-$adaptivity.
Further, a quasi-linear reproducing kernel particle method (QL-RKPM) is also formulated.
The method introduces additional sampling points in the moving least-squares function
to avoid singularity of the moment matrix, while providing controllable error in the linear
field approximation when the kernel coverage conditions in the conventional reproducing
kernel approximation of linear field are not met. This technique is employed in modeling
a variety of impact and fragment-impact problems, and is shown to provide robust and
accurate results.

The generalized and quasi-linear reproducing kernel approximations are then
combined to solve linear elastic fracture mechanics problems. The enriched approxi-
mation surrounding crack tips is obtained by an intrinsic enrichment technique using
the generalized RKPM approach to minimize the discontinuity at the interface between
enriched and non-enriched regions. The displacement discontinuity along the crack sur-
face is captured with the visibility criterion, and the quasi-linear approximation ensures
invertibility of the moment matrix even with heavily truncated supports along the crack
face. The method is applied to different fracture mechanics boundary value problems,
and is shown to be accurate and effective without the need of tedious treatments in the
existing methods.
Chapter 1

Introduction
1.1 Motivation

Numerical simulations of problems involving blast loading, very large deformations, crack propagation or material fragmentation still remain challenging for computational mechanics. While some of the common issues associated with FEM for this class of problems, such as mesh entanglement and mesh distortion necessitating constant remeshing, can be alleviated with meshfree methods by constructing the approximation space based on a set of nodes only without the need for any mesh generation and mesh quality control, other complexities remain.

In meshfree methods, arbitrary crack geometry can be represented easily by simple approaches such as the visibility criterion, and does not require any remeshing technique. Accurate representation of the singular stress field occurring at a crack tip, however, requires more sophisticated approaches based on enrichment techniques. The approximation space can be locally enriched through extrinsic enrichment techniques, but this introduces additional degrees of freedom and does not take full advantage of the approximation space construction based on a moving least-squares approximation. In contrast, intrinsic enrichment directly adds enrichment functions in the basis of the moving least-squares approximation, and does not require any additional degrees of freedom in the global system of equations. Employing intrinsic enrichments on local regions of the domain however requires particular attention, as it can give rise to a discontinuous approximation space at the subdomain interfaces.

For simulations involving brittle materials under strong dynamics, the problem may not only involve fracture but also material damage, fragmentation and debris formation. While meshfree methods coupled with a semi-Lagrangian formulation are particularly suitable for this class of problem, maintaining linear consistency in this framework is not a simple task. Enforcing the proper kernel support coverage necessary
to maintain linear basis consistency while locally reducing the order of completeness of the approximation space in the fragmented area to circumvent kernel support coverage requirement is tedious. An alternative is to employ a constant order basis approximation in the entire domain, but it often yields poor solution accuracy.

### 1.2 Objectives

The objective of this work is to develop a meshfree formulation with enhanced accuracy and robustness for fracture and material fragmentation problems. The major tasks of this dissertation are summarized as follows:

- **Development of a generalization of the RKPM construction.** For a given support size and order of consistency, the construction gives rise to an entire set of approximation functions that reproduce polynomials. Various numerical examples are presented where the generalized approximation functions can be employed to obtain unique properties, notably for $p-$refinement and oscillation diminishing applications. Examples include 1D function reproduction, 2D Poisson problem and elasticity problem.

- **Development of a quasi-linear RKPM formulation.** An approach is proposed to construct an unconditionally non-singular moment matrix, while greatly enhancing accuracy compared to constant basis approximation. Robustness and accuracy of the method are demonstrated through numerical examples including wave propagation and a Taylor bar impact problem.

- **Application of the quasi-linear RKPM formulation to circumvent the kernel support coverage requirements in problems involving impact, large deformations and material fragmentation.** The method is employed for numerous concrete perforation
problems and the evaluation of a steel plate ballistic limit. Comparisons are made with constant basis approximation and experimental data, and the proposed formulation is shown to perform well.

- Application of generalized and quasi-linear RKPM to fracture modeling. The proposed technique is shown to provide enhanced accuracy for fracture problems with complex crack configurations. Comparisons are made with traditional RKPM to illustrate the benefit of the proposed methods to reduce discontinuities at enriched subdomains interfaces.

1.3 Outline

The remainder of this dissertation is organized as follows. An overview of meshfree methods is given in Chapter 2, along with common techniques employed for fracture modeling. The construction of the reproducing kernel particle method (RKPM) is briefly reviewed in Chapter 3, with a discussion on the Lagrangian and semi-Lagrangian formulations. This construction is generalized on Chapter 4 to a broader family of approximation functions, and different properties of the obtained approximation functions are discussed, with various numerical examples where this generalization can be exploited. A quasi-linear RKPM formulation is derived in Chapter 5, and its basic properties are studied through several numerical examples. This proposed approach is then validated in Chapter 6 through a series of transient numerical simulations involving impact, large deformations, fragmentation and debris formation, and comparisons are made with available experimental data. The generalized quasi-linear RKPM approximation is then applied to fracture modeling in Chapter 7. The generalized reproducing conditions are employed to minimize the discontinuity along crack-tip-enriched subdomains, and the quasi-linear framework is exploited to guarantee moment matrix invertibility in regions
with restricted supports arising from the visibility criterion along crack interfaces. Various numerical examples are provided and compared to available reference data. Finally, a summary of this work and concluding remarks are given in Chapter 8.
Chapter 2

Literature Review
2.1 Reproducing Kernel Particle Method

Meshfree methods offer an alternative to mesh-based numerical approximation techniques that proves extremely useful for a class of problems where connectivity is difficult to maintain through the entire simulation without severe mesh distortion. Such problems include metal forming [1], large deformations analysis [2], fracture modeling [3, 4], or fragment-impact problems [5], among others.

The reproducing kernel particle method (RKPM) [6, 7] is based on a kernel function to control locality and smoothness, and a correction function to enforce polynomial reproduction up to arbitrary order. Different constructions of kernel functions can be achieved to yield unique properties in the approximation functions, such as synchronized convergence phenomenon [8, 9] where similar convergence rates can be obtained in $L_2$ and $H^1$ norms. RKPM can also be combined with wavelet theory to yield built-in multiresolution capability [10, 11]. The correction function can be constructed in different ways by varying the reproducing equations, as proposed in the implicit gradient method [12] where gradient reproducing conditions are introduced to provide strain regularization in strain localization problems. The reproducing kernel element method [13, 14] introduces approximation functions possessing interpolation property and arbitrarily high order of continuity, but suffers from high oscillations. Most techniques built on the reproducing kernel approximation relies on polynomial reproducing conditions to yield desired accuracy, but the rational nature of RK approximation functions causes difficulties when numerically integrating the Galerkin weak equation. Integration error arising from Gauss integration on background cells can deteriorate the accuracy and convergence properties of the method [15], while some nodal integration techniques suffer from spatial instability [16]. Chen et al. [17] proposed a stabilized nodal integration that preserves linear exactness in the Galerkin form. This result was later extended to arbitrary
order exactness [18], providing a consistent framework to perform domain integration of the weak form in Galerkin meshfree methods. The arbitrary order of continuity of approximation functions also allows performing point collocation directly on the strong form [19]. This approach avoids domain integration altogether, but requires higher order derivatives to be constructed, which increases the computational cost of the method. This requirement can be alleviated using a gradient reproducing approach [20], where only first order differentiation is needed for solving second order PDE’s with strong form collocation approach.

### 2.2 Lagrangian and Semi-Lagrangian RKPM Formulation

RKPM has been employed with a Lagrangian formulation for a variety of problems involving structural dynamics [21], hyperelasticity [2], nearly incompressible finite elasticity [22, 23] and plasticity [7]. In this approach, the approximation functions are built in the undeformed configuration, and can be calculated only once at the preprocessing stage of the analysis. Due to the meshfree nature of the method, materials undergoing very large deformations can be modeled without experiencing loss of accuracy and difficulties associated with mesh distortion or entanglement [7].

For extremely large deformation problems, however, the necessary mapping between the deformed and undeformed configurations can break down, especially with material fragmentation where new surface formation occur. The semi-Lagrangian formulation [24, 25] circumvents this difficulty by constructing the approximation functions directly in the deformed configuration, and no mapping function between the deformed and undeformed configurations of the material are required. Von Neumann stability analysis of the semi-Lagrangian approach with stabilized nodal integration has been
studied [24, 25]. Due to the presence of a convective term in the formulation, the stability condition is dependent on the velocity gradient between adjacent nodes [5]. The semi-Lagrangian formulation has been successfully applied to earth moving simulations [25], fragment-impact problems [5, 26] and slope stability analysis [27] among others.

2.3 Meshfree Contact Algorithms

The smoothed particle hydrodynamics (SPH) method has been applied to various hypervelocity impact problems using simple contact detection algorithms [28], but the method suffers from tensile instability [29] and inaccurate contact surface estimation [30]. For RKPM, a smooth contact algorithm [31, 1] has been developed, using the RK approximation functions to represent the contact surface, which is particularly effective for sliding contact. This approach has been applied to contacts between rubber components [23] and metal forming simulations [32]. As opposed to traditional FEM $C^0$ contact formulations, the contact surface can be represented with arbitrary order of smoothness using RK approximation functions, alleviating convergence issues related to piecewise continuous contact surface representation in FEM.

Though this approach can provide an effective contact algorithm, it requires to define contact surfaces at a preprocessing stage. For problems involving impact-perforations, material separation and fragmentation, contact surfaces are continuously evolving and are part of the solution, and hence cannot be defined a priori. A natural kernel contact algorithm based on kernel interaction has been proposed to impose the non-penetration condition [5, 33]. This approach avoids the need for a priori definition of potential contact surfaces, and a level set function [34] has been used to dynamically represent free surfaces. The kernel contact algorithm introduces a fictitious elasto-plastic law to mimic the fraction law between contacting surfaces, and it has been successfully
applied to modeling of multi-body contact and fragmentation processes. [33].

2.4 Fracture Modeling

Modeling arbitrary crack growth accurately is a very challenging task. The earlier approaches for crack modeling were the finite element method with remeshing [35], and the boundary element method [36]. FEM with remeshing requires the mesh to be made conforming with the crack surface at each step, often requiring manual intervention to keep a mesh of good quality. Moreover, the remeshing process requires interpolation of the field and state variables such as displacements, strains and stresses to the new set of nodes and Gauss points at each remeshing step, resulting in additional error in the approximation. The boundary element method avoids a large part of this remeshing process, as only the crack surface mesh needs to be rebuilt, but it can only be applied to a restricted set of problems where the adequate Green’s functions are known. Belytschko and Black [37] introduced the extended finite element method (XFEM) as an improvement of the finite element method, where enrichment functions are added to the approximation space to account for the discontinuity at the crack surfaces, and to accurately represent the asymptotic stress field at the crack tip. The finite element mesh no longer needs to be conforming to the geometry of the crack, alleviating most of the remeshing process.

Meshfree methods offer a good alternative to model crack propagation, since they don’t require any connectivity information between the nodes, and the continuity of the approximation functions can be varied easily. To account for the discontinuity in the displacement field along the crack surfaces, Belytschko et al. [38] proposed the visibility criterion. In this method, the support of the shape functions is simply truncated along the crack surface. In the early efforts, this technique was used along with a high concentration of nodes around the crack tip to provide an accurate stress field [3].
However, the visibility criterion introduces spurious discontinuities in the shape functions that cover the crack tip, which can lead to a loss of accuracy [39]. Organ et al. [40] later proposed the transparency criterion and the diffraction criterion. This approach restores continuity of the shape functions around the crack tip, but fails to capture the associated singular stress field.

Meshfree methods such as EFG and RKPM offer the freedom of constructing the desired approximation space. The typical construction involves monomials up to an arbitrary degree, but other near-tip asymptotic functions can also be added to the basis vectors, to enrich the approximation space. This approach, usually referred to as intrinsic enrichment, was introduced by Belytschko et al. [41] and Fleming et al. [39], to model crack growth. In this approach, trigonometric functions extracted from the asymptotic field around the crack tip are directly added to the bases. Including these additional functions in the entire domain can lead to a significant increase of computational cost and ill-conditioning of the moment matrices. However, the classical construction of the approximation functions based on MLS interpolation generally does not allow to use different approximation spaces in different subdomains without introducing discontinuities at their interfaces. Belytschko et al. [41] overcome this limitation by introducing a ramping function over a smoothing zone around those interfaces. Fleming et al. [39] also identified that in some specific cases, the discontinuity induced by this change of basis can be neglectable. Another approach is to augment the approximation space extrinsically. In this technique, an additional partition of unity is built locally [42], that can also be based on a MLS approximation. This partition of unity is then used to patch the required additional basis vectors, and gives rise to an extra set of unknowns to be solved. This technique can be traced back to Duarte and Oden’s $h$-$p$ cloud method [43], where all the basis functions desired in the approximation space are introduced extrinsically. To localize this partition of unity, blending functions are commonly used to ramp down the
enrichment functions away from the crack tip. However, this can lead to a loss of accuracy in the blending region, and can deteriorate the convergence rate of the method [44].
Chapter 3

Reproducing Kernel Particle Method

Formulation
3.1 Reproducing Kernel (RK) Approximation

3.1.1 Continuous Form

Consider a function $u$ over a domain $\Omega$. We consider the following convolution integral:

$$u^k(x) = \int_{\Omega} \phi_a(x - s) u(s) \, ds$$

(3.1)

where $\phi_a(x - s)$ is a kernel function of support $a$. Equation (3.1) is known as the kernel estimation of the function $u$. This kernel estimate is applied to the approximation of partial differential equations in SPH, but it fails to satisfy the completeness requirement. The main idea of the RK formulation is to introduce a correction function $C(x; x - s)$ such that the function

$$u^h(x) = \int_{\Omega} C(x; x - s) \phi_a(x - s) u(s) \, ds$$

(3.2)

is an approximation of $u$ that is consistent up to a chosen degree $N$.

Following the work of Liu et al. [6], we assume the following form for the correction function:

$$C(x; x - s) = H^T(x - s)b(x)$$

(3.3)

where $H$ is a vector containing all the basis functions:

$$H^T(x) = \begin{bmatrix} 1 & x & x^2 & \ldots & x^N \end{bmatrix}$$

(3.4)

and $b(x)$ is a set of functions that is to be determined. To obtain it, we first introduce the
Taylor expansion of $u(s)$:

$$u(s) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (x-s)^i u^{(i)}(x)$$  

where $u^{(i)}(x)$ stands for the $i$–th derivative of $u$ at point $x$. Substituting (3.5) into (3.2), we obtain

$$u^h(x) = \bar{m}_0(x)u(x) + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \bar{m}_i(x) u^{(i)}(x)$$  

where

$$\bar{m}_i(x) = \int_{\Omega} (x-s)^i H^T(x-s) b(x) \phi_a(x-s) \, ds$$  

For $u^h(x)$ to be an approximation of $u(x)$ up to order $N$, (3.6) states that we need to have $\bar{m}_0(x) = 1$ and $\forall i \in \{1, \ldots, N\} \, \bar{m}_i(x) = 0$. This set of conditions leads to the following equation:

$$M(x)b(x) = H(0)$$  

where

$$M(x) = \int_{\Omega} H(x-s) H^T(x-s) \phi_a(x-s) \, ds$$  

and

$$H(0) = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$$

The matrix $M$ is called the moment matrix in the RKPM terminology. (3.8) lets us determine the coefficients $b(x)$:

$$b(x) = M^{-1}(x)H(0)$$

We can now obtain the modified kernel function that satisfies the completeness
requirement
\[ \Psi(x; x - s) = H^T(0)M^{-1}(x)H(x - s)\phi_a(x - s) \] (3.12)

\( \Psi(x; x - s) \) is called a reproducing kernel approximation function. We can observe that
the smoothness of the approximated solution only depends on the smoothness of the
chosen kernel function: If \( \phi_a(x - s) \) is of class \( C^k \), then \( \Psi(x; x - s) \) is also of class \( C^k \).

B-Splines are commonly used as kernel functions, since they offer compact support size
and tunable order of continuity. An example of kernel function is the cubic B-Spline:

\[
\phi_a(x - s) = \begin{cases} 
\frac{2}{3} - 4z^2 + 4z^3 & \text{for } 0 \leq z_a \leq 0.5 \\
\frac{4}{3} - 4z + 4z^2 + \frac{4}{3}z^3 & \text{for } 0.5 \leq z_a \leq 1 \\
0 & \text{for } z_a > 1 
\end{cases}
\] (3.13)

It is also worth mentioning that in general, the set of basis functions used to build
the approximation space needs to be kept identical in the entire physical domain. Failure
to do so would result in discontinuities in the approximation functions. For an example, see Fig. 4.3.

### 3.1.2 Multi-Dimensional Formulation

This method can be extended to a domain in \( n_d \) dimensions. The multi-dimensional
kernel function can be built as the product of one-dimensional kernel functions:

\[
\phi_a(x - s) = \prod_{i=1}^{n_d} \phi_a(x_i - s_i) 
\] (3.14)

and the correction function is expressed as:

\[
C(x; x - s) = H^T(\theta)M^{-1}(x)H(x - s) 
\] (3.15)
where $H(x - s)$ is the multi-dimensional vector containing all the basis functions:

$$H^T(x - s) = \begin{bmatrix} 1 & x_1 - s_1 & x_2 - s_2 & \ldots & (x_{n_d} - s_{n_d})^N \end{bmatrix} \quad (3.16)$$

and the moment matrix $M(x)$ becomes:

$$M(x) = \int_\Omega H(x - s)H^T(x - s) \, ds_1 \ldots ds_{n_d} \quad (3.17)$$

The approximation functions are constructed similarly to the previous section:

$$\Psi(x; x - s) = H^T(0)M^{-1}(x)H(x - s)\phi_a(x - s) \quad (3.18)$$

### 3.1.3 Discrete Form in Multi-Dimensions

Following the work of Chen et al. [7], we can also introduce a consistent discrete formulation that still gives rise to a complete approximation function up to a chosen degree $N$. Let the domain be discretized in set of nodes $x_I$, with $I \in \{1,2,\ldots, NP\}$. The discrete moment matrix $M$ can be written as

$$M(x) = \sum_{I=1}^{NP} H(x - x_I)H^T(x - x_I)\phi_a(x - x_I) \quad (3.19)$$

At each node $x_I$ we can construct an approximation function $\Psi_I(x)$ (sometimes called shape function by analogy to FEM)

$$\Psi_I(x) = \Psi(x; x - x_I) = H^T(0)M^{-1}(x)H(x - x_I)\phi_a(x - x_I) \quad (3.20)$$
and the approximation field constructed from the discrete formulation gives

\[ u^h(x) = \sum_{I=1}^{NP} \Psi_I(x) u_I \]  

(3.21)

where the coefficients \( u_I \), with \( I \in \{1, 2, \ldots, NP\} \), are the nodal coefficients of the approximation. It is worth mentioning that the approximation functions are not nodal interpolation functions, and therefore in general we have \( u^h(x_I) = \sum_{J=1}^{NP} \Psi_J(x_I) u_J \neq u_I \).

### 3.2 Lagrangian and Semi-Lagrangian Formulations

An overview of the Lagrangian and semi-Lagrangian approximation is given in this section. Consider a body initially occupying the domain \( \Omega_X \), with undeformed natural boundary \( \Gamma^h_X \). We denote by \( \rho^0 \), \( b^0 \) and \( h^0 \) the initial density, body force expressed in the undeformed configuration and surface traction on the undeformed natural boundary, respectively. The material displacement is defined as \( u = x - X \) where \( X \) are the material coordinates and \( x \) are the spatial coordinates. The mapping function \( \varphi(X,t) \) defines the relationship between material and spatial coordinates by \( x = \varphi(X,t) \). After deformation the body occupies the domain \( \Omega_x \) with deformed natural boundary \( \Gamma^h_x \). We denote by \( \rho \), \( b \) and \( h \) the density in the deformed state, body force in the deformed configuration and surface traction on the deformed natural boundary, respectively.

#### 3.2.1 Lagrangian Formulation

The equation of motion in the Lagrangian formulation is expressed as

\[
\int_{\Omega_x} \delta u_i \rho^0 u_i \, d\Omega + \int_{\Omega_x} \delta F_{ij} P_{ji} \, d\Omega = \int_{\Omega_x} \delta u_i b_i^0 \, d\Omega + \int_{\Gamma^h_x} \delta u_i h_i^0 \, d\Gamma, 
\]  

(3.22)
where $F$ is the deformation gradient defined as $F_{ij} = \partial x_i / \partial X_j$ and $P$ is the first Piola-Kirchhoff stress tensor. The Lagrangian approximation functions $\{\Psi^L_I\}$ deform with the material, as shown in Fig. 3.1b, and are calculated in the undeformed configuration by

$$\Psi^L_I(X) = H^T(\theta)M^{-1}(X)H(X - X_I)\phi_a(X - X_I).$$ \quad (3.23)

### 3.2.2 Semi-Lagrangian Formulation

For most path-dependent materials, the Cauchy stress is employed as the stress measure, which requires to calculate the spatial derivatives of the displacement. In the Lagrangian formulation, this is obtained through the following chain rule:

$$\frac{\partial \Psi^L_I(X)}{\partial x_i} = \frac{\partial \Psi^L_I(X)}{\partial X_j} \frac{\partial X_j}{\partial x_i} = \frac{\partial \Psi^L_I(X)}{\partial X_j} F^{-1}_{ji}$$ \quad (3.24)

where $F^{-1}$ is calculated as the direct inverse of the deformation gradient $F$. This approach breaks down when the mapping function $\mathbf{x} = \phi(\mathbf{X}, t)$ can no longer be defined as a one-to-one relationship, which is typically the case in problems involving change of topology due to material separation or fragmentation, such as earth moving or material perforation processes. The Semi-Lagrangian formulation circumvents this limitation, by
expressing the equation of motion as

\[
\int_{\Omega_x} \delta u_i \rho \ddot{u}_i \, d\Omega + \int_{\Omega_x} \delta u_{(i,j)} \sigma_{ji} \, d\Omega = \int_{\Omega_x} \delta u_i b_i \, d\Omega + \int_{\Gamma^h_x} \delta u_i h_i \, d\Gamma,
\]  

(3.25)

where \(u_{(i,j)} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i)/2\) and \(\sigma\) is the Cauchy stress. As shown in Fig. 3.1c, the kernel functions follow the nodes, but do not deform with the material. The semi-Lagrangian approximation functions \(\{\Psi^{SL}_I\}\) are expressed in the deformed configuration as

\[
\Psi^{SL}_I(x) = H^T(0)M^{-1}(x)H(x - x_I)\phi_a(x - x_I).
\]

(3.26)
Chapter 4

Generalization of the Reproducing Equations
4.1 Generalized Construction of Reproducing Kernel Approximation

In this section, we introduce the concept of polynomial reproduction, and characterize a set of functions that can reproduce this function space. The original construction of RKPM approximation functions is shown to belong in this set, and a broader family of functions is explicitly constructed.

4.1.1 Function space reproduction

The original construction of RKPM is based on function reproduction rather than function space reproduction. In its original construction, a polynomial function of a given order is considered, and a convolution with a kernel function is derived, so as to exactly reproduce the given polynomial function. In this new construction, the formulation is generalized to the reproduction of the entire space of polynomial function of a given order. In what follows, reproducing conditions on the entire function space is considered, and this leaves the freedom to reproduce a member function \( f \) of this set using any other function \( g \) belonging to the same space. Though the same idea can be applied to derive reproducing conditions for broader function spaces, we will here concentrate on the polynomial space. Let \( \mathcal{P}(N) \) be the space of polynomial functions of degree \( n \) over the domain \( \Omega \). In what follows, a function \( \Psi(x; x - s) \) is said to reproduce polynomials of order \( n \) if for any function \( u \in \mathcal{P}(N) \) there exists \( \tilde{u} \in \mathcal{P}(N) \) such that

\[
    u(x) = \int_{\Omega} \Psi(x; x - s) \tilde{u}(s) \, ds \quad \forall x \in \Omega.
\]  

(4.1)
4.1.2 Preliminary result

Let \( \tilde{u} \in P(N) \). Taylor expansion of \( \tilde{u} \) gives

\[
\tilde{u}(s) = \tilde{u}(x) + \frac{(-1)^1}{1!} (x - s)^1 \tilde{u}^{(1)}(x) + \ldots + \frac{(-1)^n}{n!} (x - s)^n \tilde{u}^{(n)}(x),
\]

where \( x, s \in \Omega \) and \( \tilde{u}^{(i)} \) denotes the \( i \)-th derivative of \( \tilde{u} \). This can be expressed in a matrix form:

\[
\tilde{u}(s) = H^T(x - s) J \tilde{D}_{\tilde{u}}(x),
\]

where the following notations have been used

\[
H^T(x - s) = \begin{bmatrix}
1 & x - s & \ldots & (x - s)^n
\end{bmatrix},
\]

\[
J = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & \frac{(-1)^1}{1!} & \ldots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \frac{(-1)^n}{n!}
\end{bmatrix},
\]

\[
\tilde{D}_{\tilde{u}}(x) = \begin{bmatrix}
\tilde{u}(x) & \tilde{u}^{(1)}(x) & \ldots & \tilde{u}^{(n)}(x)
\end{bmatrix}.
\]

Multiplying (4.3) by \( H(x - s) \phi_a(x - s) \) on both sides and integrating over \( s \), we obtain:

\[
\int_{\Omega} H(x - s) \phi_a(x - s) \tilde{u}(s) \, ds = J \int_{\Omega} H(x - s) H^T(x - s) \phi_a(x - s) \, ds \tilde{D}_{\tilde{u}}(x)
\]

\[
= J M(x) \tilde{D}_{\tilde{u}}(x)
\]

\[
= Q(x) \tilde{D}_{\tilde{u}}(x),
\]
where $Q(x) = JM(x)$ and the moment matrix is defined as

$$M(x) = \int_{\Omega} H(x-s)H^T(x-s)\phi_a(x-s) \, ds.$$  \hspace{1cm} (4.10)

This yields the following equation

$$D\tilde{u}(x) = Q^{-1}(x)\int_{\Omega} H(x-s)\phi_a(x-s)\tilde{u}(s) \, ds.$$  \hspace{1cm} (4.11)

This result, which will be used in the construction of the generalized approximation functions below, simply transforms the set containing all the derivatives of a polynomial function at any position into a construction that involves only the function itself and a set of monomial basis functions that spans the space $\mathcal{P}(N)$.

### 4.1.3 Construction

Here, we intend to construct a reproducing kernel function $\Psi$ so that for a given $u \in \mathcal{P}(N)$, there exists $\tilde{u} \in \mathcal{P}(N)$ such that the following reproducing condition is satisfied:

$$u(x) = \int_{\Omega} \Psi(x;x-s)\tilde{u}(s) \, ds.$$  \hspace{1cm} (4.12)

Let $u$ and $\tilde{u}$ be expressed as

$$u(x) = \sum_{i=0}^{n} \alpha_i x^i = \alpha^T H(x),$$  \hspace{1cm} (4.13)

$$\tilde{u}(x) = \sum_{i=0}^{n} \beta_i x^i = \beta^T H(x).$$  \hspace{1cm} (4.14)

We can assume the following form for the approximation function

$$\Psi(x;x-s) = b^T(x)H(x-s)\phi_a(x-s),$$  \hspace{1cm} (4.15)
where \( b(x) \) is a vector of coefficients to be determined. Plugging this definition into (4.12) yields

\[
u(x) = \int_\Omega b^T(x) H(x-s) \phi_a(x-s) \tilde{u}(s) \, ds.
\]  

(4.16)

Combining (4.11) and (4.16) gives

\[
u(x) = b^T(x) Q(x) D\tilde{u}(x)
\]  

(4.17)

and \( D\tilde{u}(x) \) can be decomposed as:

\[
D\tilde{u}(x) = BH(x),
\]  

(4.18)

where

\[
B = \begin{bmatrix}
\beta_0 \frac{0!}{0!} & \beta_1 \frac{1!}{1!} & \cdots & \beta_{n-1} \frac{(n-1)!}{(n-1)!} & \beta_n \frac{n!}{n!} \\
\beta_1 \frac{1!}{1!} & \beta_2 \frac{2!}{2!} & \cdots & \beta_n \frac{n!}{(n-1)!} & 0 \\
\beta_2 \frac{2!}{2!} & \beta_3 \frac{3!}{3!} & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
\beta_n \frac{n!}{n!} & 0 & \cdots & \cdots & 0
\end{bmatrix}
\]  

(4.19)

We obtain

\[
\alpha^T = b^T(x) Q(x) B.
\]  

(4.20)
Since $\tilde{u}$ is of degree $n$, $\beta_n \neq 0$, hence the matrix $B$ defined in (4.19) is always invertible.

We can obtain:

$$\Psi(x;x-s) = \alpha^T B^{-1} Q^{-1}(x) H(x-s) \phi_u(x-s), \quad (4.21)$$

or

$$\Psi(x;x-s) = c^T Q^{-1}(x) H(x-s) \phi_u(x-s), \quad (4.22)$$

where

$$c^T = \alpha^T B^{-1}. \quad (4.23)$$

(4.23) can be shown equivalent to:

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} c_0 0! & c_1 1! & \ldots & c_{n-1} \frac{(n-1)!}{0!} & c_n \frac{n!}{0!} \\ 0 & c_0 1! & \ldots & c_{n-2} \frac{(n-1)!}{1!} & c_{n-1} \frac{n!}{1!} \\ 0 & 0 & \ddots & \ldots & c_{n-2} \frac{n!}{2!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & c_0 \frac{n!}{n!} & \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}, \quad (4.24)$$

with $c_0 \neq 0$. Since $u$ was chosen as an arbitrary member of the function space $\mathcal{P}(N)$, (4.24) provides a systematic construction of $\tilde{u}$, and the approximation function $\Psi$ defined in (4.22) is shown to reproduce the entire $\mathcal{P}(N)$ function space. Finally, since $c$ was also chosen arbitrarily with the only constraint that $c_0 \neq 0$, we can define a generalized
reproducing kernel approximation as:

\[
\mathcal{H} = \left\{ \Psi(x; x-s) = c^T Q^{-1}(x) \mathbf{H}(x-s) \phi_a(x-s) \mid (c \in \mathbb{R}^{n+1}, c_0 \neq 0) \right\}. \quad (4.25)
\]

By definition, every function member of the set \( \mathcal{H} \) reproduces the polynomial space \( \mathcal{P}(N) \). It follows from (4.24) that if we define \( c = \mathbf{H}(0) \), we recover the well-known reproducing equation:

\[
u(x) = \int_{\Omega} \Psi(x; x-s) u(s) \, ds \quad \forall u \in \mathcal{P}(N) \quad \forall x \in \Omega, \quad (4.26)
\]

where

\[
\Psi(x; x-s) = \mathbf{H}^T(0) Q^{-1}(x) \mathbf{H}(x-s) \phi_a(x-s). \quad (4.27)
\]

### 4.2 Discrete formulation

Let the domain \( \Omega \) be discretized with \( N_P \) nodes \( \{x_I \mid x_I \in \Omega \}_{I=1}^{N_P} \). Similarly, we can construct a discrete reproducing kernel function \( \Psi_I(x) \) so that for a given \( u \in \mathcal{P}(N) \), there exists \( \tilde{u} \in \mathcal{P}(N) \) such that the following discrete reproducing condition is satisfied:

\[
u(x) = \sum_{I=1}^{N_P} \Psi_I(x) \tilde{u}(x_I). \quad (4.28)
\]

Let \( \tilde{u} \in \mathcal{P}(N) \) be be expressed by a Taylor expansion as:

\[
\tilde{u}(x_I) = \tilde{u}(x) + \frac{(-1)^1}{1!} (x-x_I)^1 \tilde{u}^{(1)}(x) + \ldots + \frac{(-1)^n}{n!} (x-x_I)^n \tilde{u}^{(n)}(x) \quad (4.29)
\]

\[
= \mathbf{H}^T(x-x_I) \mathbf{J} \mathbf{D} \tilde{u}(x). \quad (4.30)
\]
Multiplying (4.30) by \( \mathbf{H}(x-x_I)\phi_a(x-x_I) \) on both sides and performing summation over \( I \), we obtain:

\[
\sum_{I=1}^{N_p} \mathbf{H}(x-x_I)\phi_a(x-x_I)\tilde{u}(x_I) = \mathbf{J} \sum_{I=1}^{N_p} \mathbf{H}(x-x_I)\mathbf{H}^T(x-x_I)\phi_a(x-x_I)\mathbf{D}_{\tilde{a}}(x) \quad (4.31)
\]

\[
= \mathbf{J} \tilde{\mathbf{M}}(x)\mathbf{D}_{\tilde{a}}(x) \quad (4.32)
\]

\[
= \tilde{\mathbf{Q}}(x)\mathbf{D}_{\tilde{a}}(x), \quad (4.33)
\]

where \( \tilde{\mathbf{Q}}(x) = \mathbf{J} \tilde{\mathbf{M}}(x) \) and the discrete moment matrix is defined as

\[
\tilde{\mathbf{M}}(x) = \sum_{I=1}^{N_p} \mathbf{H}(x-x_I)\mathbf{H}^T(x-x_I)\phi_a(x-x_I). \quad (4.34)
\]

The discrete approximation function is assumed to take the following form:

\[
\Psi_I(x) = \tilde{\mathbf{b}}^T(x)\mathbf{H}(x-x_I)\phi_a(x-x_I), \quad (4.35)
\]

where \( \tilde{\mathbf{b}}(x) \) is a vector of coefficients to be determined. Plugging this definition into (4.28) yields

\[
u(x) = \tilde{\mathbf{b}}^T(x) \sum_{I=1}^{N_p} \mathbf{H}(x-x_I)\phi_a(x-x_I)\tilde{u}(x_I). \quad (4.36)
\]

Combining (4.33) and (4.36) gives

\[
u(x) = \tilde{\mathbf{b}}^T(x)\tilde{\mathbf{Q}}(x)\mathbf{D}_{\tilde{a}}(x), \quad (4.37)
\]

which is now exactly identical to (4.17), leading to the similar result:

\[
\Psi_I(x) = \mathbf{c}^T \tilde{\mathbf{Q}}^{-1}(x)\mathbf{H}(x-x_I)\phi_a(x-x_I). \quad (4.38)
\]
was chosen as an arbitrary member of the function space \( \mathcal{P}(N) \), and (4.24) can be used to construct \( \tilde{u} \). The discrete approximation function \( \Psi_I(x) \) defined in (4.37) is shown to reproduce the entire \( \mathcal{P}(N) \) function space.

The derived generalization gives rise to a new set of approximation functions. For a given discretization, order of basis functions and kernel function, a series of approximation functions can be obtained by varying the vector \( c \) in the construction, as illustrated in Fig. 4.1 and Fig. 4.2 where different approximation functions with quadratic basis are constructed by varying \( c_1 \) and \( c_2 \), respectively.

As discussed above, a family of generalized reproducing kernel functions can be constructed to yield various mappings between the original functions and the reproduced functions by controlling different values of \( c \). For the example shown below, two sets of \( c \) values yield two reproducing kernels which reproduce the same quadratic function, from two different polynomials:

- for \( c^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \) we obtain the usual reproducing equations for any \( x \in \Omega \)

\[
\sum_I [\Psi_I(x) \times 1] = 1, 
\]
\[
\sum_I [\Psi_I(x) \times x_I] = x, 
\]
\[
\sum_I [\Psi_I(x) \times x_I^2] = x^2. 
\]
Figure 4.1: Generalized Approximation functions built with quadratic basis, \( e^T = [1 \ \alpha \ 0] \) where \( \alpha \) varies from \(-1\) (short dashes) to 1 (long dashes). The original construction yields the thick black curve.

- for \( e^T = [1 \ 1 \ 1] \), these reproducing equations become

\[
\sum_I [\Psi_I(x) \times 1] = 1, \tag{4.42}
\]

\[
\sum_I [\Psi_I(x) \times (x_I - 1)] = x, \tag{4.43}
\]

\[
\sum_I [\Psi_I(x) \times (x_I^2 - 2x_I)] = x^2. \tag{4.44}
\]

A family of reproducing kernels constructed by different choices of \( e \) is shown in Fig. 4.1 and Fig. 4.2.

4.3 Applications to \( p \)-Refinement and Oscillation

4.3.1 Application to \( p \)-refinement

Attempting to vary the order of approximation in the domain generally leads to discontinuities at the sub-domains interfaces. In this section we show how to take advan-
Figure 4.2: Generalized Approximation functions built with quadratic basis, $c^T = \begin{bmatrix} 1 & 0 & \alpha \end{bmatrix}$ where $\alpha$ varies from $-1$ (short dashes) to $1$ (long dashes). The original construction yields the thick black curve.

tage of the G-RKPM previously introduced to locally vary the order of approximation without discontinuity.

To demonstrate, consider a domain $\Omega = [0, 10]$, discretized with $N_p$ nodes, and split into two sub-domains: for $x \in [0, 5.5]$ a linear approximation space is used, whereas for $x \in (5.5, 10]$ a quadratic approximation space is used. In other words, the approximation field becomes

$$u^h(x) = \begin{cases} \sum_{I=1}^{N_p} \Psi^1_I u_I & \text{for } x \in [0, 5.5] \\
\sum_{I=1}^{N_p} \Psi^2_I u_I & \text{for } x \in (5.5, 10] ,
\end{cases} \quad (4.45)$$

where $\Psi^1_I$ denotes approximation functions built with a linear basis, and $\Psi^2_I$ denotes approximation functions built with a quadratic basis. Using the standard RK approximation ($c^T = H^T(0)$) to construct the two sets of approximation functions leads to a discontinuity at the interface point $x = 5.5$ as shown in Fig. 4.3.

The generalized construction introduced in this paper, however, gives us more freedom in the choice of reproducing kernel approximation function. We can solve for
the appropriate $c^T$ vector such that, at $x = 5.5$, we obtain

$$\Psi^l_1(5.5) = \Psi^l_2(5.5) \quad (4.46a)$$

$$\frac{d\Psi^l_1}{dx}(5.5) = \frac{d\Psi^l_2}{dx}(5.5) \quad (4.46b)$$

This gives rise to the approximation functions illustrated in Fig. 4.4, that are designed with $C^1$ continuity.

The impact of this result when solving PDEs is studied in a numerical example in Section 4.4.1. When certain conditions are met, this construction can be extended to a multidimensional domain. Section 4.4.2 provides numerical examples showing the performance of the method for two dimensional Poisson and linear elasticity problems.
4.3.2 Oscillation diminishing properties

In the standard RK approximation, even with a smooth kernel function, the approximation functions and their derivatives can present oscillations as illustrated in Figures 4.5a, 4.5c, 4.5e. The G-RKPM approach offers an opportunity to minimize the oscillations of the approximation functions derivatives. The derivatives of the approximation functions can be decomposed into three terms:

\[ \Psi_{I,x}(x) = e^T \tilde{Q}^{-1}(x) H x(x - x_I) \phi_{a,x}(x - x_I) \]
\[ + e^T \tilde{Q}^{-1}(x) \tilde{H} x(x - x_I) \phi_a(x - x_I) \]
\[ + e^T \tilde{Q}^{-1}(x) H(x - x_I) \phi_a(x - x_I) \] (4.47)

Undesirable oscillations can be identified to rise from the third term containing the derivative of the moment matrix. Minimizing the total variation of this term can lead to much smoother approximation functions and derivatives, as illustrated in Fig. 4.5. A direct comparison of the G-RKPM approximation functions with the standard construction can be found in Fig. 4.6.

This phenomenon can also be illustrated from a signal analysis point of view, by analyzing the Fourier transform of the approximation functions:

\[ F(\omega_1, \omega_2) = \int_{\Omega} \Psi_I(x) e^{-i(\omega_1 x_1 + \omega_2 x_2)} \, d\Omega. \] (4.48)

As discussed in [45], reproducing kernel approximation functions form low-pass filters, as illustrated in Fig. 4.7a. An important side-lobe is also visible at higher frequencies, as discussed in [46], thus aliasing. The G-RK approximation functions give much lower amplitude in side lobes and are hence less oscillatory, as shown in Fig. 4.7b. A numerical example studying the impact of this property on the solution of a boundary value problem
Figure 4.5: Oscillations in the approximation functions and their derivatives.
4.4 Numerical Examples

4.4.1 \( p \)-refinement

To demonstrate the \( p \)-refinement capabilities of the method, we will solve numerically the following boundary value problem, we will solve numerically the following boundary value problem

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= -\frac{1}{10^3}x \\
 u(0) &= 0 \\
 u(10) &= 1
\end{align*}
\] (4.49)
(a) Fourier transform of a second order complete RKPM approximation function – Amplitude.

(b) Fourier transform of a second order complete G-RKPM approximation function – Amplitude.

**Figure 4.7**: Filter properties of RKPM and G-RKPM approximation functions.
whose analytical solution is known:

\[ u(x) = \frac{1}{60} (106x - x^3). \]  \hspace{1cm} (4.50)

We will study three different measures of error, with different discretizations:

- The error on the left side of the domain, where a linear base is used,
- The error on the right side of the domain, where a quadratic base is used,
- The error on the total domain.

These three errors are shown when using the previously introduced generalization to build \( C^1 \) approximation functions shown in Fig. 4.4 (referred as “G-RKPM” in the results), and when using the traditional construction of the approximation functions, leading to a discontinuity, shown in Fig. 4.3 (referred as “RKPM” in the results).

As a reference, we also show, for these three different parts of the domain (left, right, total):

- The error we obtain when using a linear basis on the entire domain,
- The error we obtain when using a quadratic basis on the entire domain.

A complete convergence study can be found in Fig. 4.8 and Tables 4.1-4.3 for the \( L^2 \) norm, and in Fig. 4.9 as well as Tables 4.4-4.6 for the \( H^1 \) norm. A distribution of the error in the domain can also be found in Fig. 4.10. As expected, the formulation containing a strong discontinuity at \( x = 5.5 \) produces large errors and converges poorly (rates of convergence of approximately 1.25 and 0.52 in the \( L^2 \) norm and \( H^1 \) norm, respectively), whether we analyze the total error, or the error on each sub-domain separately. Using the G-RKPM formulation, however, leads to much lower errors, and asymptotic rates of convergence are obtained:
Table 4.1: Error in the $L_2$ norm, on the left part of the domain only.

<table>
<thead>
<tr>
<th>Left</th>
<th>G-RKPM</th>
<th>RKPM Discontinuous</th>
<th>RKPM 1st order</th>
<th>RKPM 2nd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$e_{L_2}$</td>
<td>rate</td>
<td>$e_{L_2}$</td>
<td>rate</td>
</tr>
<tr>
<td>1.0000</td>
<td>1.03E-03</td>
<td></td>
<td>1.15E-01</td>
<td></td>
</tr>
<tr>
<td>0.5000</td>
<td>1.57E-04</td>
<td>2.71</td>
<td>4.05E-02</td>
<td></td>
</tr>
<tr>
<td>0.2500</td>
<td>3.54E-05</td>
<td>2.15</td>
<td>1.44E-02</td>
<td>1.50</td>
</tr>
<tr>
<td>0.1250</td>
<td>8.69E-06</td>
<td>2.03</td>
<td>5.68E-03</td>
<td>1.34</td>
</tr>
<tr>
<td>0.0625</td>
<td>2.17E-06</td>
<td>2.00</td>
<td>2.45E-03</td>
<td>1.21</td>
</tr>
</tbody>
</table>

Table 4.2: Error in the $L_2$ norm, on the right part of the domain only.

<table>
<thead>
<tr>
<th>Right</th>
<th>G-RKPM</th>
<th>RKPM Discontinuous</th>
<th>RKPM 1st order</th>
<th>RKPM 2nd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$e_{L_2}$</td>
<td>rate</td>
<td>$e_{L_2}$</td>
<td>rate</td>
</tr>
<tr>
<td>1.0000</td>
<td>1.65E-03</td>
<td></td>
<td>1.24E-01</td>
<td></td>
</tr>
<tr>
<td>0.5000</td>
<td>1.43E-04</td>
<td>3.53</td>
<td>4.27E-02</td>
<td>1.53</td>
</tr>
<tr>
<td>0.2500</td>
<td>1.32E-05</td>
<td>3.44</td>
<td>1.43E-02</td>
<td>1.58</td>
</tr>
<tr>
<td>0.1250</td>
<td>1.29E-06</td>
<td>3.35</td>
<td>5.23E-03</td>
<td>1.45</td>
</tr>
<tr>
<td>0.0625</td>
<td>1.48E-07</td>
<td>3.13</td>
<td>2.10E-03</td>
<td>1.32</td>
</tr>
</tbody>
</table>

- On the left side of the domain, where the approximation space contains a linear basis only, the rate of convergence reaches its theoretical value of 2.00 in the $L_2$ norm and 1.00 in the $H^1$ norm,

- On the right side of the domain, where a quadratic basis is used, we also reach the optimal rate of convergence of 3.00 in the $L_2$ norm and 2.00 in the $H^1$ norm,

- The total error on the entire domain is still dominated by the error on the left side, as expected, and therefore also converges at a rate of 2.00 and 1.00 for $L_2$ and $H^1$ norms, respectively.

Fig. 4.10a and Fig. 4.10b also illustrate clearly that in the discontinuous RKPM approach, the discontinuity causes the error to be distributed in the entire domain for the displacement field, and concentrated at the discontinuity for the strain field, whereas in the G-RKPM formulation, the dominant error can be found in the region treated with a
Table 4.3: Error in the $L_2$ norm, on the entire domain.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$e_{L_2}$</th>
<th>rate</th>
<th>$e_{L_2}$</th>
<th>rate</th>
<th>$e_{L_2}$</th>
<th>rate</th>
<th>$e_{L_2}$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000</td>
<td>1.95E-03</td>
<td></td>
<td>1.69E-01</td>
<td></td>
<td>1.31E-02</td>
<td></td>
<td>4.22E-03</td>
<td></td>
</tr>
<tr>
<td>0.5000</td>
<td>2.13E-04</td>
<td>3.19</td>
<td>5.89E-02</td>
<td>1.52</td>
<td>2.25E-03</td>
<td>2.55</td>
<td>4.20E-04</td>
<td>3.33</td>
</tr>
<tr>
<td>0.2500</td>
<td>3.78E-05</td>
<td>2.49</td>
<td>2.03E-02</td>
<td>1.54</td>
<td>3.96E-04</td>
<td>2.51</td>
<td>4.43E-05</td>
<td>3.24</td>
</tr>
<tr>
<td>0.1250</td>
<td>8.79E-06</td>
<td>2.10</td>
<td>7.72E-03</td>
<td>1.39</td>
<td>7.11E-05</td>
<td>2.48</td>
<td>4.94E-06</td>
<td>3.16</td>
</tr>
<tr>
<td>0.0625</td>
<td>2.17E-06</td>
<td>2.02</td>
<td>3.22E-03</td>
<td>1.26</td>
<td>1.31E-05</td>
<td>2.44</td>
<td>5.78E-07</td>
<td>3.10</td>
</tr>
</tbody>
</table>

Figure 4.8: Convergence study in the $L_2$ norm for different types of approximation functions.
Table 4.4: Error in the $H^1$ norm, on the left part of the domain only.

<table>
<thead>
<tr>
<th>Left</th>
<th>G-RKPM</th>
<th>RKPM Discontinuous</th>
<th>RKPM 1st order</th>
<th>RKPM 2nd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>$e_{H^1}$</td>
<td>rate</td>
<td>$e_{H^1}$</td>
<td>rate</td>
</tr>
<tr>
<td>1.0000</td>
<td>5.15E-03</td>
<td>1.32E-01</td>
<td>1.16E-02</td>
<td>1.37E-02</td>
</tr>
<tr>
<td>0.5000</td>
<td>2.02E-03</td>
<td>6.01E-02</td>
<td>2.04E-03</td>
<td>2.50</td>
</tr>
<tr>
<td>0.2500</td>
<td>9.68E-04</td>
<td>3.58E-02</td>
<td>9.70E-04</td>
<td>1.07</td>
</tr>
<tr>
<td>0.1250</td>
<td>4.81E-04</td>
<td>2.42E-02</td>
<td>4.82E-04</td>
<td>1.01</td>
</tr>
<tr>
<td>0.0625</td>
<td>2.40E-04</td>
<td>1.69E-02</td>
<td>2.40E-04</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 4.5: Error in the $H^1$ norm, on the right part of the domain only.

<table>
<thead>
<tr>
<th>Right</th>
<th>G-RKPM</th>
<th>RKPM Discontinuous</th>
<th>RKPM 1st order</th>
<th>RKPM 2nd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>$e_{H^1}$</td>
<td>rate</td>
<td>$e_{H^1}$</td>
<td>rate</td>
</tr>
<tr>
<td>1.0000</td>
<td>6.21E-03</td>
<td>1.42E-01</td>
<td>4.80E-02</td>
<td>1.30E-02</td>
</tr>
<tr>
<td>0.5000</td>
<td>1.09E-03</td>
<td>7.45E-02</td>
<td>1.66E-02</td>
<td>1.53</td>
</tr>
<tr>
<td>0.2500</td>
<td>2.02E-04</td>
<td>4.39E-02</td>
<td>5.94E-03</td>
<td>1.48</td>
</tr>
<tr>
<td>0.1250</td>
<td>4.03E-05</td>
<td>2.95E-02</td>
<td>2.22E-03</td>
<td>1.42</td>
</tr>
<tr>
<td>0.0625</td>
<td>9.04E-06</td>
<td>2.05E-02</td>
<td>8.69E-04</td>
<td>1.35</td>
</tr>
</tbody>
</table>

Table 4.6: Error in the $H^1$ norm, on the entire domain.

<table>
<thead>
<tr>
<th>Total</th>
<th>G-RKPM</th>
<th>RKPM Discontinuous</th>
<th>RKPM 1st order</th>
<th>RKPM 2nd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>$e_{H^1}$</td>
<td>rate</td>
<td>$e_{H^1}$</td>
<td>rate</td>
</tr>
<tr>
<td>1.0000</td>
<td>8.07E-03</td>
<td>1.94E-01</td>
<td>4.94E-02</td>
<td>1.88E-02</td>
</tr>
<tr>
<td>0.5000</td>
<td>2.29E-03</td>
<td>9.57E-02</td>
<td>1.67E-02</td>
<td>1.56</td>
</tr>
<tr>
<td>0.2500</td>
<td>9.88E-04</td>
<td>5.66E-02</td>
<td>6.02E-03</td>
<td>1.47</td>
</tr>
<tr>
<td>0.1250</td>
<td>4.83E-04</td>
<td>3.81E-02</td>
<td>2.27E-03</td>
<td>1.41</td>
</tr>
<tr>
<td>0.0625</td>
<td>2.40E-04</td>
<td>2.66E-02</td>
<td>9.02E-04</td>
<td>1.33</td>
</tr>
</tbody>
</table>
smaller order of approximation, that is on the left part of the domain. Similar behavior can be observed in the case of a non-uniform node distribution, as shown in Fig. 4.10c and Fig. 4.10d.

One can also observe that the error obtained on the right side of the domain when using the G-RKPM formulation is of smaller magnitude than the one obtained when using quadratic basis in the entire domain. This is not an effect of the domain coupling itself, but is caused by the difference in the construction of the approximation functions. The purely quadratic basis analysis was performed with the traditional approximation functions construction $e^T = H^T(0)$, whereas the G-RKPM was built with the adequate $e^T$ vector such that the continuity requirements at $x = 5.5$ as stated in equation (4.46) are satisfied. In this particular case this construction gives smaller errors.

**Figure 4.9:** Convergence study in the $H^1$ norm for different types of approximation functions.
Figure 4.10: Distribution of the error in the displacement and strain fields with uniform and non-uniform discretizations, using discontinuous RKPM approximation and G-RKPM.
4.4.2 \( p \)-refinement in two dimensions

The previous example can be applied to a two dimensional domain. To illustrate, in the following we consider a square domain \( \Omega = [0, 10] \times [0, 10] \) over which we study the following Poisson problem:

\[
\begin{aligned}
\nabla^2 u &= f & \text{on } \Omega \\
u &= g & \text{on } \partial \Omega
\end{aligned}
\]  

(4.51)

where \( f \) and \( g \) are designed such that the following function \( u \) is solution to the problem:

\[
u(x,y) = \left[ \cos \left( \frac{\pi x}{5} \right) + \sin \left( \frac{\pi x}{5} \right) \right] \left[ \cos \left( \frac{\pi y}{5} \right) + \sin \left( \frac{\pi y}{5} \right) \right].
\]

(4.52)

The domain is split into two parts: \( \Omega_1 = [0,5] \times [0,10] \) where linear approximation functions are used, and \( \Omega_2 = [5,10] \times [0,10] \) where quadratic approximation functions are used.

Two approaches are studied in this example. In the first approach, the approximation functions are constructed as the product of one dimensional approximation functions in the \( x \) and \( y \) directions:

\[
\psi_I(x) = \Psi_I(x)\Psi_I(y).
\]

(4.53)

Following the same procedure as detailed above, the vector \( c^T \) is solved for to obtain continuous approximation functions along the interface between \( \Omega_1 \) and \( \Omega_2 \). This method is denoted as parametric in the figures.

In the second approach, the approximation functions are directly built following the regular two dimensional construction. To ensure continuity of the approximation field, the second order basis function aligned with the direction of the interface is included in
both subdomains, in other words the following bases are considered:

\[
H_1^T = \begin{bmatrix}
1 & x & y & y^2
\end{bmatrix} \quad \text{in } \Omega_1,
\]

\[
H_2^T = \begin{bmatrix}
1 & x & y & x^2 & xy & y^2
\end{bmatrix} \quad \text{in } \Omega_2.
\] (4.54)

Figures 4.11 and 4.12 show the performance of both methods in \(L_2\) and \(H^1\) norms respectively, compared to a direct approach with discontinuous approximation functions on the interface. The error distributions illustrated in Fig. 4.13 confirm that a spurious discontinuity at the subdomains interface creates large errors and prevents the method to converge properly, while the G-RKPM counterpart yields much lower error. It is again observed that the total error is dominated by the error in the lower order region, as expected.

This method can also be used to solve a linear elasticity problem. Here we consider a cantilever beam subjected to a parabolic shear load on one end. The problem statement is illustrated in Fig. 4.14. Under the plane stress assumption, the exact solution to the problem is:

\[
u_x = \frac{Py}{6EI} \left[ x(6L-3x) + (2+v) \left( y^2 - \frac{H^2}{4} \right) \right],
\] (4.55)

\[
u_y = -\frac{P}{6EI} \left[ (4+5v) \frac{H^2}{4} x + 3y^2v(L-x) + x^2(3L-x) \right].
\] (4.56)

where \(P\) represents the resultant of the shear stress applied to the right, and \(I = H^3/12\) is the moment of inertia of the beam. In this example, \(P\) is chosen to be \(10^3\)kN. The dimensions of the beam are \(H = 2\)m and \(L = 10\)m and the material constants are \(E = 30 \cdot 10^6\)kN/m and \(v = 0.3\). The domain \(\Omega = [0, 10] \times [-1, 1]\), is decomposed into \(\Omega_1 = [5, 10] \times [-1, 1]\) and \(\Omega_2 = [0, 5] \times [-1, 1]\), and basis described in (4.54) are used.

Again, approximations functions are designed such that they are continuous
Figure 4.11: Convergence study in the $L_2$ norm, using RKPM and G-RKPM approximation. Rates of convergence are indicated on the figure.
Figure 4.12: Convergence study in the $H^1$ norm, using RKPM and G-RKPM approximation. Rates of convergence are indicated on the figure.
(a) RKPM formulation, parametric approach, error $|u - u^h|$

(b) G-RKPM formulation, parametric approach, error $|u - u^h|$

(c) RKPM formulation, parametric approach, error $\sqrt{(u_x - u^h_x)^2 + (u_y - u^h_y)^2}$

(d) G-RKPM formulation, parametric approach, error $\sqrt{(u_x - u^h_x)^2 + (u_y - u^h_y)^2}$

(e) RKPM formulation, 2D approach, error $|u - u^h|$

(f) G-RKPM formulation, 2D approach, error $|u - u^h|$

(g) RKPM formulation, 2D approach, error $\sqrt{(u_x - u^h_x)^2 + (u_y - u^h_y)^2}$

(h) G-RKPM formulation, 2D approach, error $\sqrt{(u_x - u^h_x)^2 + (u_y - u^h_y)^2}$

Figure 4.13: Distribution of the error on the domain with parametric and 2D approaches.
along the interface between $\Omega_1$ and $\Omega_2$, and results are compared against traditional RKPM construction. Fig. 4.15 shows the convergence properties of the method in the $H^1$ norm, and error distributions in the displacement and strain fields are given in Fig. 4.16. Similarly to the previous example, optimal convergence is reached in each subdomain, and the total error is dominated by the lower order part of the approximation, whereas the direct approach leads to a discontinuity along $y = 5$ which creates very large errors and prevents the approximation to converge at expected rates.

### 4.4.3 Oscillation diminishing properties

The influence of the oscillation diminishing properties of G-RKPM when solving a boundary value problem is now studied. As an example, we consider a 2-dimensional domain $\Omega = [0, 10] \times [0, 10]$ with the following Poisson problem statement:

\[
\begin{align*}
    u_{xx} + u_{yy} &= s(x,y) \quad \text{in } \Omega \\
    u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

(4.57)

where $s(x,y)$ is chosen such that the analytical solution to this problem is

\[
u(x,y) = \sin \left( \frac{\pi x}{10} \right) \sin \left( \frac{2\pi x}{10} \right) \sin \left( \frac{\pi y}{10} \right) \sin \left( \frac{2\pi y}{10} \right). \quad (4.58)\]
Figure 4.15: Convergence study in the $H^1$ norm, using RKPM and G-RKPM approximation. Rates of convergence are indicated on the figure.
Figure 4.16: Distribution of the error on the domain using RKPM and G-RKPM approximation.

This boundary value problem is solved using quadratic basis RKPM, and the generalized formulation is compared with the original approximation functions, where G-RKPM is used to minimize oscillations in the approximation functions and their derivatives. The $L_2$ error norm is studied in Fig. 4.17, while the $H^1$ error norm is studied in Fig. 4.18. As expected, the asymptotic rates of convergence are similar for both methods. However, G-RKPM provides much better accuracy. In fact the first refinement step of G-RKPM already yields better accuracy than the third refinement step of RKPM in the $H^1$ norm.

This chapter is currently being prepared for submission for publication of the material. J.S. Chen, E. Yreux, “A generalized formulation of the reproducing kernel particle method”. The dissertation author was the primary investigator of this material.
Figure 4.17: Convergence study in the $L_2$ norm, using RKPM and G-RKPM approximation. Rates of convergence are indicated on the figure. Optimal rate is 3.0.

Figure 4.18: Convergence study in the $H^1$ norm, using RKPM and G-RKPM approximation. Rates of convergence are indicated on the figure. Optimal rate is 2.0.
Chapter 5

A Quasi-Linear RKPM Formulation
5.1 Background

RK Approximation functions as derived in Chapter 3 can be obtained with arbitrary order of completeness, given that the moment matrix defined in (3.19) is non-singular at any point in the domain. This condition is always satisfied when approximation functions are constructed with constant order basis, but it imposes restrictions on the choice of kernel support sizes for higher orders of approximation. For linear approximation in a three-dimensional space for example, each kernel function must cover at least four non-coplanar points for the moment matrix to be invertible. This is a manageable requirement in the case of a Lagrangian analysis, where the approximation is constructed in the undeformed configuration. However for a certain class of problems involving strong topology changes, the Semi-Lagrangian formulation is more suitable. In this approach, the approximation functions are constructed in the deformed configuration, providing a formulation very robust to extreme deformations. Maintaining linear basis reproduction in this situation hence requires proper kernel support coverage in the deformed configuration, which can be very difficult to achieve in simulations involving material separation or fragmentation as illustrated in Fig. 5.1. In a situation where some nodes may not have any neighboring particle, the only achievable construction is constant field reproduction, which can drastically reduce the accuracy of the numerical solution.
A simple approach to enhance the overall accuracy of the solution would be to use linear basis approximation functions wherever possible (that is, for every point covered by enough neighboring kernel functions to construct a first order moment matrix), and to reduce the approximation space to constant field reproduction only where necessary. Unfortunately directly varying the approximation space from linear to constant basis in the domain results in discontinuous approximation functions as shown in Fig. 5.2. This spurious discontinuity induces poor convergence properties and undesirable transient characteristics, defeating the purpose of introducing linear basis approximation in the first place.

An alternative to circumvent this difficulty is to initially define a set of three zones as shown in Fig. 5.3. The red region in the center represents where constant basis approximation is used, a surrounding transitioning region, shown in green, is utilized to smoothly ramp to linear order approximation functions, and the gray exterior zone is where linear basis approximation is employed. This approach restores a continuous approximation field, but requires some a priori knowledge for the user to define these three different regions properly. It also presents the drawback of reducing the order of approximation precisely in locations where an accurate solution is desirable.

The objective of this chapter is to develop a new formulation that improves the method’s robustness by obtaining an unconditionally non-singular moment matrix while
keeping close to linear order accuracy. The Quasi-Linear RKPM formulation is derived to that end in section 5.2, with some visualization of the obtained approximation functions in section 5.2.4. Numerical examples are provided in section 5.3.

5.2 Construction of RK approximation functions with Quasi-Linear Approximation Property

In this section, we show how RK approximation functions are constructed from a moving least-squares [47] approximation with polynomial basis of order $p$, and conditions for the resulting moment matrix to be nonsingular are identified in the linear case. A modified construction is then proposed, embedding an unconditionally invertible moment matrix and preserving exact reproduction of the constant basis, while introducing controllable error in the linear field.

5.2.1 Moving Least-Squares Approximation

Consider a function $u$ defined on a domain $\Omega \subset \mathbb{R}^d$ discretized by a set of nodes $\{x_I\}_{I=1}^{N_p}$. Defining $u_I = u(x_I)$, the $p$-th order moving least-squares (MLS) local
approximation of \( u \) near \( \bar{x} \) is denoted as \( u_{\bar{x}}(x) \) and given by

\[
u_{\bar{x}}(x) = H^T(\bar{x} - x)b(\bar{x}), \tag{5.1}\]

where \( \{H_i(\bar{x} - x)\}_{i=1}^n \) are the polynomial basis functions, \( n = \binom{p+d}{d} \), and \( \{b_i(\bar{x})\}_{i=1}^n \) are the corresponding coefficients obtained by minimizing the weighted moving least-squares residual

\[
r_{\bar{x}} = \sum_{x_I \in S(\bar{x})} \left[ u_{\bar{x}}^h(x_I) - u_I \right]^2 \phi_a(\bar{x} - x_I), \tag{5.2}\]

where \( \phi_a(\bar{x} - x_I) \) is a weight function of local support \( a \), and \( S(\bar{x}) = \{x_I|\phi_a(\bar{x} - x_I) \neq 0\} \) is the set of nodes covering \( \bar{x} \). Minimizing \( r_{\bar{x}} \) with respect to \( b(\bar{x}) \) gives

\[
\frac{\partial r_{\bar{x}}}{\partial b(\bar{x})} = \sum_{x_I \in S(\bar{x})} H(\bar{x} - x_I) [H^T(\bar{x} - x_I)b(\bar{x}) - u_I] \phi_a(\bar{x} - x_I) = 0. \tag{5.3}\]

We obtain

\[
b(\bar{x}) = M^{-1}(\bar{x}) \sum_{x_I \in S(\bar{x})} H(\bar{x} - x_I)\phi_a(\bar{x} - x_I)u_I, \tag{5.4}\]

where

\[
M(\bar{x}) = \sum_{x_I \in S(\bar{x})} H(\bar{x} - x_I)H^T(\bar{x} - x_I)\phi_a(\bar{x} - x_I). \tag{5.5}\]

The global RK approximation is obtained by setting \( \bar{x} \rightarrow x \) and inserting (5.4) into (5.1):

\[
u^h(x) = u_{\bar{x} \rightarrow x}^h(x) = H^T(0)M^{-1}(x) \sum_{x_I \in S(x)} H(x - x_I)\phi_a(x - x_I)u_I \tag{5.6}\]

\[
= \sum_{x_I \in S(x)} \Psi_I(x)u_I \tag{5.7}\]
where the RK approximation functions are defined as

\[ \Psi_I(x) = H^T(\theta)M^{-1}(x)H(x - x_I)\phi_u(x - x_I). \]  

(5.8)

It is easily shown that the basis functions are reproduced exactly, by verifying that

\[ \sum_I \Psi_I(x)H^T(x_I) = H(x) \] or, conversely, \[ \sum_I \Psi_I(x)H^T(x - x_I) = H(\theta). \] Indeed:

\[
\sum_{x_I \in S(x)} \Psi_I(x)H^T(x - x_I) = H^T(\theta)M^{-1}(x) \sum_{x_I \in S(x)} H(x - x_I)H^T(x - x_I)\phi_u(x - x_I)
\]

(5.9)

\[
= H^T(\theta)M^{-1}(x)M(x)
\]

(5.10)

\[
= H^T(\theta).
\]

(5.11)

5.2.2 Moment Matrix Nonsingularity

The RK approximation functions defined above can be constructed at any \( x \in \Omega \) as long as \( M(x) \) is nonsingular. (5.5) can be expressed as

\[
M(x) = \sum_{x_I \in S(x)} m_I(x)\phi_u(x - x_I),
\]

(5.12)

where \( \{m_I(x)\}_{I=1}^{N_p} \) are positive semi-definite \( n \times n \) matrices defined as

\[
m_I(x) = H(x - x_I)H^T(x - x_I).
\]

(5.13)

The moment matrix \( M(x) \) is hence a sum of rank-one matrices [48]. As a consequence, a necessary condition for \( M(x) \) to be invertible is \( \text{Card}(S(x)) \geq n \), in other words \( x \) must have at least \( n \) neighbors. Furthermore, for the case of linear basis in \( d \) dimensions (hence \( n = d + 1 \)), a sufficient condition is that the neighbors of \( x \) are not contained in an
hyperplane in $\mathbb{R}^d$ [48]. To summarize, to construct linear RK approximation functions at $\mathbf{x}$, a necessary and sufficient condition is:

- In a one-dimensional domain, $\mathbf{x}$ is covered by at least 2 nodes,
- In a two-dimensional domain, $\mathbf{x}$ is covered by at least 3 non-collinear nodes,
- In a three-dimensional domain, $\mathbf{x}$ is covered by at least 4 non-coplanar nodes.

### 5.2.3 Quasi-Linear Approximation

We now consider the case of linear basis in $d$ dimensions, which implies $n = d + 1$. For any $\mathbf{x} \in \Omega$, consider a set of sampling points $\{\mathbf{x}_k^*(\mathbf{x})\}_{k=1}^{N_S}$ not contained in an hyperplane of $\mathbb{R}^d$ (which implies $N_s \geq n$), as illustrated in Fig. 5.4. Let $\alpha \in \mathbb{R}^+$. To overcome the aforementioned limitation on the locality of approximation functions, we study the following modified weighted MLS residual:

$$
\tilde{r}_{\mathbf{x}} = r_{\mathbf{x}} + \alpha r_{\mathbf{x}}^*,
$$

where $r_{\mathbf{x}}$ is defined in (5.2) and

$$
r_{\mathbf{x}}^* = \sum_{\mathbf{x}_I \in S(\mathbf{x})} \sum_{k=1}^{N_S} \left[ u_{\mathbf{h}}^h(\mathbf{x}_k) - u_I \right]^2 \phi_\alpha(\mathbf{x} - \mathbf{x}_I),
$$

where $\mathbf{x}_k = \mathbf{x}_k^*(\mathbf{x})$. Minimizing $\tilde{r}_{\mathbf{x}}$ and following the same procedure as in section 5.2.1, we obtain

$$
\tilde{u}_h^h(\mathbf{x}) = \mathbf{H}^T(\Theta)\tilde{M}^{-1}(\mathbf{x}) \sum_{\mathbf{x}_I \in S(\mathbf{x})} \tilde{H}(\mathbf{x}, \mathbf{x}_I)\phi_\alpha(\mathbf{x} - \mathbf{x}_I)u_I
$$

with

$$
\tilde{H}(\mathbf{x}, \mathbf{x}_I) = \mathbf{H}(\mathbf{x} - \mathbf{x}_I) + \alpha \sum_{k=1}^{N_S} \mathbf{H}(\mathbf{x} - \mathbf{x}_k^*(\mathbf{x})),
$$
and
\[ \tilde{M}(x) = M(x) + \alpha M^*(x), \]  
(5.18)

with
\[ M^*(x) = \sum_{x_j \in S(x)} \sum_{k=1}^{N_s} H(x - x_k^*(x)) H^T(x - x_k^*(x)) \phi_a(x - x_j). \]  
(5.19)

The matrix \( M^*(x) \) is obviously positive semi-definite since for any \( z \in \mathbb{R}^n \)
\[ z^T M^*(x) z = \left[ \sum_{x_j \in S(x)} \phi_a(x - x_j) \right] \left[ \sum_{k=1}^{N_s} \left| H^T(x - x_k^*(x)) z \right|^2 \right] \geq 0. \]  
(5.20)

Since the kernel function is a positive function, \( \sum_{x_j \in S(x)} \phi_a(x - x_j) \) is always greater
than zero except if \( x \) does not have any neighbor, in which case there is no approximation
function to compute. Following [48] and taking the simplest case \( N_s = n \), we can deduce
that \( z^T M^*(x) z = 0 \) if and only if
\[
\begin{bmatrix}
1 & x - x_{11}^*(x) & \cdots & x - x_{1d}^*(x) \\
1 & x - x_{21}^*(x) & \cdots & x - x_{2d}^*(x) \\
\vdots & \vdots & \ddots & \vdots \\
1 & x - x_{n1}^*(x) & \cdots & x - x_{nd}^*(x)
\end{bmatrix} z = 0.
\]  
(5.21)

By definition, since the sampling points are not contained in an hyperplane in \( \mathbb{R}^d \),
the determinant of the coefficient matrix in equation (5.21) is positive, which means
\( z^T M^*(x) z = 0 \iff z = 0 \) and therefore \( M^*(x) \) is positive definite. The new moment
matrix \( \tilde{M}(x) \) is thus the sum of a positive semi-definite and a positive definite matrix,
which in turn is positive definite, therefore invertible. The Quasi-Linear approximation
functions defined as

$$\tilde{\Psi}_I(x) = H^T(0)\tilde{M}^{-1}(x)\tilde{H}(x, x_I)\phi_a(x - x_I) \tag{5.22}$$

can hence be constructed for any distribution of nodes and definition of kernel supports.

Similarly to section 5.2.1, we can study the approximation of polynomial functions by inspecting

$$\sum_{x_I \in S(x)} \tilde{\Psi}_I(x)H^T(x - x_I) = H^T(0)\tilde{M}^{-1}(x) \sum_{x_I \in S(x)} \tilde{H}(x, x_I)H^T(x - x_I)\phi_a(x - x_I). \tag{5.23}$$

Observing that

$$\sum_{x_I \in S(x)} \tilde{H}(x, x_I)H^T(x - x_I)\phi_a(x - x_I)$$

$$= \tilde{M}(x) + \alpha \sum_{x_I \in S(x)} \sum_{k=1}^{N_S} H(x - x_k^*(x)) \left[ H^T(x - x_I) - H^T(x - x_k^*(x)) \right] \phi_a(x - x_I),$$

we can rewrite (5.23) as

$$\sum_{x_I \in S(x)} \tilde{\Psi}_I(x)H^T(x - x_I) = H^T(0) + \xi^T(x), \tag{5.25}$$
where \( \xi^T(x) \) represents the error in the monomial basis approximations, and is defined by

\[
\xi^T(x) = \alpha H^T(0) \tilde{M}^{-1}(x)
\]

\[
\sum_{x_I \in \mathcal{S}(x)} \sum_{k=1}^{N_S} H(x - x_k^*(x)) \left[ H^T(x - x_I) - H^T(x - x_k^*(x)) \right] \phi_a(x - x_I).
\]

Two factors can affect the value of the error \( \xi^T(x) \): the value of the coefficient \( \alpha \), and the position of the sampling points \( \{ x_k^*(x) \}_{k=1}^{N_S} \). To quantify the influence of these two factors while keeping equations simple, we now study a one-dimensional domain. A simple choice of sampling points is \( \{ x - \varepsilon, x + \varepsilon \} \), and in this case

\[
\xi^T(x) = \alpha H^T(0) \tilde{M}^{-1}(x) \sum_{x_I \in \mathcal{S}(x)} \left\{ \begin{array}{c} 1 \\ \varepsilon \end{array} \right\} \begin{bmatrix} 0 & x - x_I - \varepsilon \\ 1 & 0 - \varepsilon \end{bmatrix} + \begin{bmatrix} 1 \\ -\varepsilon \end{bmatrix} \begin{bmatrix} 0 & x - x_I + \varepsilon \\ 0 & x - x_I \end{bmatrix} \phi_a(x - x_I), \quad (5.27)
\]

which after simplification yields

\[
\xi^T(x) = 2\alpha H^T(0) \tilde{M}^{-1}(x) \sum_{x_I \in \mathcal{S}(x)} \begin{bmatrix} 0 & x - x_I \\ 0 & -\varepsilon^2 \end{bmatrix} \phi_a(x - x_I). \quad (5.28)
\]

An important feature of the obtained approximation functions is that they maintain exact reproduction of the constant field, which can be seen by observing that \( \xi_0(x) = 0 \). Since the error in the linear field tends to zero as \( \alpha \) decreases, a good practice is to choose \( \alpha \) as small as possible while ensuring the resulting moment matrix \( \tilde{M}(x) \) is numerically invertible. It can also be deduced from (5.28) that \( \xi^T \) does not approach zero as \( \varepsilon \) tends to zero. The next section provides some examples and visualization on the impact of
these parameters on the approximation functions and the resulting error introduced in the linear field reproduction.

To summarize, provided that the sampling points \( \{x_k(x)\}_{k=1}^{N_x} \) are not contained in an hyperplane in \( \mathbb{R}^d \) and \( \alpha > 0 \), the Quasi-Linear approximation functions defined in (5.22) have the following key properties:

- The modified moment matrix (5.18) is non-singular, and therefore Quasi-Linear approximation functions can be constructed independently of the node distribution and kernel supports definition.

- The Quasi-Linear approximation functions possess zeroth order consistency, in other words they reproduce the constant basis function exactly.

- It can be deduced from (5.18) and (5.17) that as \( \alpha \) tends to zero, the Quasi-Linear approximation functions reduces to the regular linear basis approximation functions.

In the rest of this work, for any evaluation point \( x \) the sampling points are defined as

\[
x_1(x) = x - \epsilon \\
x_2(x) = x + \epsilon \\
\]

(5.29)

in a one-dimensional domain, as

\[
x_1^*(x) = \{x - \epsilon, y\} \\
x_2^*(x) = \{x + \epsilon, y\} \\
x_3^*(x) = \{x, y - \epsilon\} \\
x_4^*(x) = \{x, y + \epsilon\} \\
\]

(5.30)
in a 2-dimensional domain, and as

\[ x_1^*(x) = \{x - \varepsilon, y, z\} \quad x_3^*(x) = \{x, y - \varepsilon, z\} \quad x_5^*(x) = \{x, y, z - \varepsilon\} \]

\[ x_2^*(x) = \{x + \varepsilon, y, z\} \quad x_4^*(x) = \{x, y + \varepsilon, z\} \quad x_6^*(x) = \{x, y, z + \varepsilon\} \]

(5.31)

in a 3-dimensional domain. Unless otherwise specified the value of the parameter \( \varepsilon \) is chosen to be 10% of the nodal spacing.

### 5.2.4 Linear Reproduction

In this section we compare constant and linear basis approximation functions with the Quasi-Linear formulation constructed with various values of the parameter \( \alpha \) and different choices of sampling points, and study the error introduced in the reproduction of the linear field.

Fig. 5.5 shows different 1-D Quasi-Linear (QL) approximation functions for a normalized support size \( a = 1.1 \), compared to constant and linear basis. It is observed that for large values of \( \alpha \) the Quasi-Linear approximation function approaches the constant basis approximation functions, while for small values of \( \alpha \) it approaches the pure linear basis construction. Fig. 5.6 studies the error obtained in the reproduction of the linear function \( u(x) = x \) with different values of the coefficient \( \alpha \), compared to the error obtained with constant basis approximation functions. As expected from the study above, the error decreases as \( \alpha \) gets smaller, and can be reduced to an arbitrarily small value.

Also of interest is the following case with a normalized support of \( a = 0.8 \) illustrated in Fig. 5.7. In this situation the support is insufficient to construct linear basis approximation functions, but comparisons can still be made with the constant basis approximation. It can be observed that both constant basis and Quasi-Linear with any value of \( \alpha \) yields approximation functions with a constant value of 1 for
Figure 5.5: Different approximation functions with support size $a = 1.1$ and various values of $\alpha$.

Figure 5.6: Error in the approximation of $u(x) = x$ with normalized support $a = 1.1$ and various values of $\alpha$. 
Figure 5.7: Different approximation functions with support size \( a = 0.8 \) and various values of \( \alpha \).

\[ x \in [-0.2, 0.2], \] which is a direct consequence of the choice of support. As only a single approximation function has a non-zero value on this interval, it is the only option to preserve partition of unity (in other words, to maintain constant field reproduction). This also has consequences on the approximation error for the linear function \( u(x) = x \) studied in Fig. 5.8: Again, for \( x \in [-0.2, 0.2] \), we can only enforce reproduction of a constant field, leading to a linear error in this region. Outside of this region, the error decreases with smaller values of \( \alpha \).

To further illustrate, consider a one-dimensional domain \( \Omega = [0, 30] \) discretized with 21 nodes \( \{x_I = I + 0.025I^2\}_{I=0}^{20} \) carrying approximation functions of fixed support size \( a = 1.5 \). A linear function \( u(x) = x \) is approximated with constant basis, linear basis and Quasi-Linear basis approximation, and Fig. 5.9 shows the error \( u^h(x) - u(x) \) obtained with these different methods. On the left part of the domain \( (x < 14) \), any evaluation point is covered by at least two nodes, and linear approximation functions can be constructed. On the right hand side of the domain however \( (x > 14) \), some evaluation points are only covered by a single node, and the moment matrix at these points becomes singular, destroying the solution entirely. In contrast, constant basis approximation can
be obtained in the entire domain $\Omega$, but yields relatively large errors in the entire domain. The Quasi-Linear formulation (here employed with $\alpha = 0.001$) can also be constructed everywhere in the domain, and yields better accuracy than constant basis. As expected from the study above, the error is very small at any point covered by two nodes or more, and gets larger where only constant basis accuracy can be achieved.

5.2.5 Convergence Properties

To determine the convergence properties of the method under proper kernel supports definition, consider the following two-dimensional Poisson problem:

$$
\begin{cases}
  u_{xx} + u_{yy} = s & \text{in } \Omega = [0, 10] \times [0, 10] \\
  u = 0 & \text{on } \partial \Omega
\end{cases}
$$

(5.32)

where $s$ is chosen such that the analytical solution to the problem is

$$
u(x, y) = \sin \left( \frac{\pi x}{10} \right) \sin \left( \frac{2\pi x}{10} \right) \sin \left( \frac{\pi y}{10} \right) \sin \left( \frac{2\pi y}{10} \right).

(5.33)$$
Figure 5.9: Error in the approximation of $u(x) = x$ with a non-uniform discretization, using a fixed support $a = 1.5$ and constant basis, linear basis and Quasi-Linear with $\alpha = 0.001$.

A normalized support size of $a = 1.5$ is used for the entire set of results, and the problem is solved with the following approximation space constructions:

- Constant basis approximation functions,
- Linear basis approximation functions,
- Quasi-Linear approximation functions with various coefficients $\alpha$.

The approximation error is studied in the $H^1$ norm in Fig. 5.10. As expected from the study in section 5.2.3, large values of the parameter $\alpha$ yield results very similar to constant basis approximation functions in terms of approximation error, while smaller values of $\alpha$ give convergence properties closer to linear basis approximation. Fig. 5.10 shows that a linear rate of convergence in the $H^1$ norm can be obtained when $\alpha$ varies linearly with the nodal distance $h$, performing similarly to the pure linear basis construction.
This study confirms that the Quasi-Linear formulation improves the robustness of the method by providing an unconditionally non-singular moment matrix in the construction of the approximation functions. The accuracy is also greatly enhanced compared to constant basis approximation, as the error introduced in the linear field can be kept arbitrarily small in regions where the kernel coverage is sufficient by properly selecting the value of $\alpha$. As opposed to other methods, no user-defined regions are necessary: Quasi-Linear approximation functions provide close-to-linear basis accuracy wherever support coverage of surrounding particles is adequate, and automatically reduces to constant order approximation otherwise. The present method is hence particularly suitable for semi-Lagrangian analysis where the approximation functions are constructed in the current configuration.

It should also be noted that the computational cost involved with Quasi-Linear approximation is very similar to the cost of pure linear basis construction, as two sampling points (six in a three-dimensional space) are sufficient to ensure invertibility of the moment matrix. Furthermore, only the basis functions are evaluated at each sampling point: there is no additional evaluation of kernel functions.
5.2.6 Mass Lumping

The consistent mass matrix $M$ is given as

$$M_{IJ} = \int_{\Omega} \Psi_I(x) \Psi_J(x) \rho(x) \, d\Omega. \quad (5.34)$$

The corresponding row-sum lumped mass matrix $M^L$, is expressed by

$$M_{II}^L = \int_{\Omega} \left( \sum_K \Psi_K(x) \right) \Psi_I(x) \rho(x) \, d\Omega. \quad (5.35)$$

Since the Quasi-Linear approximation functions satisfy the partition of unity condition exactly, we have $\sum_K \Psi_K(x) = 1$, hence

$$M_{II}^L = \int_{\Omega} \Psi_I(x) \rho(x) \, d\Omega, \quad (5.36)$$

which gives the following nodal mass after nodal integration

$$m_I = \sum_K \Psi_I(x_K) \rho(x_K) \omega_K \, d\Omega, \quad (5.37)$$

where $\omega_K$ is the domain integration weight associated with node $K$.

5.2.7 Stability Analysis

The semi-Lagrangian full discrete equation is given as

$$M \mathbf{a}^n + N \mathbf{v}^n = f_{ext}^n - f_{int}^n \quad (5.38)$$
where the mass matrix $M$ and the convective term $N$ are defined as

$$M_{II} = \rho \Delta x$$  \hspace{1cm} (5.39)\]

$$N_{IJ} = \rho \Delta x \sum_L \Psi_I(x_L) \Psi_J^*(x_L)$$  \hspace{1cm} (5.40)\]

where

$$\Psi_J^*(x) = H^T(0)M^{-1}(x)H(x-x_J)\phi\left(\frac{x-x_J}{a}\right).$$  \hspace{1cm} (5.41)\]

Ignoring the external force vector, considering a central difference temporal discretization and a one-dimensional wave equation on an infinite domain with uniform nodal spacing $\Delta x$, equation (5.38) becomes

$$\rho \Delta x \left(\frac{u_{I}^{n+1} - 2u_{I}^{n} + u_{I}^{n-1}}{\Delta t^2}\right) = -f_I^{int} - \sum_J N_{IJK} \frac{u_{K}^{n+1} - u_{K}^{n-1}}{2\Delta t}, \forall I.$$  \hspace{1cm} (5.42)\]

We introduce the Fourier mode $d_I^n = \lambda^n e^{ik\Delta x}$ where $\lambda$ is the amplification factor and $k$ is the wave number. A von Neumann stability analysis is performed and the amplification factor $\lambda$ is shown to satisfy the following equation:

$$S_0 \lambda^2 + 2S_1 \lambda + S_2 = 0$$  \hspace{1cm} (5.43)\]

with

$$S_0 = 1 - A_2$$  \hspace{1cm} (5.44)\]

$$S_1 = -1 + \frac{2\Delta t^2 c^2}{\Delta x^2} A_1$$  \hspace{1cm} (5.45)\]

$$S_2 = 1 + A_2$$  \hspace{1cm} (5.46)\]

where $c$ is the wave speed, $\rho$ is the density and the coefficients $A_1$ and $A_2$ are related.
to the method utilized for domain integration. For stabilized non-conforming nodal integration [5], they can be obtained as:

\[
A_1 = \frac{1}{\beta^2} \left\{ \left[ \Psi((1 + \frac{\beta}{2})\Delta x) - \Psi((1 - \frac{\beta}{2})\Delta x) \right]^2 \sin^2(k\Delta x) \\
+ 4[\Psi((1 + \frac{\beta}{2})\Delta x) - \Psi((1 - \frac{\beta}{2})\Delta x)][\Psi((2 + \frac{\beta}{2})\Delta x) - \Psi((2 - \frac{\beta}{2})\Delta x)] \sin^2(k\Delta x) \cos(k\Delta x) \\
+ [\Psi((2 + \frac{\beta}{2})\Delta x) - \Psi((2 - \frac{\beta}{2})\Delta x)]^2 \sin^2(2k\Delta x) \right\} \tag{5.47}
\]

and

\[
A_2 = i\Delta t \left\{ \Psi(0)[\Psi^*(\Delta x) \sin(k\Delta x) + \Psi^*(2\Delta x) \sin(2k\Delta x)] \\
+ \Psi(\Delta x)[\Psi^*(\Delta x) \sin(2k\Delta x) + \Psi^*(2\Delta x)(\sin(3k\Delta x) + \sin(k\Delta x))] \\
+ \Psi(2\Delta x)[\Psi^*(\Delta x)(\sin(3k\Delta x) - \sin(k\Delta x)) + \Psi^*(2\Delta x) \sin(4k\Delta x)] \right\} \tag{5.48}
\]

where \( \beta \) is the ratio between the smoothing zone size and the nodal spacing, \( k \) is the wave number and \( \Psi(x) = \Psi_I(x_I + x) \forall I \). Fig. 5.11 gives the critical timestep as a function of the normalized support size \( R \) for various domain integration methods and approximation function construction. For SCNI [17], which corresponds to the case \( \beta = 1.0 \), and for SNNI with \( \beta = 2.0 \), the stability condition is identical for constant, linear and quasi-linear approximation functions. For SNNI with a smoothing zone size of \( \beta = 1.5 \), however, the stability condition is more restrictive for constant basis and for quasi-linear approximation functions with large coefficient \( \alpha \). As the coefficient \( \alpha \) decreases, the critical timestep increases and converges towards the linear basis approximation function results. For all of these constructions, the critical timestep increases with larger support size.
5.3 Numerical Examples

5.3.1 Wave Propagation

We now study the impact of a linear elastic bar of length $L$ with initial velocity $v_0 = 10.0 \text{ m/s}$ on a rigid wall as illustrated in Fig. 5.12 and with properties given in
Table 5.1. The analytical displacement is given as

\[
u(x, t) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1} v_0 L}{\pi^2 (2k-1)^2} \frac{v_0 L}{c} \sin \left( \frac{c}{2L} (2k-1) \pi t \right) \cos \left( \frac{c}{2L} (2k-1) \pi x \right), \quad (5.49)
\]

where \( c = \sqrt{E/\rho} \). The bar is discretized with a non-uniform distribution of nodes as shown in Fig. 5.12. The nodal spacing in the \( x \) direction is approximately 0.125 cm in the entire bar. On the free end side, the nodal spacing is also approximately 0.125 cm in the \( y \) direction, while on the fixed end side this value is increased to 0.25 cm. The support size is fixed in the entire domain to 1.5 \times 0.125 = 0.1875 cm in both directions. Three different approaches are employed to solve this problem, and compared to the analytical displacement given in (5.49):

- **Constant basis:** Approximation functions with constant order of completeness are employed in the entire domain.

- **Automatic Basis:** A simple algorithm based on the conditioning of the moment matrix is adopted. Approximation functions with linear consistency are used where sufficient support is available (on the free end side), while constant basis approximation functions are used if the moment matrix is detected to be ill-conditioned or singular (on the fixed end side).

- **Quasi-Linear:** Approximation functions with Quasi-Linear properties as derived above are employed in the entire domain.

Fig. 5.13 and Fig. 5.14 give the displacement history at the free end of the bar and at the midpoint, respectively, for these three different approaches compared to the analytical displacement history at these points. The constant basis approximation quickly accumulates large errors in the amplitude and phase of the propagating wave. The automatic basis approach also shows large errors, which is expected since about half
Table 5.1: Material properties for the wave propagation problem.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus E</td>
<td>69.0 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio ν</td>
<td>0.0</td>
</tr>
<tr>
<td>Density ρ</td>
<td>2700 kg/m³</td>
</tr>
<tr>
<td>Length L</td>
<td>10.0 cm</td>
</tr>
</tbody>
</table>

Figure 5.13: Displacement at the free end of the bar for different RKPM formulations, compared to the analytical solution.

of the domain gets approximated with constant basis. The quasi-linear approximation, however, automatically maintains close to linear consistency in the x direction, and thus provides a much more accurate approximation.

5.3.2 Taylor Bar Impact

The Quasi-Linear RKPM approximation was implemented into a 3-D dynamic code employing the semi-Lagrangian formulation. Consider a cylindrical aluminum bar impacting a rigid wall with an initial velocity of 373 m/s. The bar has an initial height of 2.346 cm and radius of 0.391 cm. The material properties are given in Table 5.2 and
hardening is defined by the yield function

\[ f(s, \bar{\varepsilon}) = \|s\| - \sqrt{\frac{2}{3}}K(\bar{\varepsilon}), \]  

(5.50)

where \( \bar{\varepsilon} \) is the equivalent plastic strain, \( s \) is the deviatoric Cauchy stress and

\[ K(\bar{\varepsilon}) = \sigma_Y (1 + 125\bar{\varepsilon})^{0.1}. \]  

(5.51)

Height and radius history obtained with constant basis, automatic basis and Quasi-Linear RKPM are presented in Fig. 5.15 and Fig. 5.16, compared with experimental data [49] and reference numerical simulations [50]. The deformed shapes after impact obtained with the three different methods are given in Fig. 5.17. Plastic strain distribution on the impact face of the bar at the end of the simulation is given in Fig. 5.18, where the connectivity between nodes has been used to obtain a meshed solid. The constant basis approximation yields
Table 5.2: Material properties of aluminum bar.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus $E$</td>
<td>78.2 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio $\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>Density $\rho$</td>
<td>2700 kg/m$^3$</td>
</tr>
<tr>
<td>Yield stress $\sigma_y$</td>
<td>0.29 GPa</td>
</tr>
</tbody>
</table>

Figure 5.15: Taylor bar radius history for Quasi-Linear, automatic and constant basis approximations, compared to final radius from reference simulations [50].

unacceptable results in terms of height and radius after impact, and strong numerical fracturing is seen in the deformed shape in Fig. 5.17a. The automatic basis approach improves the accuracy of the solution but the final height is still unsatisfactory, and some splitting is present in the impact face as shown in Fig. 5.17b. Fig. 5.18b also shows some unphysical plastic strain concentrations. Quasi-Linear approximation on the other hand gives very reasonable results both in terms of height and radius after impact, and Fig. 5.18c shows a monotonous and symmetric evolution of the equivalent plastic strain along the radial direction.

This chapter is currently being prepared for submission for publication of the material. J.S. Chen, E. Yreux, “A quasi-linear RKPM formulation”. The dissertation
Figure 5.16: Taylor bar height history for Quasi-Linear, automatic and constant basis approximations, compared to experimental final height [49].

Figure 5.17: Deformed shape after impact.
Figure 5.18: Impact face with distribution of equivalent plastic strain after full deformation.

author was the primary investigator of this material.
Chapter 6

Application of Quasi-Linear RKPM to Large Deformation and Fragment-Impact Problems
The Quasi-Linear RKPM approximation derived in the previous chapter was implemented into a 3D dynamic code employing the semi-Lagrangian formulation. A variety of test were conducted in order to validate the implementation and assess its performance, and results are summarized in this chapter. For the steel plate perforation simulations, projectile’s deformations are experimentally observed to be very small, so the contact conditions can be very well handled by the smooth contact algorithm, defining the projectile’s outer boundary as master surface and the target plate as slave nodes. For the concrete perforation tests however the bullets undergo significant deformation, and the kernel contact with level set representation of the material interface is employed.

6.1 Ballistic Perforation of Steel Plate

6.1.1 Problem Statement

A circular steel plate of 500mm diameter and 12mm thickness is subjected to an impact of a cylindrical steel projectile of 20mm diameter and 80mm height as illustrated in Fig. 6.1. The target is fully clamped along the exterior boundary, and is modeled with the Johnson and Cook constitutive equation [51] and failure criterion [52], where the
equivalent stress is defined as

\[
\sigma_{eq} = (A + B\bar{\varepsilon}^m) (1 + \dot{\varepsilon}^*)^C (1 - T^*m),
\]

(6.1)

where \( \bar{\varepsilon} \) is the equivalent plastic strain and \( \dot{\varepsilon}^* = \dot{\varepsilon}/\dot{\varepsilon}_0 \) represents the plastic strain rate normalized by a reference strain rate at which experiments were conducted to obtain the material properties. The normalized temperature \( T^* = (T - T_r)/(T_m - T_r) \) is utilized to characterize thermal softening caused by adiabatic heating. Damage accumulation is expressed as

\[
D = \sum \frac{\Delta \bar{\varepsilon}}{\varepsilon_f},
\]

(6.2)

where the failure strain \( \varepsilon_f \) is defined as

\[
\varepsilon_f = [D_1 + D_2e^{D_3\sigma^*}][1 + \dot{\varepsilon}^*]^{D_4}[1 + D_5T^*m],
\]

(6.3)

and the pressure-stress ratio \( \sigma^* \) is given by

\[
\sigma^* = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3\sigma_{eq}}.
\]

(6.4)

The projectile is modeled with a bilinear elastoplastic law with isotropic hardening. The material properties for the target plate (Weldox 460E steel) and projectile (Arne-tool steel) were characterized by Dey et al. [53] and are given in Table 6.1 and Table 6.2 respectively. Since the projectile deformations are experimentally observed to be small, the smooth contact algorithm can be employed, where the master surface is defined on the projectile and slave nodes represent the target.
Table 6.1: Material properties of the target plate.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus $E$</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio $\nu$</td>
<td>0.33</td>
</tr>
<tr>
<td>Density $\rho$</td>
<td>7850 kg/m$^3$</td>
</tr>
<tr>
<td>Specific heat capacity $C_p$</td>
<td>452 J/kgK</td>
</tr>
<tr>
<td>Reference temperature $T_r$</td>
<td>293 K</td>
</tr>
<tr>
<td>Initial temperature $T_0$</td>
<td>293 K</td>
</tr>
<tr>
<td>Melting temperature $T_m$</td>
<td>1800 K</td>
</tr>
<tr>
<td>Temperature softening coefficient $m$</td>
<td>0.893</td>
</tr>
<tr>
<td>Taylor–Quinney constant $\chi$</td>
<td>0.9</td>
</tr>
<tr>
<td>Yield stress $\sigma$</td>
<td>499 MPa</td>
</tr>
<tr>
<td>Strain hardening $B$</td>
<td>382 MPa</td>
</tr>
<tr>
<td>Strain hardening exponent $n$</td>
<td>0.458</td>
</tr>
<tr>
<td>Strain rate hardening $C$</td>
<td>0.0079</td>
</tr>
<tr>
<td>Reference strain rate $\varepsilon_0$</td>
<td>$5.0 \times 10^{-4}$ s$^{-1}$</td>
</tr>
<tr>
<td>Fracture strain coefficients $D_1$</td>
<td>0.636</td>
</tr>
<tr>
<td>$D_2$</td>
<td>1.936</td>
</tr>
<tr>
<td>$D_3$</td>
<td>-2.964</td>
</tr>
<tr>
<td>$D_4$</td>
<td>-0.014</td>
</tr>
<tr>
<td>$D_5$</td>
<td>1.014</td>
</tr>
</tbody>
</table>

Table 6.2: Material properties of the projectile.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus $E$</td>
<td>204 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio $\nu$</td>
<td>0.33</td>
</tr>
<tr>
<td>Density $\rho$</td>
<td>7850 kg/m$^3$</td>
</tr>
<tr>
<td>Yield stress $\sigma_0$</td>
<td>1900 MPa</td>
</tr>
<tr>
<td>Tangent modulus $E_t$</td>
<td>15 GPa</td>
</tr>
</tbody>
</table>
Figure 6.2: Close up view of the cross section of the steel plate after impact using constant basis and quasi-linear basis, $v_i = 303.5$ m/s. Experimental impact and exit crater size and position are shown in blue.
Figure 6.3: Close up view of the cross section of the steel plate after impact using constant basis and quasi-linear basis, $v_i = 179.4 \text{ m/s}$. Experimental impact crater size and position is shown in blue.
6.1.2 Numerical Results and Comparison to Experimental Data

Impact velocities \( v_i \) ranging from 100 m/s to 400 m/s are studied to determine the ballistic limit of the studied steel plate and projectile. Simulations are performed using both the new Quasi-Linear formulation and constant basis approximation, and results are compared to available experimental data [53]. Table 6.3 gives the numerical exit velocities versus impact velocities as well as the computed velocity reduction factors for both formulation, and these results are also illustrated in Fig. 6.4. Deduced ballistic limits are given in Table 6.4. It is apparent that the constant basis formulation tends to strongly underestimate velocity reduction factor, and the ballistic limit obtained with constant basis approximation shows a relative error of more than 40% while the Quasi-Linear formulation yields a relative error well under 10%. Deformed shapes of the steel plate after impact are also illustrated in Fig. 6.2 and Fig. 6.3 for impact velocities of 303.5 m/s and 179.4 m/s, respectively. For high-velocity impacts the deformed shapes obtained with constant and quasi-linear bases are relatively similar as shown in Fig. 6.2. For impact velocities closer to the ballistic limit however this is no longer the case. The constant basis approximation severely underestimates the deflection of the target plate, as illustrated in Fig. 6.3a where the position of the impact crater is very inaccurate compared to experimental measurements. Furthermore, the Quasi-Linear basis simulation illustrated in Fig. 6.3 correctly predicts that the projectile does not perforate through the target and the absence of crater on the exit side of the plate, whereas the constant basis approximation predicts a projectile residual velocity of more than 100 m/s and generates a relatively large crater on the exit side.
Table 6.3: Summary of numerically obtained residual velocities compared to experimental values.

<table>
<thead>
<tr>
<th>Initial Velocity (m/s)</th>
<th>Residual Velocity (m/s) and Velocity Reduction (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Experimental</td>
</tr>
<tr>
<td>399.6</td>
<td>291.3 (27.1%)</td>
</tr>
<tr>
<td>303.5</td>
<td>199.7 (34.2%)</td>
</tr>
<tr>
<td>244.2</td>
<td>132.6 (45.7%)</td>
</tr>
<tr>
<td>200.4</td>
<td>71.4 (64.4%)</td>
</tr>
<tr>
<td>189.6</td>
<td>43.7 (64.4%)</td>
</tr>
<tr>
<td>179.4</td>
<td>0.0 (100.0%)</td>
</tr>
<tr>
<td>173.7</td>
<td>0.0 (100.0%)</td>
</tr>
<tr>
<td>120.0</td>
<td>0.0 (100.0%)</td>
</tr>
<tr>
<td>110.0</td>
<td>0.0 (100.0%)</td>
</tr>
</tbody>
</table>

Figure 6.4: Obtained residual velocities for different impact velocities with constant basis and Quasi-Linear approximation, compared to experimental data.

Table 6.4: Comparison of ballistic limits obtained from experiment and numerical simulations.

<table>
<thead>
<tr>
<th></th>
<th>Ballistic Limit</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experiment</td>
<td>184.5 m/s</td>
<td>-</td>
</tr>
<tr>
<td>Quasi-Linear simulation</td>
<td>195.0 m/s</td>
<td>5.7%</td>
</tr>
<tr>
<td>Constant basis simulation</td>
<td>115.0 m/s</td>
<td>43.4%</td>
</tr>
</tbody>
</table>
Table 6.5: Material properties of steel projectiles.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus $E$</td>
<td>200.0 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio $\nu$</td>
<td>0.26</td>
</tr>
<tr>
<td>Yield stress $\sigma_y$</td>
<td>2400 MPa</td>
</tr>
<tr>
<td>Hardening Modulus $H$</td>
<td>2500 MPa</td>
</tr>
<tr>
<td>Density $\rho$</td>
<td>7806 kg/m$^3$</td>
</tr>
</tbody>
</table>

6.2 Spherical Penetrator Impacting High-Strength Concrete Panels

6.2.1 Problem Statement

To further validate the implementation of the Quasi-Linear RKPM formulation, eighteen perforation simulations (Fig. 6.5) of Cor-Tuf ultra high-strength concrete panels with various thicknesses and steel projectile sizes were performed, namely 0.5″, 1.0″ and 1.5″ thick concrete slabs, and 32, 86 and 129 grain steel projectiles. A wide range of impact velocities was used, and the results were compared to available experimental data [26]. The steel projectiles are modeled with $J_2$ plasticity with isotropic hardening, and the corresponding material parameters are listed in Table 6.5. The concrete panel is modeled using the Advanced Fundamental Concrete model [54, 26] coupled with a microcrack informed damage model [55]. Corresponding material parameters are extracted from [26] and repeated in Table 6.6. The level-set enhanced kernel contact algorithm [33] was employed to represent evolving contact surfaces.

6.2.2 Numerical Results and Comparison to Experimental Data

Available experimental data includes projectile exit velocity and panel mass loss. Table 6.7 summarizes the eighteen tests characteristics and the obtained experimental and
Table 6.6: Material properties of CorTuf concrete panels.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear modulus $G$</td>
<td>18.457 GPa</td>
</tr>
<tr>
<td>Failure surface parameter $C_1$</td>
<td>1016.3 MPa</td>
</tr>
<tr>
<td>Failure surface parameter $C_2$</td>
<td>908.65 MPa</td>
</tr>
<tr>
<td>Failure surface parameter $C_3$</td>
<td>0.0125</td>
</tr>
<tr>
<td>Failure surface parameter $C_4$</td>
<td>0.10382</td>
</tr>
<tr>
<td>Failure surface parameter $C_5$</td>
<td>792.89 MPa</td>
</tr>
<tr>
<td>Failure surface parameter $A_n$</td>
<td>$1.7345 \times 10^{-9}$ Pa$^{-1}$</td>
</tr>
<tr>
<td>Triaxial extension/compression strength ratio TXETXCR</td>
<td>0.625</td>
</tr>
<tr>
<td>Third-invariant dependent failure surface parameter PRECRIT</td>
<td>$0.177 \times 10^{22}$</td>
</tr>
<tr>
<td>Maximum tensile pressure $P_{\text{min}}$</td>
<td>6.8947 MPa</td>
</tr>
<tr>
<td>Equation of state parameter $C_6$</td>
<td>172.37 MPa</td>
</tr>
<tr>
<td>Equation of state parameter $C_7$</td>
<td>0.00781</td>
</tr>
<tr>
<td>Equation of state parameter $C$</td>
<td>7912.2 MPa</td>
</tr>
<tr>
<td>Equation of state parameter $D$</td>
<td>$-29.205$ GPa</td>
</tr>
<tr>
<td>Equation of state parameter $S$</td>
<td>187.10 GPa</td>
</tr>
<tr>
<td>Equation of state parameter $C_9$</td>
<td>77.958 GPa</td>
</tr>
<tr>
<td>Equation of state parameter $C_{10}$</td>
<td>0.24863</td>
</tr>
<tr>
<td>Damage evolution parameter $D_1$</td>
<td>$4.057 \times 10^{-10}$ Pa$^{-1}$</td>
</tr>
<tr>
<td>Tensile strength for microscale informed damage model $F_c$</td>
<td>6.8947 MPa</td>
</tr>
<tr>
<td>Density $\rho$</td>
<td>2267.4 kg/m$^3$</td>
</tr>
<tr>
<td>Mass proportional damping DAMP</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Figure 6.5: Spherical steel projectile impacting a Cor-Tuf ultra high-strength concrete panel with initial velocity $v_0$. 
numerical exit velocities as well as the associated velocity reduction factors, as shown in Fig. 6.6, Fig. 6.7 and Fig. 6.8 for 1.5″, 1.0″ and 0.5″ thick concrete slabs respectively. The overall trend shows good agreement with experimental data, but some individual tests exhibit relatively large discrepancy between experiment and numerical simulation. In general the higher range of impact velocity proves more difficult to capture accurately in the simulations.

The panel mass loss due to debris formation was measured experimentally. In the simulation, every concrete particle with a residual velocity greater or equal to 10 m/s was considered as debris. The experimental and numerical concrete panel mass loss are summarized in Table 6.8, and plots are given in Fig. 6.9, Fig. 6.10 and Fig. 6.11 for 1.5″, 1.0″ and 0.5″ slab thicknesses, respectively. Experimental data was not available for Run 5 and Run 15, hence only the numerically obtained debris weight is provided. Numerical simulation and experiment globally agree quite well, but some selected cases with high impact velocity such as Run 07 present a large disagreement. An estimate of the sensitivity of the debris weight to the impact velocity can be seen in Fig. 6.10 between Run 9 and Trial 2. Both experiments involve a 1.0″ thick concrete panel and a 86 grain projectile and the impact velocity varies only by 7%, but the experimentally measured debris weight varies by 38%.

### 6.2.3 Illustrations

Some typical deformed shapes after impact for concrete slabs of thickness 0.5″, 1.0″ and 1.5″ are shown in Fig. 6.12, Fig. 6.13 and Fig. 6.14 respectively. Similarly, typical cross-sections of crater shapes after perforation are shown in Fig. 6.15, Fig. 6.16 and Fig. 6.17 for concrete slabs of thickness 0.5″, 1.0″ and 1.5″, respectively, in which particles with velocity greater than 10.0 m/s are treated as debris and removed. The associated projectiles are represented to scale on the right of the panel for size comparison.
Table 6.7: Experimental and numerical exit velocities and velocity reduction factors.

<table>
<thead>
<tr>
<th>Panel</th>
<th>Projectile</th>
<th>Run #</th>
<th>Test #</th>
<th>Panel Thickness (in.)</th>
<th>Impact Velocity (ft/s)</th>
<th>Exit Velocity (ft/s)</th>
<th>Velocity Reduction</th>
<th>Numerical Exit Velocity (ft/s)</th>
<th>Numerical Velocity Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal 1.5'' Thick</td>
<td>129 gr 1/2'' sphere</td>
<td>4</td>
<td>090806-02</td>
<td>1 5/8</td>
<td>5636</td>
<td>2597</td>
<td>54%</td>
<td>2005</td>
<td>64%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8</td>
<td>090819-01</td>
<td>1 9/16</td>
<td>1337</td>
<td>0</td>
<td>100%</td>
<td>-118</td>
<td>100%</td>
</tr>
<tr>
<td></td>
<td>86 gr 7/16'' sphere</td>
<td>15</td>
<td>090527-06</td>
<td>1 7/16</td>
<td>2616</td>
<td>50</td>
<td>98%</td>
<td>92</td>
<td>96%</td>
</tr>
<tr>
<td>Tr 1</td>
<td></td>
<td>091013-03</td>
<td>1 9/16</td>
<td>5604</td>
<td>2246</td>
<td></td>
<td>60%</td>
<td>1608</td>
<td>71%</td>
</tr>
<tr>
<td></td>
<td>32 gr 5/16'' sphere</td>
<td>7</td>
<td>091014-02</td>
<td>1 9/16</td>
<td>6718</td>
<td>177</td>
<td>97%</td>
<td>991</td>
<td>85%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>090806-01</td>
<td>1 9/16</td>
<td>1985</td>
<td>0</td>
<td>100%</td>
<td>-171</td>
<td>100%</td>
</tr>
<tr>
<td>Nominal 1'' Thick</td>
<td>129 gr 1/2'' sphere</td>
<td>14</td>
<td>090917-01</td>
<td>1</td>
<td>1582</td>
<td>75</td>
<td>95%</td>
<td>98</td>
<td>94%</td>
</tr>
<tr>
<td></td>
<td>86 gr 7/16'' sphere</td>
<td>13</td>
<td>090904-02</td>
<td>1</td>
<td>3657</td>
<td>1787</td>
<td>51%</td>
<td>1165</td>
<td>68%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>090819-02</td>
<td>1</td>
<td>1767</td>
<td>0</td>
<td>100%</td>
<td>82</td>
<td>95%</td>
</tr>
<tr>
<td></td>
<td>32 gr 5/16'' sphere</td>
<td>12</td>
<td>090904-01</td>
<td>1</td>
<td>884</td>
<td>0</td>
<td>100%</td>
<td>-72</td>
<td>100%</td>
</tr>
<tr>
<td>Tr 2</td>
<td></td>
<td>090911-03</td>
<td>1 1/16</td>
<td>1892</td>
<td>0</td>
<td></td>
<td>100%</td>
<td>72</td>
<td>96%</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>091014-01</td>
<td>1 1/16</td>
<td>6521</td>
<td>1284</td>
<td></td>
<td>80%</td>
<td>2106</td>
<td>68%</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>090903-01</td>
<td>1 1/16</td>
<td>3015</td>
<td>80</td>
<td></td>
<td>97%</td>
<td>75</td>
<td>97%</td>
</tr>
<tr>
<td>Nominal 1/2'' Thick</td>
<td>129 gr 1/2'' sphere</td>
<td>6</td>
<td>090807-02</td>
<td>9/16</td>
<td>1516</td>
<td>654</td>
<td>57%</td>
<td>661</td>
<td>56%</td>
</tr>
<tr>
<td></td>
<td>86 gr 7/16'' sphere</td>
<td>2</td>
<td>090805-02</td>
<td>5/8</td>
<td>377</td>
<td>0</td>
<td>100%</td>
<td>0</td>
<td>100%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>090902-03</td>
<td>9/16</td>
<td>859</td>
<td>0</td>
<td>100%</td>
<td>72</td>
<td>92%</td>
</tr>
<tr>
<td></td>
<td>32 gr 5/16'' sphere</td>
<td>1</td>
<td>090805-01</td>
<td>11/16</td>
<td>2868</td>
<td>976</td>
<td>66%</td>
<td>676</td>
<td>76%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>090807-01</td>
<td>1 1/16</td>
<td>785</td>
<td>0</td>
<td>100%</td>
<td>0</td>
<td>100%</td>
</tr>
</tbody>
</table>
Figure 6.6: Experimental and numerical velocity reduction factors for the 1.5” thick concrete panel.

Figure 6.7: Experimental and numerical velocity reduction factors for the 1.0” thick concrete panel.
**Figure 6.8:** Experimental and numerical velocity reduction factors for the 0.5” thick concrete panel.

**Table 6.8:** Experimental and numerical panel debris weight after perforation process.

<table>
<thead>
<tr>
<th>Run</th>
<th>Experimental Panel Debris Weight (lb)</th>
<th>Numerical Panel Debris Weight (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.429</td>
<td>0.454</td>
</tr>
<tr>
<td>8</td>
<td>0.012</td>
<td>0.002</td>
</tr>
<tr>
<td>15</td>
<td>N/A</td>
<td>0.184</td>
</tr>
<tr>
<td>Tr 1</td>
<td>0.420</td>
<td>0.344</td>
</tr>
<tr>
<td>7</td>
<td>0.409</td>
<td>0.210</td>
</tr>
<tr>
<td>3</td>
<td>0.006</td>
<td>0.003</td>
</tr>
<tr>
<td>14</td>
<td>0.131</td>
<td>0.133</td>
</tr>
<tr>
<td>13</td>
<td>0.170</td>
<td>0.157</td>
</tr>
<tr>
<td>9</td>
<td>0.109</td>
<td>0.081</td>
</tr>
<tr>
<td>12</td>
<td>0.010</td>
<td>0.000</td>
</tr>
<tr>
<td>Tr 2</td>
<td>0.160</td>
<td>0.103</td>
</tr>
<tr>
<td>16</td>
<td>0.190</td>
<td>0.126</td>
</tr>
<tr>
<td>11</td>
<td>0.096</td>
<td>0.094</td>
</tr>
<tr>
<td>6</td>
<td>0.060</td>
<td>0.065</td>
</tr>
<tr>
<td>2</td>
<td>0.005</td>
<td>0.001</td>
</tr>
<tr>
<td>10</td>
<td>0.056</td>
<td>0.025</td>
</tr>
<tr>
<td>1</td>
<td>0.044</td>
<td>0.060</td>
</tr>
<tr>
<td>5</td>
<td>N/A</td>
<td>0.001</td>
</tr>
</tbody>
</table>
Figure 6.9: Experimental and numerical panel mass loss after impact for the 1.5” thick concrete panel. Experimental data for Run 15 is not available.

Figure 6.10: Experimental and numerical panel mass loss after impact for the 1.0” thick concrete panel.
Figure 6.11: Experimental and numerical panel mass loss after impact for the 0.5” thick concrete panel. Experimental data for Run 5 is not available.

The impact creates larger craters on the exit face than on the impact face, which coincides with experimental observations.
**Figure 6.12**: Typical deformed shape after impact of a 0.5″ thick slab: Run 06.

**Figure 6.13**: Typical deformed shape after impact of a 1.0″ thick slab: Run 16.
Figure 6.14: Typical deformed shape after impact of a 1.5" thick slab: Run 04.

Figure 6.15: Typical cross-section of crater shape on a 0.5" thick slab, impact face on the left and exit face on the right: Run 01. Experimental crater dimensions are shown in thick blue.
Figure 6.16: Typical cross-section of crater shape on a 1.0” thick slab, impact face on the left and exit face on the right: Run 13. Experimental crater dimensions are shown in thick blue.

Figure 6.17: Typical cross-section of crater shape on a 1.5” thick slab, impact face on the left and exit face on the right: Trial 1. Experimental crater dimensions are shown in thick blue.
6.3 Bullet Perforating Circular Concrete Slabs of Different Thicknesses

Cylindrical WES5000 concrete slabs of three different thicknesses were perforated using identical SAP projectiles with an impact velocity of 313 m/s. The steel projectiles are modeled with $J_2$ plasticity with isotropic hardening, and the corresponding material parameters are listed in Table 6.9. The concrete panel is modeled using the Advanced Fundamental Concrete Model [54, 26] coupled with a microcrack informed damage model [55]. Corresponding material parameters are listed in Table 6.10. The level-set enhanced kernel contact algorithm [33] was employed to represent evolving contact surfaces.

Slabs of 127 mm (Fig. 6.18), 216 mm (Fig. 6.19) and 254 mm (Fig. 6.20) thickness are considered, and exit velocities are compared to available experimental data [56]. Obtained velocity histories are shown in Fig. 6.21, and numerical exit velocities can be compared to experimental data. Velocity reduction factors were also calculated and are provided in Table 6.11. The overall trend of the simulation is in good agreement with the experiment, showing more severe velocity reduction for thicker targets. The two thinner targets present less than 5% error in the velocity reduction factor, while the numerical simulation involving the thicker 254 mm produces larger error, predicting a 76% velocity reduction factor instead of the measured 86%.

Table 6.9: Material properties of steel projectiles.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus $E$</td>
<td>205.0 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio $\nu$</td>
<td>0.26</td>
</tr>
<tr>
<td>Yield stress $\sigma_y$</td>
<td>1750 MPa</td>
</tr>
<tr>
<td>Hardening Modulus $H$</td>
<td>1000 MPa</td>
</tr>
<tr>
<td>Density $\rho$</td>
<td>7860 kg/m$^3$</td>
</tr>
</tbody>
</table>
Table 6.10: Material properties of WES5000 concrete slabs.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear modulus $G$</td>
<td>6.893 GPa</td>
</tr>
<tr>
<td>Failure surface parameter $C_1$</td>
<td>501.05 MPa</td>
</tr>
<tr>
<td>Failure surface parameter $C_2$</td>
<td>476.3 MPa</td>
</tr>
<tr>
<td>Failure surface parameter $C_3$</td>
<td>0.01</td>
</tr>
<tr>
<td>Failure surface parameter $C_4$</td>
<td>0.1</td>
</tr>
<tr>
<td>Failure surface parameter $C_5$</td>
<td>516.95 MPa</td>
</tr>
<tr>
<td>Failure surface parameter $A_n$</td>
<td>0.00324 MPa$^{-1}$</td>
</tr>
<tr>
<td>Triaxial extension/compression strength ratio TXETXCR</td>
<td>0.625</td>
</tr>
<tr>
<td>Third-invariant dependent failure surface parameter PRECRIT</td>
<td>$0.177 \times 10^{22}$</td>
</tr>
<tr>
<td>Maximum tensile pressure $P_{\text{min}}$</td>
<td>2.47 MPa</td>
</tr>
<tr>
<td>Equation of state parameter $C_6$</td>
<td>55.14 MPa</td>
</tr>
<tr>
<td>Equation of state parameter $C_7$</td>
<td>0.0025</td>
</tr>
<tr>
<td>Equation of state parameter $C$</td>
<td>4248.3 MPa</td>
</tr>
<tr>
<td>Equation of state parameter $D$</td>
<td>6196.5 MPa</td>
</tr>
<tr>
<td>Equation of state parameter $S$</td>
<td>68.237 GPa</td>
</tr>
<tr>
<td>Equation of state parameter $C_9$</td>
<td>68.237 GPa</td>
</tr>
<tr>
<td>Equation of state parameter $C_{10}$</td>
<td>0.25</td>
</tr>
<tr>
<td>Damage evolution parameter $D_1$</td>
<td>0.0006 MPa$^{-1}$</td>
</tr>
<tr>
<td>Tensile strength for microscale informed damage model $F_c$</td>
<td>10.0 MPa</td>
</tr>
<tr>
<td>Density $\rho$</td>
<td>2267 kg/m$^3$</td>
</tr>
<tr>
<td>Mass proportional damping DAMP</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Figure 6.18: Quasi-Linear RKPM modeling of impact of a 127 mm thick concrete slab with steel projectile.
**Figure 6.19**: Quasi-Linear RKPM modeling of impact of a 216 mm thick concrete slab with steel projectile.

**Figure 6.20**: Quasi-Linear RKPM modeling of impact of a 254 mm thick concrete slab with steel projectile.
Figure 6.21: Combined velocity histories of Quasi-Linear RKPM modeling for the three different thicknesses of WES 5000 concrete slabs compared to experimental data.

Table 6.11: Obtained exit velocities and velocity reduction factors for the three different thicknesses of WES5000 concrete slabs.

<table>
<thead>
<tr>
<th>Slab thickness</th>
<th>Experimental</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exit Velocity</td>
<td>Reduction Factor</td>
</tr>
<tr>
<td>127 mm</td>
<td>224 m/s</td>
<td>28%</td>
</tr>
<tr>
<td>216 mm</td>
<td>115 m/s</td>
<td>63%</td>
</tr>
<tr>
<td>254 mm</td>
<td>45 m/s</td>
<td>86%</td>
</tr>
</tbody>
</table>
Chapter 7

Fracture Modeling
7.1 Numerical Methods for Accurate Fracture Modeling

The presence of a crack in a body induces consequent changes in the solution of the boundary value problem. First of all it induces a discontinuity in the displacement field across the crack surface, that needs to be properly captured in the numerical solution, and hence requires appropriate selection of the approximation space. The other important phenomenon is the singularity in the stress and strain fields happening at the crack tip. The asymptotic solution obtained in linear elastic fracture mechanics can be embedded in the approximation to improve the accuracy of the numerical solution.

7.1.1 Visibility Criterion

The first approach, introduced by Belytschko et al. [38] in the Element-Free Galerkin method, consists of truncating the support of the approximation functions along the crack surface, as illustrated in Fig. 7.1. Under this approach, it might be necessary to increase the support size of the approximation functions located at the vicinity of the crack surface to guarantee invertibility of their associated moment matrix. The visibility criteria efficiently accounts for the discontinuity occurring across the crack surface in the displacement field, but it induces spurious discontinuities ahead of the crack tip, as can be seen in Fig. 7.1b.

The early efforts simply used a higher concentration of nodes around the crack tip, to reduce the impact of the introduced spurious discontinuities, and to improve the accuracy of the approximation in this critical region. Another way to get around this limitation is to regularize the discontinuities through a smoothing process, as proposed by Organ et al. [40] with the transparency and diffraction criteria, but extension to 3-D is
Figure 7.1: Illustration of the visibility criterion: The gray area represents the truncated support of a particular approximation function.

not straight-forward.

### 7.1.2 Extrinsic Enrichment

A method to enrich a smooth approximation to account for discontinuity and crack tip singularity is the extrinsic enrichment. It requires the construction of a local partition of unity around the crack tip, multiplied with four additional basis functions based on the asymptotic field in this region. The approximation function becomes

$$\mathbf{u}^h(x) = \sum_{I \in N} \Psi_I(x) d_I + \sum_{K \in N_{\text{tip}}} \sum_{i=1}^{N_{\text{enr}}} \Phi_i(x) \Psi_K(x) e_{iK},$$

(7.1)

where $N$ is the set containing all the nodes in the domain, and $N_{\text{tip}}$ is the set of nodes whose support covers the crack tip. $\Psi_I$ represents the RK approximation function associated with node $I$. $N_{\text{enr}}$ is the number of enrichment functions added to the approximation space. Around the crack tip region, four enrichment functions $\Phi_i$ are usually introduced, defined as

$$\Phi = \left\{ \sqrt{r} \sin \left( \frac{\theta}{2} \right), \sqrt{r} \cos \left( \frac{\theta}{2} \right), \sqrt{r} \sin \left( \frac{\theta}{2} \right) \sin(\theta), \sqrt{r} \cos \left( \frac{\theta}{2} \right) \sin(\theta) \right\},$$

(7.2)
where \((r, \theta)\) represents polar coordinates in a coordinate system centered at the crack tip and aligned with the crack path, as illustrated in Fig. 7.2. The coefficients to be solved are \(\{d_I\}\), that represent the usual nodal unknowns for the smooth field, and \(\{e_{ik}\}\), that represents the unknowns associated with the enriched field. As can be seen from the second term on the right hand side of (7.1), a local partition of unity built by the RK approximation is introduced around the crack tip.

Outside of the enriched region surrounding the crack tip, the discontinuity along the crack surface is captured using the visibility criterion: the support of the surrounding approximation functions is truncated along the crack surface. Another technique is to add additional enrichment functions in the approximation space (7.1), involving Heaviside functions along the crack surface.

In (7.1), the enrichment field around the crack tip is introduced only for nodes whose support covers the crack tip. In practice, this means that the enriched region gets smaller and smaller as the discretization is refined, yielding suboptimal convergence rates. In theory it is desirable to keep the size of the enriched region constant as the discretization is refined. This would lead to optimal convergence, but can unfortunately cause very severe ill-conditioning of the global stiffness matrix [57, 58]. A specifically designed local preconditioner can be constructed to alleviate this issue [58], but this
induces a non negligible additional cost.

7.1.3 Intrinsic Enrichment

Another approach is to use the very general construction of the reproducing kernel approximation functions to include other bases than polynomial functions. As detailed in chapter 3, RK approximation is built based on a vector containing the different basis functions necessary in the approximation space, as represented in equation (3.4). The idea is to modify the approximation functions at the vicinity of the crack tip, which gives the following expression for the approximation functions:

\[
\Psi_f(x) = \begin{cases} 
\Psi_{ef}(x, r, \theta) & \text{at the vicinity of the crack tip} \\
\tilde{\Psi}_f(x) & \text{everywhere else in the domain},
\end{cases}
\]

(7.3)

where \(\tilde{\Psi}_f(x)\) represents the usual RK approximation function built with polynomial basis, and \(\Psi_{ef}(x, r, \theta)\) represents the modified approximation functions, where the enrichments detailed in (7.2) are directly added in the basis vector, and the construction of the modified approximation functions \(\Psi_{ef}\) is similar to the procedure detailed in chapter 3. For example in the linear basis case, enriched approximation functions are defined as:

\[
\Psi_{ef}(x, r, \theta) = H_e^T(\theta, r, \theta) M_e^{-1}(x, r, \theta) H_e(x - x_f, r, \theta) \phi_a(x - x_I),
\]

(7.4)

where

\[
H_e^T(\theta, r, \theta) = \\
\begin{bmatrix}
1 & 0 & 0 & \sqrt{r} \sin\left(\frac{\theta}{2}\right) & \sqrt{r} \cos\left(\frac{\theta}{2}\right) & \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta) & \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta)
\end{bmatrix},
\]

(7.5)
\[ H_e^T(x - x_I, r, \theta) = \]
\[
\begin{bmatrix}
1 & x - x_I & y - y_I & \sqrt{r}\sin\left(\frac{\theta}{2}\right) & \sqrt{r}\cos\left(\frac{\theta}{2}\right) & \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta) & \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta)
\end{bmatrix},
\]

(7.6)

and

\[ M_e(x, r, \theta) = \sum_J H_e(x - x_J, r, \theta) H_e^T(x - x_J, r, \theta) \phi_a(x - x_J). \]

(7.7)

7.1.4 Stress Intensity Factor

One of the main characteristics that needs to be defined when modeling crack growth is the crack propagation criteria. Assuming we already have a pre-crack in a domain, it is important to know under which conditions this crack will extend, and if it does, in which direction it should propagate. A useful concept to characterize this is the strain energy release rate, often denoted \( G \), which represents the energy dissipated per unit surface of newly created crack. A fracture criterion \( G_c \) can be defined as a material property that is independent of the applied loading, geometry and boundary conditions, such that if the condition \( G > G_c \) is met, the crack starts growing. The direction of propagation itself can be determined by a different criteria such as the maximum hoop strain direction.

It is thus important to obtain the strain energy release rate accurately, since it is so central to the crack propagation mechanism. A very popular way to calculate this energetic quantity is the use of the \( J \)–integral. This concept was proposed by Rice [59], who proved the existence of a path independent integral to obtain the strain energy release rate without having to evaluate stresses at the immediate vicinity of the crack tip. The
Figure 7.3: Example of path $\Gamma$ that can be used to compute the J-integral.

The J-integral is defined as

$$J = \int_{\Gamma} \left( W n_1 - \sigma_{ij} n_i \frac{\partial u_j}{\partial x_1} \right) ds,$$

(7.8)

where $n$ represents the normal vector to the contour $\Gamma$ as represented in Fig. 7.3, $x_1$ is the direction of the crack orientation, and $n_1$ is the component of $n$ that is parallel to $x_1$. $W$ is the strain energy density function, defined by

$$W(\varepsilon_{kl}) = \int_0^{\varepsilon_{kl}} \sigma_{ij} d\varepsilon_{ij}.$$  

(7.9)

In both the linear elastic case and if the material experiments small-scale yielding at the crack tip [60], the $J-$integral is actually equal to the strain energy release rate $G$.

Although the $J-$integral is shown to be path independent in the continuum mechanics framework, some variations are observed in the numerical solution. To reduce this numerical path dependency, a common technique is to transform the contour integral given in (7.8) into a surface (or volume in the three-dimensional case) integral, as derived in [61]. We obtain

$$J = \int_A \left( \sigma_{ij} \frac{\partial u_j}{\partial x_1} - W \delta_{1j} \right) \frac{\partial q}{\partial x_j} dA,$$

(7.10)

where $A$ is the area enclosed by the chosen contour $\Gamma$, and $q$ can be any continuous function that is equal to zero on $\Gamma$, and equal to one at the crack tip. As can be seen
from (7.10), only the parts of the domain $A$ where $q$ is not constant have a contribution to the integral. It is hence good practice to define a subdomain of $A$, centered on the crack tip, on which the function $q$ is constant and equal to one. On the rest of the domain $A$, $q$ can be defined as a function ramping down to zero on $\Gamma$. This approach avoids evaluating stresses and strains at the vicinity of the crack tip, which are typically of lower accuracy due to the crack tip singularity.

7.2 Quasi-Linear RKPM for Robust Crack Modeling

As covered in the previous section, a natural way to include the discontinuity along the crack surface in the approximation space is through the visibility criterion: regions close to the crack tips can be treated with explicit or implicit enrichment, while the rest of the crack surface is accounted for by truncating the supports of neighboring particles along the crack. For general crack geometries, it can be difficult to maintain linear basis approximation functions, as the supports of particles along the crack interface can be truncated to the point where not enough neighboring nodes are present to construct a first order moment matrix. The Quasi-Linear RKPM framework provides a robust yet accurate solution to this problem.

7.3 Generalized RKPM for Accurate Implicit Enrichment

The RKPM construction relies on using polynomial bases the approximation space. This framework can also be used to include other non polynomial special functions, which is the approach taken with the implicit enrichment technique. For the sake of computational efficiency and numerical stability, these enrichment functions are only
included in regions surrounding crack tips, while other parts of the domain are represented with simple polynomial bases. This creates a numerical interface between enriched and non-enriched regions along which approximation functions can be non-conforming, leading to a discontinuity in the approximation. The Generalized RKPM framework introduced in Chapter 4 is used to construct intrinsically enriched approximation functions at the vicinity of the crack tip while minimizing this discontinuity. In the domain away from crack tips, regular RK approximation functions $\tilde{\Psi}_I(x)$ defined as

$$
\tilde{\Psi}_I(x) = H^T(\theta) M^{-1}(x) H(x - x_I) \phi_a(x - x_I)
$$

are employed, and for each region around the $i$-th crack tip, intrinsically enriched G-RKPM approximation functions are constructed as:

$$
\Psi_{e,i}(x, r_i, \theta_i) = \tilde{c}_i^T(r_i, \theta_i) M^{-1}_e(x, r_i, \theta_i) H_e(x - x_I, r_i, \theta_i) \phi_a(x - x_I),
$$

where $H_e$ and $M_e$ are defined in (7.6) and (7.7) respectively, and

$$
\tilde{c}_i^T(r, \theta) =
\begin{bmatrix}
c_{i0} & c_{i1} & c_{i2} & c_{i3} \sqrt{r} \sin\left(\frac{\theta}{2}\right) & c_{i4} \sqrt{r} \cos\left(\frac{\theta}{2}\right) & c_{i5} \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta) & c_{i6} \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta)
\end{bmatrix}.
$$

The coefficients $c_i$ are solved to minimize the least square functional $\Pi_i(\Psi_{e,i})$ quantifying the discontinuity on the subdomain interface $\Gamma$, yielding $c_i^*$ defined as

$$
c_i^* = \min_{c_i} \Pi = \int_{\Gamma} \left\{ \sum_{I} \left[ \Psi_{I}^{e,i}(x) - \tilde{\Psi}_I(x) \right]^2 \right\} d\Gamma.
$$
Fig. 7.4: Example of discontinuity at the interface between linear basis approximation function and crack tip enriched approximation function.

Fig. 7.4 illustrates the strong discontinuity at the subdomains interface arising from regular RKPM approximation. This discontinuity is greatly reduced in Fig. 7.5, constructed with generalized RKPM according to (7.14).

7.4 Study of a finite crack embedded in an infinite plate

In this section, we consider an finite crack of length 2a, embedded in an infinite plate subjected to a remotely applied uniform traction $\sigma_\infty$, as illustrated in Fig. 7.6. The analytical solution of this problem is given by

$$2\mu u = \frac{\kappa - 1}{2} \sigma_\infty (r_1 r_2)^{1/2} \cos \left( \frac{\phi_1 + \phi_2}{2} \right) - \frac{\sigma_\infty r^2}{(r_1 r_2)^{1/2}} \sin \phi \sin \left( \phi - \frac{\phi_1 + \phi_2}{2} \right), \quad (7.15a)$$

$$2\mu v = \frac{\kappa + 1}{2} \sigma_\infty (r_1 r_2)^{1/2} \sin \left( \frac{\phi_1 + \phi_2}{2} \right) - \frac{\sigma_\infty r^2}{(r_1 r_2)^{1/2}} \sin \phi \cos \left( \phi - \frac{\phi_1 + \phi_2}{2} \right), \quad (7.15b)$$
Figure 7.5: Example of interface between linear basis approximation function and crack tip enriched approximation function, using Generalized RKPM.

\[
\sigma_{xx} = \sigma_\infty \left[ \frac{r}{(r_1 r_2)^{1/2}} \cos \left( \frac{\phi - \phi_1 + \phi_2}{2} \right) - \frac{a^2 r}{(r_1 r_2)^{3/2}} \sin \phi \sin \left( \frac{3 (\phi_1 + \phi_2)}{2} \right) \right], \quad (7.15c)
\]

\[
\sigma_{yy} = \sigma_\infty \left[ \frac{r}{(r_1 r_2)^{1/2}} \cos \left( \frac{\phi - \phi_1 + \phi_2}{2} \right) + \frac{a^2 r}{(r_1 r_2)^{3/2}} \sin \phi \sin \left( \frac{3 (\phi_1 + \phi_2)}{2} \right) \right], \quad (7.15d)
\]

\[
\sigma_{xy} = \sigma_\infty \frac{a^2 r}{(r_1 r_2)^{3/2}} \sin \phi \cos \left( \frac{3 (\phi_1 + \phi_2)}{2} \right), \quad (7.15e)
\]

where \( \kappa = \frac{3 - \nu}{1 + \nu} \) under the plane stress assumption. The variables \( \phi, \phi_1, \phi_2, r, r_1, r_2 \) are defined in Fig. 7.6, \( \lambda \) and \( \mu \) are the Lamé constants, and \( \nu \) is the Poisson’s ratio. This problem involving an infinite domain can be transformed into a finite domain problem by using the analytical solutions expanded in (7.15) to prescribe essential or natural boundary conditions on the finite domain of study. It is also worth mentioning that the problem is symmetric with respect to the \( x \) and \( y \) axis. In the following analysis, only the half domain with symmetry with respect to the \( y \) axis is employed, as illustrated in Fig. 7.7. The problem is studied with both traditional RKPM and generalized RKPM with minimization of the interface discontinuity as presented in section 7.3. Error analysis in the \( L_2 \) and \( H^1 \) norms are provided in Table 7.1 and Table 7.2 using RKPM and G-RKPM,
respectively, and are illustrated in Fig. 7.8 for the $L_2$ norm and Fig. 7.9 for the $H^1$ norm. The normalized stress intensity factor is defined as

$$ F_I = \frac{K_I}{\sigma_\infty \sqrt{\pi a}}, \quad (7.16) $$

where $K_I$ is the stress intensity factor and $\sigma$ is the applied biaxial loading, and the results are listed in Table 7.3 and illustrated in Fig. 7.10. The analytical normalized stress intensity factor is given as $F_I = 0.7071$. Both methods yield expected convergence properties in the $L_2$ and $H^1$ norms of the error in the displacement field, though G-RKPM provides a significant increase of accuracy, especially in the $H^1$ norm. The major contribution of GRPKM can be seen in the stress intensity factor calculation: the strong discontinuity at the subdomains interface causes the RKPM analysis to inaccurately predict the stress intensity factor, and refining the discretization does not reduce the error. Results obtained with G-RKPM however provide a more reliable estimation, and the relative error uniformly decreases as the discretization is refined.

**Figure 7.6**: Finite crack embedded in an infinite plate, subjected to a remote load $\sigma_\infty$. 
Figure 7.7: Deformed shape of the studied finite dimensional plate containing an edge crack.

Table 7.1: Error analysis in the $L_2$ and $H_1$ norms using intrinsically enriched RKPM.

<table>
<thead>
<tr>
<th># nodes</th>
<th>$h$</th>
<th>$L_2$ error $e_{L_2}$</th>
<th>rate</th>
<th>$H^1$ error $e_{H^1}$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>676</td>
<td>40.00</td>
<td>$5.85 \times 10^{-3}$</td>
<td>–</td>
<td>$1.42 \times 10^{-5}$</td>
<td>–</td>
</tr>
<tr>
<td>1296</td>
<td>28.57</td>
<td>$4.13 \times 10^{-3}$</td>
<td>1.03</td>
<td>$1.13 \times 10^{-5}$</td>
<td>0.66</td>
</tr>
<tr>
<td>3364</td>
<td>17.54</td>
<td>$2.49 \times 10^{-3}$</td>
<td>1.04</td>
<td>$8.37 \times 10^{-6}$</td>
<td>0.62</td>
</tr>
<tr>
<td>4356</td>
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<td>1.06</td>
<td>$7.71 \times 10^{-6}$</td>
<td>0.62</td>
</tr>
<tr>
<td>5776</td>
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<td>1.02</td>
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</tr>
<tr>
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<tr>
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<td>0.62</td>
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</tbody>
</table>

Table 7.2: Error analysis in the $L_2$ and $H_1$ norms using intrinsically enriched generalized RKPM.

<table>
<thead>
<tr>
<th># nodes</th>
<th>$h$</th>
<th>$L_2$ error $e_{L_2}$</th>
<th>rate</th>
<th>$H^1$ error $e_{H^1}$</th>
<th>rate</th>
</tr>
</thead>
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<td>–</td>
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<td>$2.14 \times 10^{-3}$</td>
<td>1.07</td>
<td>$8.37 \times 10^{-6}$</td>
<td>0.64</td>
</tr>
<tr>
<td>4356</td>
<td>15.38</td>
<td>$1.87 \times 10^{-3}$</td>
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<td>$7.71 \times 10^{-6}$</td>
<td>0.62</td>
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<td>1.03</td>
<td>$6.13 \times 10^{-6}$</td>
<td>0.56</td>
</tr>
<tr>
<td>15876</td>
<td>8.00</td>
<td>$9.47 \times 10^{-4}$</td>
<td>1.03</td>
<td>$5.47 \times 10^{-6}$</td>
<td>0.50</td>
</tr>
<tr>
<td>24336</td>
<td>6.45</td>
<td>$7.64 \times 10^{-4}$</td>
<td>1.00</td>
<td>$4.78 \times 10^{-6}$</td>
<td>0.60</td>
</tr>
</tbody>
</table>
Table 7.3: Convergence of the normalized stress intensity factor using intrinsically enriched RKPM and generalized RKPM.

<table>
<thead>
<tr>
<th># nodes</th>
<th>$h$</th>
<th>$F_I$</th>
<th>error</th>
<th>$F_I$</th>
<th>error</th>
</tr>
</thead>
<tbody>
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<td>2.49%</td>
</tr>
<tr>
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<td>4.19%</td>
<td>0.7192</td>
<td>1.72%</td>
</tr>
<tr>
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<td>17.54</td>
<td>0.6718</td>
<td>4.99%</td>
<td>0.7149</td>
<td>1.11%</td>
</tr>
<tr>
<td>4356</td>
<td>15.38</td>
<td>0.6713</td>
<td>5.07%</td>
<td>0.7145</td>
<td>1.05%</td>
</tr>
<tr>
<td>5776</td>
<td>13.33</td>
<td>0.6699</td>
<td>5.26%</td>
<td>0.7140</td>
<td>0.97%</td>
</tr>
<tr>
<td>10000</td>
<td>10.10</td>
<td>0.6683</td>
<td>5.50%</td>
<td>0.7125</td>
<td>0.76%</td>
</tr>
<tr>
<td>15876</td>
<td>8.00</td>
<td>0.6668</td>
<td>5.69%</td>
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<td>24336</td>
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<td>0.6667</td>
<td>5.71%</td>
<td>0.7108</td>
<td>0.52%</td>
</tr>
</tbody>
</table>

Figure 7.8: Convergence study in the $L_2$ norm using RKPM and G-RKPM. Errors and rates are provided in Tables 7.1 and 7.2.
Figure 7.9: Convergence study in the $H^1$ norm using RKPM and G-RKPM. Errors and rates are provided in Tables 7.1 and 7.2.

Figure 7.10: Obtained normalized stress intensity factor $F_I$ using RKPM and G-RKPM, compared to analytical value. Errors are provided in Table 7.3.
Figure 7.11: Convergence study of the normalized stress intensity factor $F_I$ using RKPM and G-RKPM. Errors are provided in Table 7.3.

7.5 Modeling of Multiple Cracks Using Generalized Quasi-Linear RKPM

In the following numerical examples, the visibility criterion is employed to enforce the strong discontinuity along the crack surfaces, and intrinsic enrichment is used around each crack tip. Quasi-Linear RKPM is utilized in the entire domain to obtain a robust construction of approximation functions despite support truncation along the crack face. The generalization of the RKPM framework is also introduced to minimize the discontinuity along the interfaces between enriched and non-enriched regions of the domain. The effect of the latter can be visualized by comparing Fig. 7.4 and Fig. 7.5. These two figures represent an approximation function whose support covers part of the enriched subdomain interface using RKPM and Generalized RKPM respectively. In this case, the additional liberty provided by the generalization allows to greatly reduce the spurious approximation function discontinuity along the interfaces between enriched and
unenriched domains.

7.5.1 Cross-shaped cracks in a finite plate

A finite square plate of dimensions $2L \times 2L$ with a cross-shaped crack under biaxial loading is modeled, as shown in Fig. 7.12. A discretization of $36 \times 36$ nodes is employed, and for various ratios of crack length $a$ over plate dimension $L$ the stress intensity factor is computed using the domain form of the J-Integral presented above. The enriched subdomains dimensions are defined as the minimum size such that every node whose support overlaps with the crack tip belongs to the enriched region. Normalized stress intensity factors defined as

$$F_I = \frac{K_I}{\sigma \sqrt{\pi a}}$$  \hspace{1cm} (7.17)

are given in Table 7.4 and shown in Fig. 7.13, where $K_I$ is the stress intensity factor and $\sigma$ is the applied biaxial loading. Both QL-RKPM and Generalized QL-RKPM with minimized spurious discontinuity are compared to reference solutions [62, 63]. Though QL-RKPM provides a reasonable trend as $a/L$ varies, the magnitude of the stress intensity factor is underestimated over the entire range of $a/L$ ratio, while the generalized quasi-linear RK formulation yields more accurate stress intensity factor calculations. Distributions of displacement field and stress fields around one crack tip are shown in Fig. 7.14 and Fig. 7.15 respectively, with both QL-RKPM and Generalized QL-RKPM, and it can be observed that the spurious discontinuity is typically more pronounced when the Generalized formulation is not employed, and distributions of displacements and stress along the line $x = L/2$ are given in Fig. 7.16, Fig. 7.17 and Fig. 7.18 for the case $a/L = 0.5$. 

Figure 7.12: Cross-shaped crack – Problem statement (left) and deformed shape (right). Evaluation points in the intrinsically enriched region are shown in red.

Figure 7.13: Cross-shaped crack – Normalized stress intensity factors for various $a/L$ ratios.
Table 7.4: Cross-shaped crack – Normalized stress intensity factors for various $a/L$ ratios.

<table>
<thead>
<tr>
<th>$a/L$</th>
<th>Reference</th>
<th>QL-RKPM</th>
<th>G-QL-RKPM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.915</td>
<td>0.883</td>
<td>0.922</td>
</tr>
<tr>
<td>0.4</td>
<td>0.957</td>
<td>0.914</td>
<td>0.965</td>
</tr>
<tr>
<td>0.5</td>
<td>1.025</td>
<td>0.980</td>
<td>1.036</td>
</tr>
<tr>
<td>0.6</td>
<td>1.135</td>
<td>1.085</td>
<td>1.146</td>
</tr>
<tr>
<td>0.7</td>
<td>1.317</td>
<td>1.260</td>
<td>1.331</td>
</tr>
<tr>
<td>0.8</td>
<td>1.639</td>
<td>1.581</td>
<td>1.649</td>
</tr>
</tbody>
</table>

(a) QL-RKPM, $u_x$ distribution. (b) G-QL-RKPM, $u_x$ distribution.
(c) QL-RKPM, $u_y$ distribution. (d) G-QL-RKPM, $u_y$ distribution.

Figure 7.14: Cross-shaped crack – Displacement field distribution around crack tip located at $(-250.0, 0.0)$. 
Figure 7.15: Cross-shaped crack – Stress field distribution around crack tip located at $(-250.0, 0.0)$. 

(a) QL-RKPM, $\sigma_{xx}$ distribution. 
(b) G-QL-RKPM, $\sigma_{xx}$ distribution. 
(c) QL-RKPM, $\sigma_{yy}$ distribution. 
(d) G-QL-RKPM, $\sigma_{yy}$ distribution. 
(e) QL-RKPM, $\sigma_{xy}$ distribution. 
(f) G-QL-RKPM, $\sigma_{xy}$ distribution.
Figure 7.16: Cross-shaped crack – Displacement $u_x(x, y)$ along the line $x = 250.0$ as obtained with QL-RKPM and G-QL-RKPM. The enriched region is shown in gray.

Figure 7.17: Cross-shaped crack – Stress $\sigma_{xx}(x, y)$ along the line $x = 250.0$ as obtained with QL-RKPM and G-QL-RKPM. The enriched region is shown in gray.
Figure 7.18: Cross-shaped crack – Stress $\sigma_{yy}(x,y)$ along the line $x = 250.0$ as obtained with QL-RKPM and G-QL-RKPM. The enriched region is shown in gray.

7.5.2 Star-shaped cracks in a finite plate

Similarly to the previous example, a finite square plate of dimensions $2L \times 2L$ under biaxial loading is modeled with a $36 \times 36$ nodes discretization. A star-shaped crack as shown in Fig. 7.19 is studied, where the angle between each crack branch is set as $\pi/3$, and the length of each branch is $a$. Again, the plate is analyzed under various of ratios $a/L$ using RKPM and G-RKPM. Table 7.5 and Fig. 7.20 show the normalized stress intensity factors as defined in (7.17) for each case at crack tip $A$, compared to reference solutions [62, 63]. Similarly to the cross-shaped crack, the QL-RKPM results show a correct trend but tend to underestimate the stress intensity factor, while G-QL-RKPM agrees very well with the reference solution. Distributions of displacements and stress along the line $x = L/2$ are given in Fig. 7.21, Fig. 7.22 and Fig. 7.23 for the case $a/L = 0.5$. 
Figure 7.19: Star-shaped crack – Problem statement (left) and deformed shape (right). Evaluation points in the intrinsically enriched region are shown in red.

Figure 7.20: Star-shaped crack – Normalized stress intensity factors at crack tip A for various $a/L$ ratios.
Table 7.5: Star-shaped crack – Normalized stress intensity factors at crack tip A for various $a/L$ ratios.

<table>
<thead>
<tr>
<th>$a/L$</th>
<th>Normalized Stress Intensity Factor $F_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reference</td>
</tr>
<tr>
<td>0.3</td>
<td>0.785</td>
</tr>
<tr>
<td>0.4</td>
<td>0.825</td>
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<tr>
<td>0.5</td>
<td>0.881</td>
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<td>0.6</td>
<td>0.976</td>
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<tr>
<td>0.7</td>
<td>1.114</td>
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<tr>
<td>0.8</td>
<td>1.358</td>
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</table>

Figure 7.21: Star-shaped crack – Displacement $u_x(x,y)$ along the line $x = 250.0$ as obtained with QL-RKPM and G-QL-RKPM. The enriched region is shown in gray.
Figure 7.22: Star-shaped crack – Stress $\sigma_{xx}(x,y)$ along the line $x = 250.0$ as obtained with QL-RKPM and G-QL-RKPM. The enriched region is shown in gray.

Figure 7.23: Star-shaped crack – Stress $\sigma_{yy}(x,y)$ along the line $x = 250.0$ as obtained with QL-RKPM and G-QL-RKPM. The enriched region is shown in gray.
Chapter 8

Conclusion


8.1 Conclusions

The approximation functions commonly employed in RKPM achieve arbitrary order completeness by enforcing reproduction of the monomial basis functions. This condition is extended with the proposed generalized construction, leading to an entire family of approximation functions for a given nodal distribution, kernel support size and order of completeness. This generalization can be exploited to introduce desirable properties in the approximation space, as demonstrated through $p$-refinement and oscillation diminishing capabilities.

The semi-Lagrangian formulation is effective in handling extremely large deformations and material fragmentation, by reconstructing the approximation functions in the deformed configuration. Maintaining linear accuracy in these simulations, however, can be very challenging as it requires constantly updating the kernel support size to prevent the moment matrix to become singular. This can be unrealistic to achieve in fragmented areas of the domain, but locally lowering the order of approximation to constant basis introduces undesirable discontinuities in the approximation space. A quasi-linear RKPM formulation has been proposed precisely to circumvent this difficulty, and the resulting construction has been demonstrated to be as robust as constant basis approximation, while greatly enhancing the accuracy of the solution. This technique has been applied to various impact and fragmentation problems, and has been showed to provide satisfying results.

The generalized and quasi-linear RKPM construction has been applied to fracture mechanics modeling using intrinsic crack tip enrichment. The visibility criterion is employed along the crack surface to account for the discontinuous displacement field, and asymptotic solutions around the crack tip are added to the basis functions forming the approximation space around the crack tip. The quasi-linear formulation ensures
moment matrix nonsingularity despite the kernel support truncation originating from the visibility criterion. The generalized RKPM is used to construct the crack-tip-enriched approximation functions such that the discontinuity at the interface between enriched and non-enriched domains is minimal. The method has been applied to various crack geometries, and comparisons have been made with reference solutions. It has also been shown that the generalized RKPM results give enhanced accuracy in terms of displacement and stress fields, and stress intensity factor.

8.2 Recommendations for Future Research

Directions for future research are recommended as follows:

- Application of quasi-linear RKPM to extreme events simulation, including material deformation and fragmentation under blast loading, landslide simulation under earthquake loading, etc.

- Extension of the proposed quasi-linear RKPM formulation to higher order polynomial completeness. This work focuses on improving the robustness of semi-Lagrangian simulations while providing close-to-linear consistency. A similar approach can be used to obtain robust construction of higher order approximation by carefully choosing the set of sampling points.

- Application of generalized RKPM to crack propagation using intrinsic enrichment. The proposed framework can be applied to intrinsic crack tip enrichment for either quasi-static crack propagation or dynamic simulation.
Bibliography


