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Author
Oery, Aniko

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Essays on Dynamic Microeconomic Theory

by

Aniko Öry

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in

Economics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Benjamin E. Hermalin, Co-chair
Professor William Fuchs, Co-chair
Professor David S. Ahn
Professor Ganesh Iyer

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Essays on Dynamic Microeconomic Theory

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Abstract

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Doctor of Philosophy in Economics

University of California, Berkeley

Professor Benjamin E. Hermalin, Co-chair

Professor William Fuchs, Co-chair

This dissertation studies three different dynamic environments and trade-offs that arise from the dynamic structure of the problem.

The first chapter considers a dynamic pricing problem. The Internet allows sellers to track “window shoppers,” consumers who look but do not buy, and to lure them back later by targeting them with an advertised sale. This new technology thus facilitates intertemporal price discrimination, but simultaneously makes it too easy for a seller to undercut her regular price. Because buyers know they could be lured back, the seller is forced to set a lower regular price. Advertising costs can, therefore, serve as a form of commitment: a seller can actually benefit from higher costs of advertising. Based on this framework, the impact of advertising costs on prices, profits, and welfare are analyzed using a dynamic pricing model. Furthermore, it is demonstrated how buyers’ time preferences give rise to price fluctuation or an everyday-low-price in equilibrium.

The second chapter originates from a joint work with William Fuchs and Andrzej Skrypacz. We analyze a dynamic market with adverse selection and correlated values. Uninformed buyers compete inter- and intra-temporarily for a good that is sold by an informed seller who is suffering a liquidity shock. We contrast a transparent (public price offers) with an opaque (private price offers) information structure. For infrequent trading, all equilibria with private offers are in pure strategies and coincide with equilibria with public offers. If the frequency of trade exceeds a certain threshold, then pure strategy equilibria cannot be sustained with private offers. We characterize the set of mixed-strategy equilibria with private offers if the seller has two opportunities to trade before the deadline and show that any equilibrium with private offers Pareto-dominates the unique pure strategy equilibrium with public offers. We provide a strong intuition for why these results extend to more general settings.

The third chapter is based on joint work with Michèle Müller-Itten and it is concerned with dynamic affirmative action policies. We provide a dynamic overlapping generations
framework to analyze the costs and benefits of affirmative action policies if mentoring complementarities are present. The old population boosts the young population’s productivity through mentoring, assuming the mentoring boost is increasing in the fraction of mentors from the same group. In such a framework, the main trade-off is between using the most able workers and making the optimal use of the mentoring complementaries. A balanced labor force is always a steady state. We establish conditions under which this steady state is welfare optimal and stable. Affirmative action is more likely to be welfare-enhancing in markets with an under-supply of workers because it can increase the total labor force. This effect is the largest with a subsidy in favor of minority workers because no majority workers will be crowded out. We also establish conditions under which drastic affirmative action policies should be implemented initially and terminated after one period. Finally, we present some simulations of the effect of different policies for a continuous time version of the model.
To my parents Yumiko Kemmoku-Öry and Zsolt Öry.
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Chapter 1

Consumers on a Leash: Advertised Sales and Intertemporal Price Discrimination

1.1 Introduction

The advent of tracking via the use of “cookies” by online sellers allows those sellers to keep “window shoppers,” consumers who look but do not buy, on a leash. Sellers can reel in window shoppers, who are likely to be responsive to a discounted price, through a targeted email announcing a discounted price. The consequences, however, of this kind of tracking and targeting on the dynamics of prices, profits, and welfare are ambiguous. While it allows sellers to intertemporally price discriminate, it also hinders commitment to future prices because sellers are always tempted to offer a sale when they learn about the interest of a shopper.

This paper proposes a dynamic pricing model in which a monopolist uses targeted sales in order to recall window shoppers and in which the costs of such sales (advertising costs) endogenously provide the monopolist with some ability to commit to high prices for an extended period of time. Specifically, a seller can set prices aimed at high-valuation buyers today, knowing that she can collect low-valuation buyers later via a targeted sale. The profitability of this strategy is partially undermined if the high-valuation buyers, anticipating a future sale, decide to wait. Consequently, the seller faces a trade-off between having frequent sales, allowing her to sell to low-valuation buyers with less delay, and maintaining a high regular price. Finally, if the seller cannot commit to her frequency of sales, she is exposed to the logic of the Coase conjecture (Coase 1972). In particular, were advertising costless, the seller would like to immediately offer a low price to a buyer after he has revealed his low valuation by not buying. In that case, only a low price can be sustained in equilibrium.

1See also Stokey 1979, Bulow 1982, Fudenberg, Levine, and Tirole 1985, Gul, Sonnenschein, and Wilson 1986
CHAPTER 1. CONSUMERS ON A LEASH: ADVERTISED SALES AND INTERTEMPORAL PRICE DISCRIMINATION

How valuable advertising costs are as a commitment device to the seller and how profitable price discrimination is compared to a constant low price depends critically on buyers’ time preferences. While the cost of advertising determines the frequency of sales, buyers’ time preferences determine the regular price that induces high-valuation buyers to buy. For example, if buyers are extremely impatient, then the seller can extract the entire surplus from high-valuation buyers by charging a high regular price regardless of the frequency of sales. In that case, commitment is of little value. In contrast, with very patient buyers, the regular price can be set at a profitable level only if the seller has very infrequent sales. Then, infinite advertising costs become profit-maximizing, but moderate levels of advertising costs are of no value to the seller as a commitment device. For intermediately patient buyers, moderate levels of advertising costs provide the seller with a level of commitment that is optimal to the seller.

To formalize these ideas, I consider buyers to arrive randomly over time. Past buyers’ arrivals and purchasing decisions are the seller’s private information. Although buyers are forward-looking and rational, they are only actively in the market upon arrival and when the seller “reactivates” them through costly advertising. Furthermore, buyers do not observe past price paths and sales, so they assign the same probability to all possible equilibrium histories given the observed price. Put differently, buyers know the frequency of sales but not the exact timing of the next sale.

I focus on stationary equilibria in which high-valuation buyers buy upon arrival: hence, the number of low-valuation buyers who have not bought is the relevant state variable. I show that the seller either chooses a constant low price (everyday-low-price or EDLP equilibrium) or a high regular price with occasional sales (advertising equilibrium). In an advertising equilibrium, the seller accumulates a fixed number of low-valuation buyers before having an advertised sale. I call this number the “cutoff-demand”. The constant regular price must make high-valuation buyers indifferent between buying and waiting. The regular price and cutoff-demand uniquely describe the equilibrium price path.

The equilibrium frequency of sales in an advertising equilibrium is determined by two conditions: the seller cannot have an incentive to hold a sale before the cutoff-demand is reached, nor any incentive to wait after it is reached. These conditions are independent of the arrival of high-valuation buyers and the regular price. For any given advertising cost, a (generically) unique cutoff-demand satisfies these conditions. In particular, sales are less frequent if advertising costs are high. Moreover, I show that an advertising equilibrium exists if and only if the frequency of sales supported by the cost of advertising results in a high enough regular price to support profits in excess of an EDLP strategy. An EDLP equilibrium always exists.

After characterizing the equilibrium outcome, I elaborate on the role of buyers’ time preferences and the value of advertising as a commitment device. To that end, I first consider the full commitment benchmark, in which the seller can commit to a frequency of sales at time zero. In that case, the seller only faces a trade-off between frequent sales and a high
regular price. Buyers’ time preferences specify how sensitive the regular price is to changes in the frequency of sales. The optimal frequency of sales is first decreasing in the discount rate of buyers and then increasing. On the one hand, if buyers are infinitely impatient, the seller can perfectly price discriminate and prefers very frequent sales. On the other hand, if buyers have a finite discount rate, less frequent sales allow the seller to increase the price significantly. If buyers are, however, very patient, the regular price is close to the low-valuation of buyers even with a delay in trade, such that the seller prefers very frequent sales or no sales at all.

Without ex-ante commitment, the seller’s benefit from advertising costs as a commitment device is thus critically affected by buyers’ time preferences. I distinguish between the following three cases:

1. If buyers are relatively patient, an equilibrium with periodic sales exists only for large enough advertising costs. The profit-maximizing advertising cost is infinite.

2. For intermediate levels of buyer patience, low advertising costs can increase the seller’s profit significantly because she can commit to a lower frequency of sales and, hence, raise the regular price sufficiently to make price discrimination profitable. However, for intermediate advertising costs, the cost of advertising outweighs the benefit from increasing the regular price, so the unique equilibrium is an EDLP equilibrium. For high advertising costs, an equilibrium with sales exists and the sellers profits converge to the static monopoly profit. The seller prefers small or infinite advertising costs.

3. For highly impatient buyers, a positive advertising cost that gives enough commitment power to not have sales after each arrival suffices to sustain periodic sales in equilibrium. As buyers become more impatient, the seller’s profit-maximizing advertising cost decreases.

Hence, profits are non-monotonic in advertising costs. In the region in which finite advertising costs are optimal, the optimal advertising cost is first increasing and then decreasing in the impatience of buyers.

Finally, I show how market size (i.e., the rate at which buyers arrive) influences the frequency of sales. This is important from an empirical perspective. It also has normative implications because social welfare is increasing in the frequency of sales. The average time between two sales is decreasing in the market size, even though more buyers are being reactivated at each sale. Hence, the regular price is lower in large markets and consumers capture more surplus. At the same time, in high-volume, low-ticket markets such as headphones, sales are less frequent than in low-volume, high-ticket markets such as furniture, if both markets are about the same size in terms of revenues.

These results not only have implications for prices in online markets, but also for the budget firms want to allocate to email advertising campaigns. While it is unrealistic for sellers to commit to infinite or very large advertising costs, sellers have an incentive to
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commit to moderate amounts. Depending on the time preferences of her customers, a seller can benefit from developing a reputation for fancy or elaborate - thus expensive - email advertising. Related to this finding, I show that the seller always prefers a fixed advertising cost to a variable advertising cost that depends on the number of people contacted.\(^2\) The significance of email advertising in general is, in part, reflected by the 2 billion dollars that US firms are forecast to spend on it in 2014.\(^3\) Online retail sales in general accounted for 231 billion dollars in 2012 and is predicted to reach 370 billion dollars by 2017.\(^4\)

More broadly, this paper helps to understand the long-run equilibrium implications of the seller’s ability to recall buyers. While targeting per se will always benefit consumers, if the static monopoly price is high, it can hurt or benefit the seller. It is a well-known fact that the ability to track and target reduces advertising costs by resolving an old dilemma of advertising, first pinpointed by John Wanamaker with his famous quote:

“Half the money I spend on advertising is wasted. The trouble is I don know which half.”\(^5\)

However, if buyers are relatively impatient, it is better for a firm not to collect customers’ contact information. Nevertheless, if buyers are sufficiently patient, the seller earns more than static monopoly profits with some intermediate advertising costs.

This paper is mostly related to the intertemporal price discrimination papers by Conlisk, Gerstner, and Sobel 1984 and Sobel 1991, in which sellers have limited commitment for exogenous reasons. In their models, the frequency of sales is determined by the length of a period rather than the cost of advertising. Sobel 1991 shows that as period length goes to zero, stationary equilibria satisfy the Coase conjecture. This paper shows that advertising costs may permit sellers to escape the logic of the Coase conjecture by endogenously creating limited commitment. Moreover, my model allows for a detailed analysis of the interplay of the value for commitment and time preferences. Thus, this paper extends our understanding of intertemporal price discrimination in general.

The idea that advertising can serve to activate buyers has been used in static models of advertising, such as Iyer, Soberman, and Villas-Boas 2005. They consider oligopolistic

\(^2\)This suggests that in connection with sales, firms are willing to pay more to an advertising platform that has fixed rates rather than per-click or per-view rates as commonly used. But for the pricing of banner advertising, intermediary agents are involved, such that asymmetric information and moral hazard can play an important role.


\(^5\)See for example Hoffman and Novak 2000.
markets where several sellers compete for buyers through advertising. In my setup, advertising instead creates competition between the seller and her future self by making buyers long-lived, allowing her to intertemporally price discriminate. A detailed discussion of the related literature can be found in section 1.6.

The paper is organized as follows. In Section 2.2, I introduce the model and equilibrium notion. Section 1.3 characterizes all stationary equilibria in which high-valuation buyers buy upon arrival. Using these results, I present some comparative statics about the frequency of sales and advertising costs in Section 1.4. In Section 1.5, I investigate the role of buyers’ time preferences on the equilibrium outcome and the benefit of advertising costs as a commitment device to the seller. Finally, in Section 1.6, I discuss some assumptions and results and relate the paper to the relevant literature. Section 1.7 concludes the analysis.

1.2 Model

Basics

A monopolist (she) sells a homogeneous good over time. Time is continuous. For simplicity, and in order to abstract away from inventory considerations, I assume that the seller’s marginal cost is equal to zero.

 Buyers arrive according to a Poisson process with arrival rate \( \lambda \). Each buyer (he) wants, at most, one unit of the good in his lifetime and his valuation is his private information.\(^6\) Valuations of buyers are independently distributed and independent of arrivals. They are high, \( v_H \), with probability \( \pi \) and otherwise low, \( v_L \), where \( v_H > v_L > 0 \). Hence, low-valuation buyers arrive according to a Poisson process with an arrival rate \( \lambda(1 - \pi) \) and high-valuation buyers arrive according to an independent Poisson process with arrival rate \( \lambda\pi \). The seller privately observes arrivals of the buyers and whether a buyer has bought or not.

Buyers observe the price when they arrive, but they do not observe past or future prices.\(^7\) They believe that all arrival times are equally likely.\(^8\) Hence, all buyers have the same belief about the upcoming price path. Upon arrival, a buyer decides whether to buy or wait. A buyer who waits can only buy if he comes back later at a search cost \( c > 0 \) or if “reactivated” by the seller. The seller can inform window shoppers who have previously visited, but not bought, about a sale via a targeted advertising at a cost \( C_A > 0 \). For now, I assume that the advertising cost is a fixed cost no matter how many buyers are being reactivated. I discuss the implications of advertising costs being variable in Section 1.6. Call a point in time at which the seller chooses to advertise an \textit{advertised sales period}. All other times are \textit{regular periods}. The seller chooses her price and advertising strategy at each point in time.

---

\(^{6}\)I discuss in Section 1.6 how the assumption that each buyer demands only a single unit can be relaxed.

\(^{7}\)If buyers can observe past price paths, the qualitative results of the paper will not change. In that case, the regular price is not constant, but continuously decreasing over time until the seller has an advertised sale.

\(^{8}\)I formalize this when I introduce the equilibrium notion.
The seller’s discount rate is \( r_s \) and the buyers’ \( r_b \). If the seller sells to \( N_t \) buyers in period \( t \) at a price \( p_t \), then her profit at time \( t \) is \( \Pi_t = p_t \cdot N_t \) if she does not advertise, and \( \Pi_t = p_t \cdot N_t - C_A \) if she advertises. Her expected discounted profits are hence,

\[
\Pi = \mathbb{E} \left[ \sum_{t : N_t \neq 0} e^{-r_s t} \cdot \Pi_t \right].
\]

A buyer with valuation \( v \) who buys \( \tau \) periods after arrived, receives an expected payoff

\[
U = \mathbb{E} \left[ e^{-r_b \tau} \cdot (v - p_\tau) \right]
\]

relative to the time of arrival if he does not search and \( U = \mathbb{E} \left[ e^{-r_b \tau} \cdot (v - p_\tau - c) \right] \) otherwise. Note that \( \tau \) and \( N_t \) are random variables and the seller’s pricing strategy can be contingent on the information at time \( t \).

**Equilibrium**

I focus on stationary equilibria with the seller’s belief about the pool of inactive buyers in the market being the state variable satisfying the following assumptions:

- If buyers are indifferent between buying and not buying, they buy. (A1)
- Buyers do not change their beliefs about the state variable after observing an unexpected price. (A2)
- All equilibrium prices are greater than or equal to \( v_L \). (A3)

Assumption (A1) essentially restricts equilibria to pure-strategy equilibria. Assumption (A2) can be rationalized by buyers that believe that the seller deviates from the equilibrium price path with very small probability similar to “trembling hand perfect equilibria.” In other words, the seller cannot signal to buyers the current state of the world through the current price. Assumption (A3) is satisfied in all limiting equilibria of the corresponding discrete-time version of the game as the length of the period goes to zero which follows from the argument used in discrete-time bargaining models with incomplete information, such as in Fudenberg, Levine, and Tirole 1985.  

Because of assumptions (A1) and (A3), the unique advertised sales price is given by \( v_L \). Assumption (A2) implies that, in any period in which only high-valuation buyers buy, the price must be equal to the highest price acceptable to high-valuation buyers. Finally, in stationary equilibria, seller and buyers cannot coordinate search and sales times in the following sense: they cannot agree to trade at a discounted price \( t \) periods after a buyer’s

\footnote{For non-stationary equilibria, this property does not hold and the set of equilibria is very rich: Sobel 1991 shows a folk theorem for such equilibria in a discrete-time model for a durable goods monopolist with arriving buyers.}
arrival time, where \( t \) is independent of the state variable. Consequently, on equilibrium path, buyers do not search. Thus, a seller’s ability to track makes search of buyers moot. Nevertheless, search can play a critical role out of equilibrium.

On the equilibrium path, the only payoff-relevant state of the game is the number, \( n \), of buyers who have arrived but not yet bought. These are low-valuation buyers in equilibria satisfying assumptions (A1)-(A3). It is sufficient to focus on strategies of the seller that depend only on the equilibrium state variable \( n \), because one can define out-of-equilibrium beliefs for the buyers such that the seller cannot profitably deviate to a price that neither buyer type would accept. Suppose a high-valuation buyer who observes a price he will not accept believes that he is the only buyer who has seen this deviation. Then, in a continuation game, he only buys during an advertised sale at a price that is lower than the highest price he would have accepted upon arrival. Because future buyers do not observe past deviations by the seller, there is a continuation equilibrium in which the seller moves back to the equilibrium path by treating the buyer like a low-valuation buyer or by advertising a price that is acceptable to him if he had high valuation.

Thus, it is natural to focus on stationary equilibria in which the seller’s equilibrium strategy depends only on \( n \) in all periods. Let the seller’s pricing strategy \( p : \mathbb{N} \to \mathbb{R}_+ \), and advertising strategy \( \sigma : \mathbb{N} \to \{0,1\} \) be functions of the state variable \( n \), where \( \sigma(n) = 1 \) means the seller holds a sale as soon as \( n \) buyers have “accumulated.” Buyers of both types choose an acceptance rule of whether to buy or wait upon arrival given each price \( p \).

Moreover, buyers’ beliefs about the state variable \( n \in \mathbb{N}_0 = \{0,1,2,\ldots\} \) are represented by a distribution \( \mu \) on \( \mathbb{N} \). I refer to the number of buyers who have arrived but not bought that triggers a sale

\[
N = \inf\{n : \sigma(n) = 1\}
\]

as the cutoff-demand. Then, equilibrium strategies \((p,\sigma)\) of the seller that result in a cutoff-demand \( N \), acceptance rule and beliefs \( \mu \) of the buyers, constitute a stationary equilibrium if and only if the following holds:

1. **Seller’s profit maximization:** given buyers’ acceptance rules, \( p(n) \) maximizes the seller’s expected continuation payoff for any \( n \).

2. **Buyers’ utility maximization:** given their belief \( \mu \), high-valuation buyers buy whenever their utility from buying today \( v - p \) is larger than the expected utility of waiting

\[
\mathbb{E}[e^{-r_b \tau}] \cdot (v_H - v_L)
\]

\[\text{10This is in sharp contrast to the literature on obfuscation in online markets in order to increase search costs of buyers, such as Ellison and Ellison 2009 and Ellison and Wolitzky 2012. Search models in oligopolistic setups such as Stahl 1989 seem particularly applicable to brick-and-mortar markets.}\]

\[\text{11On equilibrium path, all buyers buy during a sale, so the issue is the purchase decision upon arrival.}\]
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where $\tau$ is the waiting time until the next sale derived from $\mu$ and the sellers strategy using Bayes’ rule. They wait whenever the utility of waiting is higher. Low-valuation buyers buy if and only if $p \leq v_L$. Finally, each buyer prefers not to search.

3. **Beliefs:** upon arrival, buyers’ equilibrium beliefs $\mu$ must be consistent with the seller’s and other buyers’ strategies in the following sense. Each strategy of the seller and equilibrium acceptance rule of buyers induce a Markov process on $\mathbb{N}$ that increases by one after every arrival of a low-valuation buyer as long as $n < N$ and drops to zero after an arrival of a low-valuation buyer for $n > N$. Then, a buyer’s belief about the state space must be the stationary distribution of this Markov chain

$$
\mu(n) = \begin{cases} 
\frac{1}{N} & \text{for } n \in \{0, \ldots, N-1\} \\
0 & \text{otherwise}
\end{cases}
$$

whenever it exists (i.e., $N < \infty$) and arbitrary otherwise. If the buyer does not purchase, he updates his beliefs according to Bayes’ rule; hence $\mu$ is the uniform distribution on $\{1, \ldots, N\}$.

4. **No out-of-equilibrium signaling:** Whenever a buyer observes an out-of-equilibrium price, his belief about the state of the world does not change.

Conditions 1 and 2 are standard. Condition 3 entails that, in equilibrium, buyers’ beliefs about the state are consistent with the seller’s and other buyers’ strategies. Due to the Markovian structure of the equilibrium, these beliefs must coincide with the stationary equilibrium of the process describing the evolution of states. Condition 4 guarantees that assumption (A2) is satisfied.

Because buyers do not observe the state variable, all regular periods appear the same to buyers upon arrival. In every regular period, the seller either chooses the unique highest price $p_H(N)$ that induces purchase by high-valuation buyers only or a price equal to $v_L$. If she chooses a price of $v_L$, the state variable does not change. Hence, in equilibrium, she either prefers to choose $v_L$ in all periods or a regular price $p_H(N)$ in all regular periods. All equilibria are fully characterized by the frequency of sales given by the cutoff-demand $N$ and the regular price $p_H(N)$.

1.3 Characterization of the Stationary Equilibrium

In this section, I characterize all stationary equilibria.

Preliminaries

First, note that given $C_A < \infty$, $N = \infty$ can only be an equilibrium outcome if the price is $v_L$ as the following lemma shows.
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Lemma 1. In an equilibrium with \( N = \infty \), the price must be constant and equal to \( v_L \). Hence, the seller’s profit is given by

\[
\Pi_L = \frac{\lambda}{r_s} \cdot v_L.
\]

Proof. Let \( t^\lambda_n \) be the random variable describing the \( n \)-th arrival time of a Poisson process with arrival rate \( \lambda \). It can be shown that \( t^\lambda_n \) is distributed according to a gamma distribution \( \Gamma \left( n, \frac{1}{\lambda} \right) \). The corresponding moment generating function is given by

\[
E \left[ e^{st^\lambda_n} \right] = \left( 1 - \frac{s}{\lambda} \right)^{-n}.
\] (1.1)

In any equilibrium in which the seller never advertises, all buyers must buy upon arrival. If not, so if some low-valuation buyers do not buy with a positive probability, then the state variable \( n \) would reach any arbitrarily high level with positive probability. As a result, the instantaneous profit that the seller can make by advertising \( n \cdot v_L - C_A \) can become arbitrarily high with positive probability. Hence, having an advertised sale would be a profitable deviation for the seller for large enough \( n \). Finally, note that all buyers buy upon arrival if and only if the price is constant and equal to \( p = v_L \). Hence, the seller’s profit is given by

\[
\Pi_L = \sum_{i=1}^{\infty} E \left[ e^{-r_s t^\lambda_i} \right] \cdot v_L = \left( 1 + \frac{r_s}{\lambda} \right)^{-1} \cdot v_L = \frac{\lambda}{r_s} v_L.
\]

As a result, an equilibrium can be one of only two kinds:

**Everyday-low-price equilibrium:** \( N = \infty \) and a constant price of \( v_L \).

**Advertising equilibrium:** \( N < \infty \) with a regular price \( p_H(N) \in (v_L, v_H) \).

The structure of an advertising equilibrium with cutoff-demand \( N = 3 \) is illustrated in Figure 1.1. Black circles represent the random arrival times of low-valuation buyers and white circles arrival times of high-valuation buyers. The state variable \( n \) increases by one after arrivals of low-valuation buyers. When the cutoff-demand is reached (i.e., \( n = N \)) the state drops to \( n = 0 \) and a new cycle starts. The price path is given by a regular price \( p_H(3) \) that is accepted by all high-valuation buyers and advertised sales at price \( v_L \) at which all \( N \) low-valuation buyers buy.

In the following, I focus on properties of the cutoff-demand \( N \) and regular price \( p_H(N) \) of advertising equilibria.

**Buyers’ beliefs.** In equilibrium, buyers hold correct beliefs about \( N \), but they do not know when the last sale has happened. Consequently, they do not know the number of buyers, \( n \), who have yet to buy and they assign equal probability to the states \( n \in \{0, \ldots, N-1\} \). As a result, their belief about the waiting time until the next sale, is the same. The distribution of the equilibrium expected waiting time \( \tau \) can be characterized by the following lemma.
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Figure 1.1: Illustration of a price path with cutoff-demand \( N = 3 \).

**Lemma 2.** Given the seller’s equilibrium cutoff-demand \( N < \infty \), a buyer believes in equilibrium that the waiting time until the next sale, \( \tau \), conditional on not buying upon arrival is a uniform mixture of gamma distributions \( \Gamma \left( N - (n + 1), \frac{1}{\lambda(1-\pi)} \right) \) with \( n \in \{0, \ldots, N - 1\} \).

The intuition for this result is that given the current state is \( n \), if a buyer does not accept the regular price, he has to wait for \( N - (n + 1) \) low-valuation buyers to arrive because the seller thinks that he has low-valuation. Hence, he has to wait for time \( t \frac{\lambda(1-\pi)}{N - (n+1)} \) until the next sale. By equilibrium condition 3, a buyer assigns equal probability to states \( n \in \{0, \ldots, N - 1\} \).

**Buyers’ indifference rule.** Given the equilibrium cutoff-demand \( N \), a high-valuation buyer is willing to pay at most \( p_H(N) \) satisfying

\[
\frac{v_H - p_H(N)}{v_H - v_L} = \mathbb{E}[e^{-rb\tau}] \cdot (v_H - v_L). \tag{1.2}
\]

That is, at a price \( p_H(N) \), high-valuation buyers are indifferent between buying today and waiting for the next sale.

Using Lemma 2 and expression (1.1), for \( N < \infty \), I can calculate the expected discount to a sale:

\[
\mathbb{E}[e^{-rb\tau}] = \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E} \left[ e^{-rb\tau \frac{1}{N(N-(n+1))}} \right] = \frac{1}{N} \cdot \frac{1 - \left( 1 + \frac{r_b}{\lambda(1-\pi)} \right)^{-N}}{1 - \left( 1 + \frac{r_b}{\lambda(1-\pi)} \right)^{-1}}
\]

for \( N \geq 1 \). For \( N \in \{0,1\} \), the expected waiting time is zero, so the buyers expected discount factor is 1.

\(^{12}\)A uniform mixture of distributions \( F_1, F_2, \ldots, F_n \) is a distribution of a random variable that is distributed according to \( F_1, F_2, \ldots, F_n \) with equal probability.
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Seller’s profits. First, the seller’s expected discount factor between the arrival of two buyers is given by

$$E \left[ e^{-r_s t^\lambda (1-\pi)} \right] = (1 + R_s)^{-1}$$

where

$$R_s = \frac{r_s}{\lambda(1-\pi)}$$

is the seller’s arrival-adjusted discount rate. This is the average discount rate between the arrival of two low-valuation buyers. Furthermore, given $n$ inactive buyers in the market, denote by $\Pi(n; N)$ the expected continuation profit if the seller can adhere to a plan to have a sale after unrealized demand $N$. Then, with $n$ inactive low-valuation buyers, the seller’s expected continuation profit is given by

$$\Pi(n; N) = \frac{\lambda\pi}{r_s} \cdot p_H(N) + (1 + R_s)^{-1} \cdot \left( \Pi(n + 1; N) - \frac{\lambda\pi}{r_s} \cdot p_r \right)$$

for $n < N$ and $\Pi(N; N) = N \cdot v_L + \Pi(0; N) - C_A$.

Equilibrium Characterization

In this subsection, I characterize all stationary equilibria and derive conditions under which an advertising equilibrium exists. Whenever it exists, it is generically unique.

First, I derive necessary conditions for the cutoff-demand $N$ in an advertising equilibrium. Profit maximization by the seller guarantees that $N < \infty$ constitutes an equilibrium if and only if, for all $n < N$,

$$\Pi(n; N) > \max \{ n \cdot v_L + \Pi(0; N) - C_A, \Pi_L \}$$

and, for $n > N$ and all $m \geq 1$,

$$n v_L + \Pi(0; N) - C_A > \max \left\{ \left(1 - (1 + R_s)^{-m} \right) \frac{\lambda\pi}{r_s} p_H(N) + (1 + R_s)^{-m} \left((n + m) v_L + \Pi(0; N) - C_A\right), \Pi_L \right\}.$$ 

In particular, the seller never wishes to have an advertised sale before the $N$-th low-valuation buyer has arrived and she cannot wish to wait for more low-valuation buyers to arrive. These conditions place upper and lower bounds on the advertising cost $C_A$ consistent with cutoff-demand being $N$. 
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Lemma 3. (i) Given a cutoff-demand $N$, a seller does not want to drop the price before period $N$ if and only if

$$C_A \geq \left(N - \left(1 - (1 + R_s)^{-N}\right) \cdot (1 + R_s^{-1})\right) \cdot v_L \equiv \underline{C}(N).$$

(1.3)

(ii) Given a cutoff-demand $N$, a seller does not want to postpone an advertised sale in period $N$ if and only if

$$C_A \leq \left(N - \left(1 - (1 + R_s)^{-N}\right) \cdot R_s^{-1}\right) \cdot v_L \equiv \overline{C}(N).$$

(1.4)

(iii) Generically, there is a unique integer $N^*(C_A)$ that satisfies (1.3) and (1.4).

Intuitively, if advertising costs are low, the seller is tempted to have a sale when only a few inactive buyers have accumulated, while if advertising costs are high, she wants to have a sale only when many inactive buyers have accumulated. The bounds on the advertising cost are independent of the value of the regular price and depend only on the seller’s arrival-adjusted discount rate $R_s$ and the cutoff-demand $N$. In particular, buyers’ time preferences are irrelevant.

The functions $\overline{C}, \underline{C}$ are both increasing in $N$ and $\underline{C}(N) < \overline{C}(N)$ for all $N$. Let the largest number that satisfies (1.3) be $\overline{N}$ and the smallest number that satisfies (1.4) be $\underline{N}$. In other words,

$$\underline{C}(N) = C_A = \overline{C}(N).$$

The uniqueness of an integer $N^*(C_A) \in [\underline{N}, \overline{N}]$ follows from the observation that $\overline{N} - \underline{N} = 1$. When $\underline{N}, \overline{N}$ are integers, there are two $N^*(C_A)$ that satisfy this condition. Note also that $\overline{N} - \underline{N} = 1$ implies that $\overline{C}(N) = \overline{C}(N + 1)$ for all $N$. Figure 1.3 illustrates $\overline{C}, \underline{C}, \underline{N}, \overline{N}$. In that case, only $N = 28$ satisfies both (1.3) and (1.4).

These conditions are the key to characterizing the stationary equilibria when the seller cannot commit to prices nor the frequency of her sales. From now on, let the search cost be small enough relative to $C_A$ to satisfy

$$\frac{\lambda \pi c}{r_s} \leq \frac{\lambda (1 - \pi)}{r_s} v_L - \max_{N} \frac{(1 + R_s)^{-N-1}}{1 - (1 + R_s)^{-N}} \cdot (N v_L - C_A).$$

(1.5)

This condition guarantees the existence of EDLP equilibria. After observing an out-of-equilibrium price, buyers believe that the state is unchanged $n = 0$. The following continuation equilibrium can be supported after a deviation: high-valuation buyers only accept prices $p \leq v_L + c$ and the seller returns to an EDLP after the first deviation. First, it is immediate that given an EDLP strategy of the seller, the buyers strategy is optimal. Moreover, given the high-valuation buyers’ strategy, the seller cannot make higher profits than with EDLP.
by condition (1.5). Given this continuation equilibrium, a deviation is not profitable for the seller because (1.5) implies that $\pi(v_L + c) < v_L$. In online markets, search costs are likely to be small compared to advertising costs.\footnote{If condition (1.5) is not satisfied, an EDLP equilibrium cannot be sustained.}

The following proposition characterizes the (generically) unique equilibrium.

**Proposition 1.** Let (1.5) be satisfied. Then, there is always an EDLP equilibrium. There exists a generically unique advertising equilibrium with cutoff-demand $N^*(C_A)$ satisfying (1.3) and (1.4) if and only if

$$\Pi(0; N^*(C_A)) > v_L \cdot \frac{\lambda}{r_s} \equiv \Pi_L. \quad (1.6)$$

In that case, the regular price must be equal to $p_H(N^*(C_A))$.

From now on, I refer to $\Pi(0; N^*(C_A))$ as the profit from price discrimination, given cutoff-demand $N^*(C_A)$. If (1.6) is satisfied, the advertising equilibrium is the more natural equilibrium to consider because the seller makes greater profits than in an EDLP equilibrium. An advertising equilibrium is constructed in two steps:

1. First, the cutoff-demand $N^*(C_A)$ that satisfies (1.3) and (1.4) need to be found.
2. Then, $\Pi(0; N^*(C_A))$ and $\Pi_L$ must be compared.

I discuss implications of the first step in Section 1.4 in form of corollaries. In Section 1.5, I show how equilibria depend on advertising costs $C_A$ and buyers’ time preferences taking into account the second step. The second step captures, in particular, how valuable the cost of advertising is for the seller as a commitment device.
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Remark 1. Instead of imposing assumptions on the search cost of buyers, I can also impose different out-of-equilibrium beliefs in order to sustain EDLP equilibria if \( \Pi(0; N^*(C_A)) < \frac{\nu_L \cdot \lambda}{r_s} \equiv \Pi_L \). For example, this is true if buyers who face a price \( p_H(N^*(C_A)) > p > \nu_L \) believe that the states are distributed uniformly on \( \{0, \ldots, N^*(C_A) - 1\} \) and the seller is setting this high price \( p \), unacceptable to low-valuation buyers, in every period with periodic sales according to cutoff-demand \( N^*(C_A) \). For all other deviations, buyers do not change beliefs about the state space.

Remark 2. In a model with a continuous inflow of buyers conditions (1.3) and (1.4) can be summarized in one equation, where the upper and lower bound of \( C_A \) coincide.

1.4 Comparative Statics: Advertising Cost and Market Size

Using the equilibrium characterization of Proposition 1, I now relate my model to two well-known benchmark models: the model by Sobel 1991, in which sellers lack any commitment power, that results in the Coasean outcome and a model in which the seller can fully commit to a constant price. Then, I analyze the effect of the arrival rate \( \lambda \) on the frequency of sales. I will refer to \( \lambda \) as the market size from now on.

An immediate implication of conditions (1.3) and (1.4) is that if advertising is costless \( (C_A = 0) \), then the unique equilibrium is an EDLP equilibrium. The reason is that the seller wants to drop the price as soon as a buyer does not buy. For \( C_A > \overline{C}(1) \), however, advertising costs serve as a commitment device for the seller. In particular, the higher that cost, the higher \( N^*(C_A) \), hence the less frequent are sales in an advertising equilibrium. By (1.3) and (1.4), every advertising cost in \((C(N^*(C_A)), \overline{C}(N^*(C_A)))\) is mapped to a unique cutoff-demand \( N^*(C_A) \). As \( C_A \) becomes infinitely large, the static monopoly profit can be sustained. More precisely, \( \Pi(0; N^*(C_A)) \) converges to

\[
\Pi_H \equiv \frac{\lambda \pi}{r_s} \cdot \nu_H,
\]

which is the profit the seller makes if she charges \( \nu_H \) in every period without ever advertising. These results are summarized in the following corollary.

Corollary 1. (i) (Coase conjecture) For \( C_A < \overline{C}(1) \), the only equilibrium is an EDLP equilibrium.
(ii) The cutoff-demand \( N^*(C_A) \) is increasing in advertising costs \( C_A \).
(iii) For \( C_A = \infty \), the static monopoly profit can be sustained at any point in time.

Hence, my model encompasses two well-known benchmark cases: (i) specifies the Coasean equilibrium outcome for monopolist without any commitment power; and (iii) captures a
monopolist who can fully commit to a constant price. In the following, the goal is to obtain a better understanding of the intermediate case, when $C(1) < C_A < \infty$.

As previously noted, the equilibrium cutoff-demand $N^*(C_A)$ is independent of the buyers’ discount rates $r_b$ and $v_H$ that only affect the profit from price discrimination through the regular price $p_H(N^*(C_A))$ given by (1.2). $N^*(C_A)$ is a decreasing function of the arrival-adjusted discount rate $R_s = \frac{r_s}{\lambda(1-\pi)}$ and $v_L$. Hence, an increase in the market size $\lambda$ is similar in effect to having a more patient seller.

These observations imply that, everything else the same, in larger markets (i.e., with high $\lambda$) the cutoff-demand $N^*(C_A)$ is large, that is the good is sold to more buyers per sale. However, buyers are also accumulated faster. I show that the average time between sales is decreasing in the market size $\lambda$. Hence, popular products should be on sale more frequently than less popular products. More frequent sales decrease the delay in trade with low-valuation buyers, but they also force the seller to lower the regular price.

In two markets of the same size in terms of revenues (i.e., with same $v_L\lambda$), the market with frequent arrivals and smaller valuations has less frequent sales than the one with less frequent arrivals, but higher valuations. In other words, the model predicts that high-volume, low-ticket goods, such as headphones or groceries, should be on sale less frequently than low-volume, high-ticket goods, such sofas from a specific brand, if the average revenues are approximately the same. The following corollary summarizes these insights.

Corollary 2. (i) The advertising equilibrium cutoff-demand $N^*(C_A)$ is decreasing in the arrival-adjusted discount rate $R_s$ and in $v_L$. Furthermore, $\lim_{R_s \to \infty} N^*(C_A) = \frac{C_A}{v_L}$ and $\lim_{R_s \to 0} N^*(C_A) = \infty$.
(ii) A higher arrival rate decreases the average time between two sales.
(iii) If the arrival rate increases proportionally to a decrease in $v_L$ (i.e., $\lambda v_L$ is constant), then the average time between two sales increases.

The intuition for the second part of (i) is that if the arrival-adjusted discount rate $R_s$ is extremely large, the average cost per customer $\frac{C_A}{N^*(C_A)}$ is approximately equal to the sale price $v_L$. For small $R_s$, average costs per customer converge to zero, such that the seller can make high profits during an advertised sale. While an increase in the arrival rate increases the revenues from a sale, delaying this revenue is more costly. Hence, sales are more frequent. (iii) follows from the observation that for high-volume, low-ticket goods, waiting for more low-valuation buyers to arrive is less costly than in low-volume, high-ticket markets.

Finally, for higher advertising costs, the average advertising cost per reactivated buyer $\frac{C_A}{N^*(C_A)}$ is higher. In other words, as the cost of advertising increases, the number of buyers that can be reactivated in equilibrium increases less than proportionally. Hence, a dollar of advertising cost is more valuable as a commitment device if total advertising costs are low than if they are high. The intuition is that, for high $N^*(C_A)$, the revenue during an advertised sales period is higher, so further delay of trade is more costly than if $N^*(C_A)$ is small.
Corollary 3. The average advertising cost per buyer \( \frac{C_A N^2}{N! C_A} \) is increasing in \( C_A \).

Graphically, this follows from the convexity of \( C \) and \( C \) in Figure 1.3.

Proof. This lemma follows by rewriting inequalities (1.3) and (1.4) as

\[
\sum_{i=0}^{N-1} \left( 1 - (1 + R_s)^{-i} \right) \cdot v_L \geq C_A \geq \sum_{i=1}^{N} \left( 1 - (1 + R_s)^{-i} \right) \cdot v_L.
\]

Hence, in equilibrium, the advertising cost must equal the cost of waiting for \( N \) low-valuation buyers to arrive. The cost of waiting for the \( N \)-th arrival is always higher than the cost of waiting for the \( (N-1) \)-th arrival.

All in all, my model allows to make testable qualitative predictions about markets with in which intertemporal price discrimination plays a major role. In particular, it shows how in different markets the frequency of sales, which is a crucial for the entire price level, differs.

1.5 Comparative Statics: Commitment and Buyers’ Time Preferences

In this section, I further elaborate the role of advertising costs as a commitment device and how it is related to buyers’ time preferences. I first highlight the trade-off coming from intertemporal price discrimination (similar to Sobel 1991) by assuming that the seller can commit to a fixed cutoff-demand \( N \). Then, I examine the actual model in which advertising costs endogenously generate limited commitment.

Full Commitment Benchmark

As a benchmark, it is useful to analyze the situation in which the seller can commit to a cutoff-demand \( N \) in period zero, that is I assume the seller can attain any profit \( \Pi^{FC}(N) \equiv \Pi(0; N) \). Then, the highest profit the seller can attain maximizes

\[
\Pi^{FC}(N) \equiv \Pi(0; N) = \frac{\lambda \pi}{\nu_s} \cdot p_H(N) + \frac{(1 + R_s)^{-N}}{1 - (1 + R_s)^{-N}} \cdot (Nv_L - C_A) \quad (1.7)
\]

where

- \( \lambda \) is the total expected revenue from \( v_H \)-buyers,
- \( \pi \) is the price elasticity of demand for \( v_H \)-buyers,
- \( p_H(N) \) is the price of \( v_H \)-buyers for \( N \) arrivals,
- \( \nu_s \) is the total expected revenue from \( v_L \)-buyers,
- \( v_L \) is the price of \( v_L \)-buyers for \( N \) arrivals,
- \( C_A \) is the cost of advertising per buyer.

The full commitment benchmark provides a lower bound on the profit the seller can attain in the actual model.
with respect to $N$.\footnote{Note that the optimal advertising strategy of the seller if she can fully commit to a history-dependent pricing and advertising strategy, is not necessary a function of the state variable $n$. The seller might want to have deterministic times between two sales. The reason is that if the advertising strategy depends on $n$, buyers can bring forward sales by rejecting an offer and therefore have a higher incentive to do so.} The revenue from high-valuation buyers can be calculated analogously to $\Pi_L$, while the revenue from low-valuation buyers $\pi_L(N)$ is recursively given by

$$\pi_L(N) = \mathbb{E} \left[ e^{-r_s \cdot \lambda (1-\pi)} \cdot (N \cdot v_L - C_A + \pi_L(N)) \right],$$

such that $\pi_L(N) = \frac{(1+r_s)^{-N}}{1-(1+r_s)^{-\lambda}} \cdot (N \cdot v_L - C_A)$. The following lemma summarizes the trade-off faced by the seller.

**Lemma 4.** (i) **Cost of frequent sales:** the regular price $p_H(N)$ and the expected discounted costs from advertising $\frac{(1+r_s)^{-N}}{1-(1+r_s)^{-\lambda}} \cdot C_A$ are increasing and concave in the cutoff-demand $N$.

(ii) **Benefit of frequent sales:** expected discounted revenues from low-valuation buyers $\frac{(1+r_s)^{-N}}{1-(1+r_s)^{-\lambda}} \cdot N \cdot v_L$ are decreasing and convex in the cutoff-demand $N$.

With frequent sales, i.e. if the cutoff-demand $N$ is low, the seller can pick up low-valuation buyers with less delay which increases her expected profits. This is, however, anticipated by high-valuation buyers. Consequently, the seller has to set a lower regular price $p_H(N)$ in order to satisfy (1.2). Furthermore, the cost of advertising makes intertemporal price discrimination costly. These effects are smaller for large cutoff-demands $N$, because the marginal change is further in future and thus discounted more.

Which of the two effects dominates depends on buyers’ time preferences. A natural and common benchmark is to assume $r_s = r_b$. In that case, a well-known result is that it is optimal for the seller to set the price equal to the static monopoly price.\footnote{E.g. Conlisk, Gerstner, and Sobel 1984, Sobel 1991}

**Lemma 5.** If the seller and buyer share the same discount rates ($r_s = r_b$), then it is always optimal for the seller to choose the static monopoly price in every period (i.e., $v_H$ if $v_H \pi \geq v_L$ and $v_L$ otherwise).

Intuitively, if buyers and the seller have the same time preferences given by discount rate $r$, then they have approximately the same expected discount to sale $\delta$.\footnote{In my model, the buyer’s expected discount to sale is slightly different because buyers do not know the price history.} Let us compare the profit that the seller can make if she has sales with a frequency that results in an expected discount to sale $\delta$ with the profit that she can make with a constant monopoly price. First, consider the case $v_H \pi \geq v_L$. Then the seller gains less than $\delta v_L \frac{(1-\pi)\lambda}{\tau}$ because she can sell to low-valuation buyers. However, she has to drop the regular price by $\delta (v_H - v_L)$, that...
is, she loses $\delta \frac{\lambda (v_H - v_L)}{r}$ in profits. Hence, in total, the seller loses more than $\delta \lambda \frac{v_H - v_L}{r} > 0$. If $v_L > \pi v_H$, then the seller gains by the increased price she can charge to high-valuation buyers $(1 - \delta)(v_H - v_L)$, which increases profits by $(1 - \delta)\frac{\pi \lambda (v_H - v_L)}{r}$. On the contrary, she loses profits by more than $(1 - \delta)\lambda \frac{v_L}{r}$ because she has to delay trade with low-valuation buyers. Hence, the total loss in profits from having sales is greater than $(1 - \delta)\lambda \frac{v_L}{r}$. The formal proof can be found in the appendix. Figure 1.3 illustrates profits from price discrimination $\Pi^{FC}(N)$ as a function of cutoff-demand $N$ for $\pi v_H > v_L$. The blue dashed line represents $\Pi_H$ and the red dotted line represents $\Pi_L$. $\Pi^{FC}(N)$ is always lower than $\Pi_H$, but it converges to $\Pi_H$ from below, because with infrequent sales, the profit advertised sales converges to the static monopoly profit.

![Figure 1.3: Profit as a function of cutoff-demand $\Pi^{FC}(N)$ with $v_L = 50$, $\pi = 0.5$, $v_H = 110$, $C_A = 100$, $\lambda = 2000$, $r_b = r_s = 0.5$.](image)

A more interesting and, arguably, more plausible situation is that the seller is more patient than the buyers, that is, $r_b > r_s$.\(^{17}\) If buyers are sufficiently impatient relative to the seller, then profits can be increased above static monopoly profits.\(^{18}\) In that case, the optimal cutoff-demand is finite, which has interesting implications for the profit-maximizing outcome. These insights will be useful for the analysis of the profit from price discrimination $\Pi(0; N^*(C_A))$ that will be discussed in the next section.

\(^{17}\)Assuming different discount rates of buyers and the seller seems to be a natural assumption. On the one hand, firms usually face lower market interest rates than individuals. On the other hand, it has been shown by many experimental and field studies that individuals’ time preferences are represented by relatively high discount rates. See for example, Coller and Williams 1999 or Andreoni and Sprenger 2012.

\(^{18}\)Landsberger and Meilijson 1985 shows this in a model with finite horizon and without an influx of buyers after period 0 but he does not quantify it. Similarly, Sobel 1984 briefly note that if buyers are more impatient than the seller, the seller can take advantage of the difference in time preferences.
Lemma 6. (i) As the cutoff-demand approaches infinity, profits $\Pi^{FC}(N)$ approach $\Pi_H$ from below.

(ii) The regular price $p_H(N)$ and seller’s profit $\Pi^{FC}(N)$ are increasing in buyers’ discount rate $r_b$.

(iii) The seller’s profit is decreasing in advertising cost $C_A$.

(iv) Given parameters $v_H, v_L, \pi, \lambda, C_A, r_s$ there exists a $r_b > 0$, such that for all $r_b \geq r_b$, there exists a cutoff-demand $N$ for which the seller makes higher profits than if she chooses the static monopoly price forever ($\max_N \Pi^{FC}(N) > \Pi_H$) if and only if

$$\frac{\lambda}{r_s} \cdot (v_L - \pi v_H) < \left[ \max_{N \geq 0} \frac{(1 + R_s)^{-N}}{1 - (1 + R_s)^{-N}} \cdot (Nv_L - C_A) \right].$$

(1.8)

For $C_A = 0$, this condition becomes $\pi v_H > 0$ and as $C_A \to \infty$, it converges to $v_L \leq \pi v_H$.

It is straightforward that as sales become very infrequent (i.e., $N \to \infty$), $p_H(N)$ approaches $v_H$ and the profits from low-valuation buyers vanish, as already shown in Lemma 4. More interestingly, this limit is always approached from below, that is, for large $N$, the profit loss due to a lower regular price outweighs the gain from frequent sales.

Intertemporal price discrimination is more profitable in markets with impatient buyers. As the buyers’ discount rate increases, the seller can increase the regular price $p_H(N)$ because buyers care less about future sales. Hence, consumer surplus drops and firm’s profits increase. In fact, price discrimination can increase profits far above the static monopoly profits as illustrated in Figure 1.3. (iii) is straightforward because advertising costs do not benefit the seller in any way.

Finally, (iv) specifies a condition on parameters that guarantees that price discrimination is profitable for some discount rates $r_b$ of buyers. With sufficiently large $r_b$, the seller can always charge a regular price that is arbitrarily close to $v_H$. Hence, as long as $\pi v_H \geq v_L$, it is always profitable for her to price discriminate for large enough $r_b$. Remarkably, price discrimination can also be profitable if $\pi v_H < v_L$. In fact, if advertising is free, price discrimination is always profitable for sufficiently impatient buyers. This is demonstrated in Figure 1.4. The inequality in (iv) implies that, in this case, the monopoly profit from high-valuation types $\frac{\lambda v_H}{r_s}$ must make up for the cost of delay in trade with low-valuation buyers and advertising

$$\frac{\lambda v_L}{r_s} - \max_{N \geq 0} \frac{1}{(1 + R_s)^N} \cdot (Nv_L - C_A).$$

In contrast, as high $C_A \to \infty$, the price discrimination can be profitable only if $v_L < \pi v_H$. Next, I consider some properties of the locally profit-maximizing cutoff-demand $\Pi^{FC}(N)$. Graphically, one can see that $\Pi^{FC}(N)$ is either always increasing everywhere or it has has a single local maximum (and minimum) as illustrated in Figures 1.3, 1.4, and 1.5. I do not prove that there is always a unique local maximum is, however, because the fact that the derivatives of $\Pi$ are intractable analytically.
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Figure 1.4: Profit as a function of cutoff-demand $N$ with $r_s = 0.5$, $r_b = 5$, $v_L = 50$, $\pi = 0.5$, $v_H = 90$, $C_A = 100$, $\lambda = 2000$, i.e. $v_L > \pi v_H$.

Lemma 7. (i) Any locally profit-maximizing cutoff-demand is increasing in $r_b$ for small levels of $r_b$ and decreasing everywhere else.
(ii) Any locally profit-maximizing cutoff-demand is increasing in advertising costs $C_A$.

The intuition for Lemma 7 (i) can be understood as follows. For simplicity, let us assume $C_A = 0$. Buyers’ time preferences only affect profits through the regular price $p_H(N)$. In particular, buyers’ time preferences determine the sensitivity of $p_H(N)$ to changes in $N$, that is, they affect the marginal benefit (MB) of increasing $N$. The marginal cost (MC) of increasing $N$ is given by the cost of delay and is independent of buyers’ time preferences. The MB of increasing $N$ by one is small if buyers are extremely patient or extremely impatient, but larger in between. Moreover, this marginal effect is smaller for an increase in $N$ further in the future (i.e., for large $N$) than early on (i.e., for small $N$). The seller can benefit from waiting for an additional low-valuation buyer as long as the MB is larger than the MC of delay. For very patient and very impatient buyers, the marginal benefit of delay exceeds the marginal cost of delay only for small cutoff-demands, such that the locally profit-maximizing cutoff-demand is small. In contrast, for intermediately patient buyers, higher cutoff-demands become optimal. Hence, the locally profit-maximizing cutoff-demand is first increasing and then decreasing in $r_b$ and is largest in between.

The limiting cases when buyers are extremely impatient ($r_b \to \infty$) and when buyers are extremely patient ($r_b \to 0$) are particularly interesting. If buyers are extremely impatient, then the seller can charge a regular price close to $v_H$ (as long as $N > 1$). Hence, the benefit of frequent sales completely dominates, so an increase in $N$ decreases profits. It is, thus, optimal for the seller to choose the smallest cutoff-demand. In the other extreme, for very patient buyers, the regular price $p_H(N)$ is close to $v_L$, so the seller can only benefit from extremely large $N$. Small delays are not beneficial for the seller at all, so the local maximum of $\Pi^{FC}(N)$ is zero.
These results are useful to prove similar properties for $\Pi(0; N^*(C_A))$ as a function of $C_A$ in the next subsection. In order to illustrate these results, Figure 1.5 shows that the profit-maximizing cutoff-demand is increasing from $r_b = 0.55$ to $r_b = 1$, but decreasing from $r_b = 1.3$ to $r_b = 5$.

![Figure 1.5: Profit as a function of cutoff-demand](image)

Figure 1.5: Profit as a function of cutoff-demand $N$ for $v_L = 50$, $\pi = 0.5$, $v_H = 100$, $C_A = 10$, $\lambda = 2000$, $r_s = 0.5$

![Figure 1.6: Implication of interaction of $C_A$ and $r_b$ on profit from price discrimination](image)

Figure 1.6: Implication of interaction of $C_A$ and $r_b$ on profit from price discrimination
Finally, the frequency of sales is decreasing in advertising cost $C_A$, because it is more costly to have advertised sales with higher $C_A$. Figure 1.6 summarizes all results. It depicts the region in which the seller prefers a constant monopoly price $v_H$ for different consumer impatience and advertising cost. If it is infinitely costly to recall buyers, then the repeated static monopoly outcome is profit-maximizing. On the other extreme, if it is costless to recall buyers and if they are myopic (i.e., $r_b = \infty$), then the seller can perfectly price discriminate. For intermediate values, it is harder to price-discriminate profitably the higher $C_A$ and the smaller $r_b$. However, consumers receive more surplus if $r_b$ is low. Hence, as shown in Proposition 7, the effect on welfare is generally not monotone in $r_b$, because the delay in trade with low-valuation buyers can increase or decrease in $r_b$. Finally, the greater the gap $v_L - \pi v_H$, the harder it becomes to profitably price discriminate. In particular, if $v_L - \pi v_H > 0$ and advertising costs $C_A$ are above the threshold given by (1.8), price discrimination is not profitable for any $r_b$. If $v_H \pi > v_L$, for any finite advertising cost $C_A$, price discrimination is profitable for sufficiently impatient buyers.

**Endogenous Commitment**

In this subsection, I show how the trade-off discussed in the previous subsection affects how much the seller benefits from advertising costs as a commitment device. To this end, I consider the profit from price discrimination, $\Pi(0; N^*(C_A))$, as a function of $C_A$. By Proposition 1, each $C_A$ pins down a (generically) unique $N^*(C_A)$ and also affects the function $\Pi(0; N)$ directly. Then, I show how buyers’ discount rates $r_b$ affect the equilibrium outcome. In particular, I show under which conditions there exists an advertising equilibrium.

Denote the seller’s profit if the advertising cost is $C_A = \overline{C}(N)$, i.e., just high enough to sustain cutoff-demand $N$, by

$$\overline{\Pi}(N) \equiv \Pi(0; N^*(\overline{C}(N))) = \frac{\lambda \pi}{r_s} p_H(N) + \left(1 + R_s^{-1}\right) (1 + R_s)^{-N} v_L \quad (1.9)$$

and the seller’s profit if the advertising cost is $C_A = \overline{C}(N)$ and cutoff-demand $N$ is sustained by

$$\overline{\Pi}(N) \equiv \Pi(0; N^*(\overline{C}(N))) = \frac{\lambda \pi}{r_s} p_H(N) + R_s^{-1} (1 + R_s)^{-N} v_L. \quad (1.10)$$

A change in advertising cost $C_A$ can have two different effects:

1. If the cost $C_A$ is in $(\overline{C}(N), \overline{C}(N+1))$ for an integer $N$, then a small increase in the advertising cost does not help to sustain longer cutoff-demands. Hence, profits decrease linearly.

2. If the cost $C_A$ is $\overline{C}(N) = \overline{C}(N+1)$ for an integer $N$, then a small increase in the advertising cost results in a longer cutoff-demand $N + 1$. Hence, profits jump from $\overline{\Pi}(N)$ to $\overline{\Pi}(N+1)$. This effect can be positive or negative depending on the shape of $\overline{\Pi}$ and $\overline{\Pi}$. 

More precisely, the correspondence that maps to each advertising cost the equilibrium profits in an advertising equilibrium $\Pi(C_A)$ is given by

$$\Pi(C_A) = \begin{cases} 
\{ \Pi(N), \Pi(N-1) \} & \text{if } C_A = \underline{C}(N) = \overline{C}(N-1) \\
\Pi(N) - \frac{(1+R_s)^r}{1-(1+R_s)^{r}} (C_A - \underline{C}(N)) & \text{if } C_A \in (\underline{C}(N), \overline{C}(N)) 
\end{cases}$$

By Proposition 1, an advertising equilibrium exists if and only if $\Pi(C_A) \geq \Pi_L$.

Figure 1.5 illustrates $\Pi(C_A)$. Moreover, it shows that $\Pi$ and $\overline{\Pi}$ define upper and lower bounds of $\Pi(C_A)$ given by

$$\Pi(\overline{C}^{-1}(C_A)) \leq \Pi(C_A) \leq \Pi(\overline{C}^{-1}(C_A)).$$

The inverse of the functions $\overline{C}$ and $\underline{C}$ are well-defined because they are increasing as shown in Section 1.3. Furthermore, on the grid

$$\mathcal{C} = \{ C_A | C_A = \underline{C}(N) \text{ with } N \in \{1, 2, 3, \ldots \} \},$$

the bounds are binding, that is

$$\Pi(\overline{C}^{-1}(C_A)) = \Pi(C_A) = \Pi(\overline{C}^{-1}(C_A)).$$

Hence, I focus the analysis of $\Pi(C_A)$ on the grid $\mathcal{C}$ where $C_A$ acts as an endogenous commitment device. To this end, it is sufficient to analyze the functions $\Pi$ and $\overline{\Pi}$ since their properties carry on to $\Pi$ on the grid $\{ \underline{C}(N) : N \in \{1, 2, 3, \ldots \} \}$.

The following lemma summarizes some properties of $\overline{\Pi}$. Since $\Pi(N) = \Pi(N) - \frac{v_H}{(1+R_s)^N}$, I do not analyze $\overline{\Pi}$ separately.
Lemma 8. (i) $\Pi(1) = \Pi_L$ and $\lim_{N \to \infty} \Pi(N) = \Pi_H$.
(ii) $\Pi$ has at most one local maximum and at most one local minimum.
(iii) $\Pi(N)$ and $\Pi'(N)|_{N=1}$ are increasing in $r_b$. $\Pi'(N)|_{N=1}$ is negative for small $r_b$.

First in the limit, the monopolist makes profits $\Pi_L$ and $\Pi_H$ that correspond to constant prices. For intermediate cutoff-demands, there is at most one local maximum and if the seller is sufficiently patient compared to buyers (in particular if $r_s < r_b$), then there is also at most one local minimum. This specifies the shape of $\Pi$ and how the seller’s trade-offs are resolved. Figure 1.8 illustrates $\Pi$ for different levels of discount rates of buyers.

![Figure 1.8: Highest equilibrium profits with cutoff-demand $N^*(C_A)$ for parameter values $r_s = 0.5$, $v_H = 110$, $v_L = 50$, $\pi = 0.5$, $\lambda = 1000$ and different discount rates](image)

The discount rate plays an important role for the existence of advertising equilibria. With relatively patient buyers, small $C_A$ means that it is not possible sustain long enough cycle lengths to make price discrimination profitable for the seller because the seller has to leave too much surplus to high-valuation buyers (i.e., $\Pi'(N)|_{N=1} < 0$). In that case, $\Pi$ is first decreasing and then increasing. Hence, for small advertising costs $C_A$, the unique equilibrium is an EDLP equilibrium because $\Pi(N^*(C_A)) < \Pi_L$. However, as $C_A$ increases, profits converge to $\Pi_H$. Thus, an advertising equilibrium exists, as long as $\pi v_H \geq v_L$. In contrast, with impatient buyers, even small $C_A$ can sustain cutoff-demands that make price discrimination profitable ($\Pi'(N)|_{N=1} > 0$).

If $\pi v_H > v_L$ and for some intermediate values of $r_b$, the non-monotonicity of the profit function creates interesting comparative statics: for low $C_A$, the seller wishes to price discriminate because she can increase the regular price sufficiently to make it profitable. However, in an intermediate region of advertising costs, the cost of delaying trade with low-valuation buyers dominates the benefit of sustaining longer cutoff-demands. Hence, the seller prefers to set a constant price of $v_L$ and not to have sales. For sufficiently high $C_A$, profits converge...
to the static monopoly profit, so temporary sales are retained from time to time again. Figure 1.9 illustrates $\Pi(C_A)$ for the same parameters as used for figure 1.8. It illustrates how properties of $\Pi$ carry over to $\Pi$.

![Figure 1.9: Highest equilibrium profits with cutoff-demand $N^*(C_A)$ for parameter values $r_s = 0.5$, $v_H = 110$, $v_L = 50$, $\pi = 0.5$, $\lambda = 1000$ and different discount rates](image)

Figure 1.9 also offers a calibration for equilibria given different advertising costs and buyers’ time preferences. The seller’s discount rate is set to be equal to $r_s = 0.5$ which corresponds to a discount factor of 0.61 per period (e.g. a year). If advertising costs are for example 6000, then if buyers have the same discount rate as the seller, the unique equilibrium is an EDLP equilibrium. In contrast, if the buyers’ discount rate is $r_b = 0.87$ (discount factor of 0.42), there is an advertising equilibrium and the seller makes profits close to $\Pi_L$. If the buyers’ discount rate is $r_b = 1.6$ (discount factor of 0.2), the makes even higher profits than $\Pi_H$. This illustrates that with modest advertising costs, the seller’s profits are very sensitive to differences in buyers’ time preferences.

I define two thresholds

\[ T_0(r_b) = \inf\{C_A : \Pi(C_A) < \Pi_L\} \]

and

\[ T_1(r_b) = \sup\{C_A : \Pi(C_A) < \Pi_L\} \]

where $\sup\emptyset = \inf\emptyset = 0$. The following proposition follows from Lemma 8.

**Proposition 2.** *(i)* An advertising equilibrium exists given an advertising cost $C_A \in C$ if and only if

\[ C_A \notin (T_0(r_b), T_1(r_b)). \]
(ii) \( T_1(r_b) \) is decreasing in \( r_b \) and \( \lim_{r_b \to 0} T_1(r_b) = \infty \).

(iii) \( T_0(r_b) \) is increasing in \( r_b \) as long as \( 0 < T_0(r_b) < T_1(r_b) \). \( T_0(r_b) = 0 \) for small \( r_b \).

(iv) The length of the interval \((T_0(r_b), T_1(r_b))\) is decreasing in \( r_b \). If \( \pi_H > \pi_L \), \( T_0(r_b) = T_1(r_b) = 0 \) for large enough \( r_b \). If \( \pi_H \leq \pi_L \), \( T_1(r_b) = \infty \) for all \( r_b \).

This proposition characterizes the set parameters \( r_b \) and \( C_A \) for which advertising equilibria do not exist. If the static monopoly price is high (i.e., \( \pi_H \pi > \pi_L \)), then there are essentially three different cases:

1. For small discount rates \( r_b \), \( T_0(r_b) = 0 < T_1(r_b) \), that is advertising equilibria exist only for large enough advertising costs \( C_A \).

2. For intermediate discount rates \( r_b \), \( 0 < T_0(r_b) < T_1(r_b) < \infty \), that is advertising equilibria do not exist only for some intermediate advertising costs \( C_A \). The intuition is that for intermediate advertising costs, the induced frequency of sales does not increase the regular price enough to outweigh the cost of delay and advertising.

3. For large enough discount rates \( r_b \), \( T_0(r_b) = T_1(r_b) = 0 \), that is advertising equilibria always exist as long as \( C_A \geq C(1) \).

In Figure 1.8 these three different cases for \( \pi v_H > \pi_L \) are illustrated. Indeed, for \( r_b = 0.5 \), \( \Pi(N) \) is first smaller than \( \Pi_L \) and then greater. For \( r_b = 0.86 \), \( \Pi(N) \) is first greater, then smaller and then again greater than \( \Pi_L \). For \( r_b = 1.7 \), \( \Pi(N) > \Pi_L \) always holds.

If the static monopoly price is low (i.e., \( \pi_H \pi \leq \pi_L \)), then all equilibria are EDLP equilibria for small \( r_b \). For large enough \( r_b \), advertising equilibria exist only for small enough advertising costs \( C_A \). For large advertising cost, the induced profit is always close to, but smaller than \( \Pi_H \leq \Pi_L \).

Next, I characterize the optimal advertising cost for the seller if she cannot commit. If the seller can choose \( C_A \), she faces a trade-off between decreasing profits if \( C_A \) is high and increasing commitment possibilities. Hence, a seller potentially wants to have an intermediate advertising cost. For example, in Figure 1.9, the seller would mostly prefer an advertising cost of approximately 8000. As can be seen in Figure 1.8, this would result in an average frequency of sales of one to two times per period.

**Proposition 3.** (i) Let \( \pi_H \pi > \pi_L \). The profit-maximizing advertising cost is \( C_A^* = \infty \) for sufficiently patient buyers. Otherwise, \( C_A^* < \infty \) is quasi-concave in \( r_b \).

(ii) Let \( \pi_H \pi \leq \pi_L \). Then, if there are advertising equilibria for \( r_b \), \( C_A^* < \infty \) is quasi-concave in \( r_b \).

(iii) As \( r_b \to \infty \), the profit-maximizing advertising cost \( C_A^* \) converges to \( C(1) \).

The optimal advertising cost \( C_A^* \) shows to what advertising costs the sellers would optimally want to commit. For example, a firm can benefit significantly, if it follows a policy that ensures relatively high advertising costs. This can be done by hiring expensive designers and computer specialists who among other tasks design and manage an advertising campaign.
The intuition and proof of Proposition 3 follows from Lemma 7. Recall, that buyers’ time preferences determine the sensitivity of the regular price $p_H(N)$ to changes in cutoff-demand $N$. Hence, it determines how valuable different levels of commitment are to the seller and what the marginal benefit (MB) of additional commitment (i.e., of less frequent sales) to the seller is. It is crucial that for very impatient or very patient buyers, the MB of additional commitment is small, while for intermediately patient buyers it is relatively large. The marginal cost (MC) of additional commitment comes from the cost of advertising and the cost of delay, which are both independent of buyers’ time preferences. Hence, for very patient or very impatient buyers, the MB can exceed the MC of commitment only for small levels of commitment. In contrast, if buyers are intermediately patient, the seller can benefit from increasing advertising costs $C_A$ further. For extremely patient buyers, the seller is best-off with infinite advertising costs. The locally-optimal advertising cost is, however, very small.

Hence, moderate advertising costs are only valuable for the seller as a commitment device if buyers are intermediately patient. This is seems to be a plausible assumption in many markets. The literature on intertemporal price discrimination has, however, mostly focused on either myopic buyers ($r_b \to \infty$) or the case in which buyers and seller have the same time preferences. With almost myopic buyers, the seller can almost perfectly price discriminate with very frequent sales, while with very patient buyers only an EDLP can be sustained.

Figure 1.10 summarizes the properties of equilibria if $v_H > v_L$. The boundaries of the regions are dashed because the boundaries are “thick” and belong in parts to both of the regions. The yellow region represents the parameters for which the unique equilibrium is an EDLP equilibrium, while for all other parameters, advertising can be sustained in equilibrium. The red region illustrates advertising equilibria for which the seller can make higher profits than with a constant static monopoly price. The solid blue line represents the profit-maximizing advertising cost. The dotted extension of this line illustrates the local maximum of the profit function $\Pi(C_A)$.

For patient buyers ($r_b$ small), small advertising costs imply EDLP equilibria, while large advertising costs result in an advertising equilibrium. In contrast, if buyers are intermediately patient, EDLP equilibria are sustained for intermediate advertising costs, but for small and large advertising costs, the unique equilibrium is always an advertising equilibrium. The seller prefers infinite advertising costs, but a local maximum of $\Pi(C_A)$ is attained for small advertising costs. Hence, given that infinite advertising costs are hard to sustain, in markets with intermediately patient buyers, sellers have an incentive to invest into a decrease in advertising costs that result in relatively frequent sales.

For patient enough buyers, all equilibria are advertising equilibria except if advertising costs are too small to sustain a cutoff-demand greater than 1 (Coase conjecture). A seller

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19This is because $C_A$ is not always on the grid $\{C(N) : N \in \{1, 2, 3, \ldots \}\}$. 
has an incentive to commit to intermediate advertising costs and to have less frequent sales. However, as buyers become very patient, commitment becomes less valuable and small advertising costs become optimal.

For welfare, lower advertising cost are always better in markets with sufficiently impatient and sufficiently patient buyers. For intermediate impatient levels, intermediate levels of advertising can be optimal because they result in an EDLP equilibrium.

Figure 1.11 summarizes the equilibrium properties for $v_H \pi \leq v_L$. In that case, price discrimination is never profitable for large advertising costs $C_A$ because, as $C_A \to \infty$, profits are close to $\Pi_H$, which is less than $\Pi_L$ by assumption. However, if buyers are sufficiently impatient, small advertising costs result in advertising equilibria.

All in all, for sufficiently impatient buyers, equilibria look very similar to the case with $v_H \pi \geq v_L$, while with relatively patient buyers, the equilibria differ dramatically. The reason is that with sufficiently impatient buyers, the seller can benefit from price discrimination even if the static monopoly price is low.

1.6 Discussion and Related Literature

In this section, I first discuss and relax two assumptions included in the model: fixed advertising cost and unit demand of the seller. In addition, I explain how my model is robust
to generalizations. Subsequently, I relate the paper to the existing literature. To this end, I discuss existing dynamic pricing models with full and no commitment and argue why in online markets full commitment is hard to attain. Finally, I relate my paper to other pricing and marketing literature.

Extensions

Variable Costs

Usually, the fixed cost of advertising consists of the cost of designing the advertising email while the variable cost of sending the email is close to zero. If the firm has to pay an advertising platform per view or per click, then the cost of advertising can have a variable component.

If there is a variable component of advertising costs that depends on the number $N$ of low-valuation buyers reached, the analysis can be generalized easily. In particular, with a linear cost function $C_A(N) = C_A + c_A \cdot N$, $\bar{N}$ and $\bar{N}$ can be equivalently derived from (1.3) and (1.4) with $v_L$ being reduced to $v_L - c_A$. The seller always prefers to have fixed costs to variable costs. In order to see this, note that the same $N^*(C_A)$ results from advertising costs of the form $C_A(N) = C_A + c_A \cdot N$ as long as $\frac{C_A}{v_L - c_A}$ is constant. Hence, the profits of the seller
are only affected in sales periods. The profit during a sale is given by

\[ N^* (C_A) \cdot (v_L - c_A) - C_A = (v_L - c_A) \cdot \left( N^* (C_A) - \frac{C_A}{v_L - c_A} \right). \]

Thus, the firm prefers not to face variable costs but rather a fixed advertising cost because it is a cheaper commitment device. In particular, if \( C_A \) is zero, the Coase conjecture holds. This can have implications for pricing of advertising platforms. Firms are willing to pay more to an advertising platform that has fixed rates rather than per-click or per-view rates as commonly used.

**Multi-Unit Demand**

The assumption that buyers demand only a single unit of the good in their lifetime can be relaxed without changing the results as long as buyers do not have an incentive to stockpile the good for future use during a sale. This is the case if the good is not storable, such as groceries or if buyers do not take calendar time into account even if they have learned it during a previous sales period. The latter is a plausible assumption if buyers demand a good very infrequently.

There is a recent theoretical and empirical literature on stockpiling of storable goods. Dudine, Hendel, and Lizzeri 2006 show that for storable goods prices are higher and welfare lower with commitment than without commitment. They consider a model with finite time and linear storage cost. Hendel and Nevo 2013 show that buyers anticipate future needs for soft-drinks, and buy big quantities during a sale for future consumption. Other recent empirical studies on storable goods markets have been conducted by Hendel and Nevo 2006b and Hendel and Nevo 2006a.

If buyers demand several units of the good at a time, but only once in their lifetime, results remain qualitatively unchanged. However, given an advertising cost, the seller can commit to a higher cutoff-demand as can be seen from the necessary conditions (1.3) and (1.4).

**Related Literature**

**Dynamic Pricing and Commitment**

Classic papers on dynamic pricing by Stokey 1979, Conlisk, Gerstner, and Sobel 1984, and Sobel 1991 introduce the idea of intertemporal price discrimination with limited commitment. In these papers, the level of commitment is exogenously given and buyers and sellers are assumed to be equally patient.\(^{20}\) Hence, the frequency of sales is determined by the period

\(^{20}\)This assumption is usually motivated by perfect capital markets, but there are many reasons to assume that buyers and sellers do not share the same time preferences as discussed before.
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length and if the seller is assumed to be able to change the price at any point in time, the unique equilibrium outcome results in the Coase conjecture. This paper extends this classical theory in two ways: it investigates the trade-off between the value of commitment and the cost of commitment, given by the advertising cost, and it highlights the impact of buyers’ time preferences. Furthermore, in my model, buyers do not automatically see all prices, but have to search or get informed by the seller.

Many recent papers from the dynamic mechanism design literature, such as Board 2008 and Garrett 2012 in contrast assume full commitment power by the seller. Board 2008 focuses on seasonal sales that are driven by demand fluctuations over time, rather than sales solely driven by the incentive to intertemporally price discriminate. Such sales are predicted by customers and in his model, the seller has to drop the price slowly to abate the effect of buyers delaying purchase in order to wait for a lower price, but increases the price quickly after a sale. This asymmetry between increases and decreases induces a higher total price level and, hence, reduces consumer surplus and social welfare. In contrast, in my model, buyers benefit from sales due to the inability of the seller to commit. Garrett 2012 also assumes full commitment by the seller, but considers a stationary setup as in this paper. In his model, cyclical price paths with slowly decreasing prices and upward jumps after a sale are obtained because buyers’ preferences change over time. Besbes and Lobel 2012 take a very different approach by investigating intertemporal price discrimination in a setup without exponential discounting. Instead buyers have heterogeneous finite willingness to wait and the monopolist maximizes the long-run average revenue. The optimal price path is more complicated and incorporates nested sales with largest discount at the end of a cycle.

A common justification for commitment is that in many setups a best-price provision strategy can serve as a commitment device for sellers and yield the same allocation as the best strategy under full commitment as first noted by Butz 1990. This is a reasonable argument in traditional mortar-and-brick stores. In online markets, however, this logic does not apply because the seller can send coupons to buyers (or have instantaneous sales as in my model) which are not seen by buyers who have bought. This possibility makes commitment through best-price provision harder. Nevertheless, some retailers, such as airline companies, have automatized their pricing using complicated algorithms. This can help sellers to commit. For many consumption goods, this automation is, however, not feasible because price offers require an attractive design. This design cannot be easily standardized if one thinks of the product not as a single good but as a product group (such as lamps for example) that is put on sale periodically.

The literature that assumes no commitment power has focused on sellers with a finite inventory and a sales deadline. In that case, the deadline and scarcity of products can provide the seller with sufficient commitment power to guarantee high regular prices and fire sales from time to time. Hörner and Samuelson 2011 study a variant of this setup in which the seller has only a single unit of the good and Dilme and Li 2013 deal with the multi-unit case. If the seller has many units, she is choosing prices and quantities to have on sale simultaneously. Similarly to my model, the equilibrium price path contains sales of
large quantities of the good similarly to my model. However, in their model, this is done so as to guarantee higher prices later on by making the good scarcer. Instead, my model considers a monopolist who can produce arbitrary many units, albeit at zero cost.

**Other Theories of Pricing**

The literature on price fluctuation that does not rely on intertemporal price discrimination focuses on the seller’s incentive to discriminate between different types of buyers in a static oligopolistic setups (e.g., Shilony 1977, Varian 1980, Salop and Stiglitz 1982, Sobel 1984). They derive mixed pricing strategies as an equilibrium outcome. In Varian 1980 and Salop and Stiglitz 1982, sellers price discriminate between informed and uninformed customers, facing a decreasing average cost curves. In equilibrium, firms mix among prices according to a smooth distribution. The firm with the lowest price can sell to all types of consumers, while all other firms sell only to a fraction of uninformed consumers. One drawback of these models is that they do not result in a regular price and sales prices. With loss-averse customers, Heidhues and Köszegi 2012 and Rosato 2013 show that even in the absence of competition, mixed-strategy equilibria can lead to a single regular price and sales prices.

I do not consider competitive markets, but I believe that the results remain qualitatively similar if buyers switch between firms with some probability. Prices and frequencies of sales would be pushed down, but the comparative statics should remain similar. If, however, buyers can find out about other firms with a positive search cost, interesting new trade-offs can arise. This is beyond the scope of the present paper and left for future research. Intuitively, with competition, the monopolist is not only competing with its future self, but also with another firm.

This paper also relates to the literature on behavior-based price discrimination. These papers are mainly concerned with products that are repeatedly purchased. In that case, sellers infer from buyers purchasing decision that they have a high valuation. For example, Villas-Boas 2004 considers an overlapping generation model in which buyers live for two periods and the seller can charge two different prices, one for customers who have previously bought and one for new customers. In equilibrium, the price path for new customers is alternating between a high and low price and previous customers always face a high price. A detailed review of the literature on behavior-based price discrimination can be found in Fudenberg and Villas-Boas 2012 and Fudenberg and Villas-Boas 2006. In contrast to these models that focus on sellers learning only from customers who actually buy, in my model, the monopolist learns from the fact that a customer has not bought, assuming she can identify “window shoppers.”

**Theories of Advertising**

Advertising has been interpreted as a technology to signal the quality of the good (Nelson 1970) or to persuade consumers and change their taste for the good (Dixit and Norman 1978). A comprehensive review of this classical literature on advertising can be found in
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Bagwell 2007. In contrast, I focus on informative advertising. Anderson and Renault 2013 and Anderson and Renault 2006 are also concerned with informative advertising, but they consider a static setup in which the content of informative advertising depends on the search cost that buyers need to pay in order to visit the monopolist. In my model, however, the only information that buyers can learn is the discounted price.

Recently, the focus of this field has shifted towards targeted advertising in online markets, but with a focus on static models. Iyer, Soberman, and Villas-Boas 2005 consider a model with competing firms who can target advertising and pricing to different groups. In equilibrium, firms advertise more to consumers who have a strong preference for their product. In contrast to my model, targeted advertising always increases profits. Bergemann and Bonatti 2011 compare the role of targeted advertising in offline versus that in online markets in a competitive environment with many sellers and many advertising markets (media). They show that an increase in targeting ability leads to an improvement in consumer-product matches, but also to a higher market-power of firms. Finally, in their model, the price of advertising is determined endogenously in equilibrium and is first increasing and then decreasing in targeting capacity. In my environment, however, it seems to be natural to assume that advertising costs are given by the cost of creating the advertising because sellers do not buy advertising space from third parties.

Levin and Milgrom 2010 argue that, even though targeting improves the matching between buyers and advertisers, this benefit has to be traded off with the mutual adverse-selection problems it can create between the advertisers and advertising platforms. In contrast, the present paper examines dynamic effects of advertisements that targets low-valuation buyers in order to inform them about the price and in order to activate them. To the best of my knowledge, this role of advertising is novel to the literature.

Finally, in the marketing literature, EDLP and promotional pricing are the two most important marketing and pricing strategies. Most papers consider oligopolistic setups with differences in customer types. Lal and Rao 1997 explain the coexistence of these two strategies by a game theoretic model in an oligopolistic setup, where buyers with low search cost (cherry pickers) prefer to buy from firms who engage in promotional pricing while buyers with high search cost (time constrained consumers) prefer EDLP. Bell and Lattin 1998 argue that EDLP attracts large basket consumers who prefer a low average price while small basket customers like promotional pricing. 21 I offer an alternative rationale for EDLP.

21Prominent retailers engaging in EDLP include Walmart, Home Depot, Lowe’s, Trader Joe’s. JC Penny switched back to a promotional pricing strategy after revenues had dropped significantly. One common explanation for their failure is that people make an inferences about the quality of a good through the regular price and that this plays a greater role for apparel.
1.7 Conclusion

This paper studies a dynamic monopolist who can engage in targeted advertising, but cannot commit to future pricing strategies. The ability to track and target can be a two-edged sword because it allows the seller to intertemporally price discriminate, but it also exposes the seller to the consequences of the Coase conjecture. Costly advertising can benefit the seller because advertising costs act as a deterrent to having overly frequent sales in equilibrium. Depending on the time preferences of buyers and the costs of advertising, there exists an equilibrium with regular high prices and occasional sales. An everyday-low-price equilibrium can always be sustained but yields lower profits than the advertising equilibrium, if the latter exists.

The benefit of the model is that the level of commitment is endogenously determined by the cost of advertising. I derive implications for the frequency of sales, profits, demand for advertising and welfare. The analysis shows that the frequency of sales and the resulting regular price level can be crucially affected by the advertising costs. Sales are more frequent with low advertising costs if price discrimination is profitable compared to EDLP. Because profits are non-monotonic in advertising costs, the existence of an advertising equilibrium is not monotonic in advertising cost. Moreover, the monopolist benefits from different intermediate levels of advertising costs, depending on the impatience of the buyers because buyers’ impatience determines the sensitivity of the regular price to changes in the frequency of sales.

This setup is relevant for various online markets and advertising platforms. First, firms and policy makers should be aware of the Coasean force and its implications. My model (as all game-theoretic models) hinges on the assumption that buyers correctly anticipate the frequency of sales. Hence, in the short term, firms might benefit or suffer from wrong beliefs of buyers about the frequency of sales. For example, some retailers offer sales on items in a wish list that have not been bought yet or others have increasing frequencies of sales rather than constant average frequencies of sales. In such markets, as buyers learn about the strategy of the firm, the outcome might converge to equilibria described in this paper in the long-run.
Chapter 2

Transparency and Distressed Sales under Asymmetric Information

2.1 Introduction

A public policy response to the recent financial crisis has been regulatory changes (some enacted, some still under consideration) aimed at improving the transparency with which financial securities are traded. For example, a stated goal of the Dodd-Frank Act of 2010 is to increase transparency in the financial system. The European Commission is considering revisions to the Markets in Financial Instruments Directive (MiFID), in part to improve the transparency of European financial markets. Such actions reflect a widely held belief that transparency is welfare enhancing because it is necessary for perfect competition, it decreases uncertainty, and it increases public trust. Yet, there are a number of nuances concerning transparency and the question of whether transparency enhances efficiency is correspondingly complicated. Indeed, as we show, in settings relevant to this public-policy debate, transparency actually has negative effects.

We consider a problem of an owner of an indivisible durable asset who suffers a liquidity shock and study the role of price transparency. Due to the liquidity shock, the seller’s present value of the good drops to a lower level than the true value of the good. Hence, she would like to sell the asset to a buyer not facing a liquidity shock. The problem is that usually the owner of the asset has better information about its quality. Any potential buyer therefore faces an adverse selection problem. As first stressed by Akerlof 1970, if there is only one opportunity to trade, the buyer is only willing to pay his expected valuation of the asset. However, high seller types may not want to accept this price, if the adverse selection problem is sufficiently strong, even though there are positive gains from trade for all types. In a dynamic setting, in which sellers get several chances to sell their good, this logic of a lemons market leads to inefficient delay in trade. The amount of delay is affected by whether buyers observe past price offers.

We show how the price disclosure policy influences the dynamics of trade and efficiency
in a dynamic lemon’s market. We consider two opposite information structures, one in which buyers observe past price offers and one in which they do not. First, we show that the disclosure policy only plays a role if there are frequent opportunities to trade. Second, if trade opportunities are frequent and past offers are not observable, buyers randomize between several price offers such that price realizations can look very volatile. Finally, we show that more trade takes place when past offers are not observed and that the opaque regime Pareto dominates the transparent one.

We examine a game theoretic model in discrete time with a long-lived, privately-informed seller and a competitive market of buyers in every period (modeled as a number of short-lived buyers competing in prices in every period). We study a finite horizon model. In our base case, at the deadline, we assume that the seller’s type is publicly revealed. This could arise because there are some studies needed to determine the quality of the asset but these studies take time to be conducted. Examples for such studies include recent stress tests of financial institutions or geological surveys of land with mining or oil potential where the results are revealed at some future date. Similarly, an entrepreneur might be able to prove the value of his idea after some deadline. In an extension we allow for a more general specification where there is a loss of surplus at the deadline. For example, if the deadline is reached the opportunity to trade disappears. This can create a deadline effect in which a larger mass trades just at the deadline. With public offers, this leads endogenously to a trading impasse (illiquidity) before the deadline. With private offers there is no trading impasse and hence the efficiency differential between both information regimes increases.

What makes the markets operate differently in these two information regimes? In a transparent market, buyers can observe all previous price offers and thereby learn about the quality of the good through two channels: the number of rejected offers (time on the market) and the price levels that have been rejected by the seller. By rejecting a high offer, the seller can send a strong signal to future buyers that she is of a high type. For example, in transparent exchanges, sellers try to influence prices by taking advantage of the observability of order books. In contrast, in an opaque market, in which buyers cannot observe previously rejected prices, the seller signals only via delay. Hence, with publicly observable offers, the seller has a higher incentive to reject high offers than with private offers. This difference in seller’s responses to price offers drives the differences in equilibrium dynamics that we describe in this paper.

Pure strategy PBEs with public offers always exist and they coincide with PBEs with private offers if the discounting between two periods is large. If, however, the discounting between two periods is small enough there are no pure strategy PBEs with private offers.\footnote{This could be driven by either a low interest rate or by the high frequency of the offers.} Intuitively, the difference between the two information structures is larger in high-frequency markets because the seller has a stronger incentive to signal to tomorrow’s buyers by rejecting today’s price in order to drive up future prices.
CHAPTER 2. TRANSPARENCY AND DISTRESSED SALES UNDER ASYMMETRIC INFORMATION

The reason for non-existence of pure strategy equilibria with private offers when trade is frequent can be best understood as follows. Consider a pure-strategy equilibrium with public offers. In such an equilibrium, prices are relatively low since the temptation of a seller to use a rejection of a high price as a signal is high when the time between offers is small. If one buyer was now able to make a secret offer, a higher price offer is accepted by more seller types such that it can become a profitable deviation. Then, the equilibrium with public offers is not an equilibrium with private offers. One might guess that the equilibrium might then simply have deterministically higher offers. In any pure strategy equilibrium with private offers, however, beliefs about remaining buyers in the market are correct at all times and buyers always receive zero profits. Hence, this must also be an equilibrium with public offers as deviations are less profitable with public offers. Consequently, in an equilibrium with private offers, there must be mixing between low offers and high offers. The possibility that the previous period offer was low, increases the adverse selection problem the next period and conditional on arriving to the next period dampens the beliefs about the pool of types in the market. This allows for the average offers in each period to be higher as a rejection is not such a strong signal to future buyers.

Due to multiplicities and the complicated structures of mixed-strategy equilibria with private offers, welfare analysis in the general setting is hard or next to impossible and most likely does not help to understand the driving forces. Hence, to provide intuition about welfare consequences of information disclosure, we focus our analysis on a situation with two periods. In this setting, there is a unique PBE with public offers, but multiple mixed-strategy equilibria with private offers. We characterize all (mixed-strategy) equilibria and show that expected trade in each period is constant across equilibria. This paves the way to welfare comparisons. We show that any equilibrium with private offers is Pareto superior. That is, for all seller types, it generates a higher expected welfare than the unique PBE with public offers. The result follows from the fact that with private offers, although, as explained before, the buyers mix between high (real) offers and low-ball offers (i.e., non-offers that are rejected by all seller types) on average the expected price is higher. As a result, because the sellers are risk-neutral, those sellers, who had trade in the second period if they were in the public offers environment, are clearly better off. Sellers who had traded in the first period in a public offers environment can at least receive the expected first period price with private offers which makes them better off as well. Consequently, all sellers are better off with private offers.

Related Literature

The closest paper to ours is Hörner and Vieille 2009 (HV from now on). They are also interested in comparing the trading dynamics with public versus private offers. Our model differs from their setup in two dimensions. Most importantly, while we assume intra period competition, HV consider a model with a single buyer in each period. Secondly, we focus

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2 Although this might seem obvious, with a monopolistic seller this need not be true as for example in Kaya and Liu 2012.
our analysis on a finite horizon model while they have an infinite horizon. The latter leads to differences in the solution methods but is not crucial for the results.\footnote{One can verify this by solving a finite horizon version of HV.}

The lack of intra period competition introduces Diamond Paradox effects (see Diamond 1971) because HV’s model can be interpreted as a search model in which buyers compete, but the seller faces some search friction. As a result, the equilibrium in HV with public offers is, in their own words, quite “paradoxical” since the first offer is rejected with positive probability and all other offers are rejected with probability 1. Instead, the equilibrium with public offers in our model has a positive probability of agreement in each period and slowly more and more types eventually trade.

HV focus their analysis on very large discount factors. Their main result is to show that with private offers there is eventually more trade. They show that the equilibria with private offers are in mixed strategies and have the property that there is eventually trade with probability one. They do not have a general analysis about the relative efficiency between both information regimes but state that if there are only two types of informed seller then the private offers equilibrium has higher welfare. In general it is not obvious if their model would lead to an efficiency ranking. It is possible that the endogenous trading impasse that arises with public offers is actually valuable since it serves as a commitment device, sellers know that they either trade in the first period or never again. Indeed, in separate but related work, Fuchs and Skrzypacz 2014 show that under some fairly mild regularity conditions efficiency is actually enhanced when the privately informed seller is exogenously restricted to only one opportunity to sell. With intra period competition, we show in a two period linear model, that even in the first period there is more trade with private offers. Indeed, we show that efficiency gains are driven by higher prices and more trade period by period rather than a tradeoff between more initial trade and higher trade in the long run like in HV.

Another interesting prior comparison between private and public offers goes back to Swinkels 1999. He looks at a dynamic version of the Spence signaling model where potential employers are allowed to make private offers to the “students” at any time. Swinkels shows that in this case the unique equilibrium outcome is a pooling equilibrium with all students being hired at time 0. This, he points out, is in direct contrast to the result in Nöldeke and Van Damme 1990, who show that, with public offers, the unique equilibrium to survive the NWBR refinement is a separating equilibrium where the high types go to school just long enough to credibly separate themselves from the low types. The main difference between both these papers and ours is similar to the difference between Spence and Akerlof. As in the latter, the adverse selection problem is stronger in our model and hence the buyers even with private offers would not be willing to buy at the price necessary to get all sellers to sell.\footnote{This is also what leads there to be delay in the bargaining model by Deneckere and Liang 2006.}

Our result about (non)existence of pure strategy equilibria in the private offers case is related to the result in Kremer and Skrzypacz 2007 who study a dynamic version of the
education signaling model with private offers in a finite horizon model with the type being (partially or fully) revealed in the last period. They show that there do not exist fully separating equilibria in a game with a continuum of types or with a finite number of types if the length of periods is short enough. We extend their reasoning to show that in our model with interdependent valuations pure strategy equilibria do not exist if the discount factor is high enough or the periods are short enough.

More recently, Kim 2012 compares three different information structures in a continuous time setting in which many sellers and buyers, who arrive over time at a constant rate, match randomly. In every match, the buyer makes a price offer that the seller can accept or reject. The type space of the seller is binary. Instead of looking at observability of past offers, steady state equilibria in settings in which buyers do not observe any past histories are compared with settings in which the time on the market or the number of past matches can be observed by buyers. The welfare ordering is not as clear cut as in our paper. It is shown that with small frictions, it is optimal if only the time on the market is observable while with large frictions the welfare ordering can be reversed.

For repeated first-price auctions, Bergemann and Hörner 2010 consider three different disclosure regimes and they show that if bidders learn privately about their win, welfare is maximized and information is eventually revealed. There are also many papers that explore the role of transparency different from price transparency. These models are mostly in static environments. Pancs 2011 illustrates how it can be optimal for stock exchange to allow for iceberg or hidden offers introducing latent buyers who can be attracted by such offers. Likewise, Buti and Rindi 2011 show why traders with different preferences choose different levels of information disclosure when they make offers.

Besides our contribution regarding the implications of transparency, our paper also contributes to the literature on dynamic lemons markets in general. One of the most recent works by Deneckere and Liang 2006 considers an infinite horizon bargaining situation, i.e. one long-lived buyer and one long-lived seller, with correlated valuations. They show that even in the limit as the discount factor goes to one, there can be an inefficient delay of trade unlike predicted by the Coase conjecture.\(^5\) Janssen and Roy 2002 obtain similar results with a dynamic competitive lemons market with discrete time, infinite horizon and a continuum of buyers and sellers. While in their model both market sides compete, we assume that there is only one seller. Unlike most previous papers that consider slightly different market structures, we are able to almost fully characterize equilibria in mixed strategies with private offers. This makes it possible to understand these kinds of equilibria in more detail. For example, we show that non-offers in period one are always part of an equilibrium.

A number of recent papers work directly in continuous time and rather than modeling buyers as strategic they assume there is some competitive equilibrium price path. This paper is a valuable complement since it provides a valid strategic foundation to those papers. For example one can understand the No Deals Condition in Daley and Green 2012 as arising

\(^5\)See also Fuchs and Skrzypacz 2013.
CHAPTER 2. TRANSPARENCY AND DISTRESSED SALES UNDER ASYMMETRIC INFORMATION

from private offers and the Market Clearing condition in Fuchs and Skrzypacz 2014 as arising in a setting with public offers.

2.2 Model and Preliminaries

General Setup

A seller has an asset that she values at \( c \) which is her private information and uniformly distributed on \([0, 1]\). In Section 2.6 we extend our analysis to more general distributions. One can think of the asset giving an expected cash flow each period and \( c \) being its present value for the seller. There is a finite amount of time to trade, which we normalize to 1, and it is divided into \( N \in \{1, 2, \ldots \} \) periods of length \( \Delta \). At each time \( t \in \{0, \Delta, 2\Delta, \ldots, 1 - \Delta\} \), two short lived buyers, whom we label by \( i = 1, 2 \), make simultaneous price offers \( p_i^t \) to purchase the asset. The value of the asset for the buyers is given by \( v(c) = Ac + B \) with \( A, B > 0 \) and \( A + B = 1 \), i.e., \( v(1) = 1 \). Hence, gains from trade \( v(c) - c \) are strictly positive for all \( c \in [0, 1) \). The game ends as soon as the good has been sold. If trade has not taken place by time 1, then information is revealed to buyers before the last buyers make their offers, so the seller receives a surplus \( v(c) - c \). We also consider the case when the seller receives a surplus of \( \alpha(v(c) - c) \) as an extension in Section 2.5. This captures the case \( \alpha = 0 \), when there is no opportunity of trade after time 1. We show that most of our results are robust for general \( \alpha \).

The seller discounts payoffs according to a discount rate \( r > 0 \). Hence, her period-to-period discount factor is given by \( \delta = e^{-r\Delta} \). All players are risk neutral. Given the seller’s type is \( c \) and agreement over a price \( p \) is reached in period \( t \), the seller’s time 0 present value payoff is given by \( (1 - \delta^t)c + \delta^tp \); a buyer’s payoff is \( v(c) - p \) if he gets the good and 0 otherwise. Without loss of generality, we restrict prices to be in \([0, v(1)]\), since it is a dominant strategy for the seller to reject any negative price, and for any buyer it is a dominated strategy to offer any price higher than \( v(1) \) that has a positive probability of being accepted.

We explore two different information structures. In the public offers case, we assume that period \( t \) buyers observe all past rejected offers \( \{p_s\}_{s=0}^{t-\Delta} \). A period \( t \) buyer’s strategy, \( \rho_i^{B_t} \), maps a history of prices to a probability measure on \([0, v(1)]\). A strategy for the seller is a sequence of acceptance decisions \( (\rho_i^S)^{1-\Delta} \) that depend on her type as well as on past

\(^6\)Alternatively, and mathematically equivalently, \( c \) can be thought as the cost of producing the asset.

\(^7\)The analysis is the same if there are more than two buyers since the buyers compete in a Bertrand fashion.

\(^8\)We assume linearity for tractability, as we show in Section 2.6 that most results can be extended to continuous and increasing \( v(c) \).

\(^9\)We assume \( v(1) = 1 \) only to rule out the possibility of trade ending before the last period. This allows us to avoid making assumptions about off-equilibrium beliefs if the seller does not sell by \( t \) even though in equilibrium he is supposed to. If \( v(1) > 1 \) but \( r \) is small enough so that not all types trade in equilibrium, our analysis still applies.
and current offers. That is, $\rho^S_t : [0, 1] \times [0, v(1)]^{2 \times n} \rightarrow \{(\eta_1, \eta_2) : \eta_1 + \eta_2 \leq 1\}$, where $\eta_i$ represents the probability with which the seller accepts buyer $i$'s offer in period $t$.

With private offers, we assume that period $t$ buyers are aware that the seller has rejected all offers in periods $s < t$ but, crucially, the buyers do not know what these offers actually were. The generation $t$ buyers can condition their offers on their information set. We denote the possibly mixed strategies of generation $t$ buyers by $\sigma^B_t$ which is a probability measure on $[0, v(1)]$. With a slight abuse of notation we also denote the $[0, v(1)]$-valued random variable that represents buyer $i$'s strategy by $\sigma^B_i$. A strategy for the seller is of the same form as with public offers, but we will denote them in the private offer setting by $\sigma^S_t$. We assume that the seller responses are buyer identity independent that is, conditional on receiving the same price offer, she will treat both buyers equally. Finally, we denote the distribution of seller types, that period $t$ buyers believe to be facing, by $F_t(c)$.

### Equilibrium Notion

We will be interested in characterizing the appropriately extended notion of perfect Bayesian equilibrium (PBE) for our setting. Essentially, this entails a sequence of pricing strategies for the two buyers, $\{(\rho^B_1, \rho^B_2)\}_t$ and $\{(\sigma^B_1, \sigma^B_2)\}_t$, for public and private offers, respectively, acceptance rules $\{\rho^S_t\}_t$ and $\{\sigma^S_t\}_t$, respectively, and the buyers' beliefs $\{F_t\}_t$ satisfying the following three conditions:

1) Any price offer in the support of $\rho^B_i$ and $\sigma^B_i$ must maximize the buyer's payoff conditional on the seller's acceptance rule, the other buyer's strategy $\rho^B_j$ and $\sigma^B_j$, respectively, and his belief $F_t(c)$.

2) The buyer's beliefs are updated according to Bayes rule taking the seller's and the other buyer's strategies as given. Hence, in particular, $F_0(c) = c$. If the seller accepts an offer that no remaining seller type was supposed to accept in equilibrium, we do not need to worry how beliefs evolve off-equilibrium since the game ends in that case. Moreover, by the assumption that $v(1) = 1$, there is a mass of sellers that does not trade before time 1. This implies we do not have to consider the off-equilibrium case in which all types should have traded but the seller is still around.

3) The seller's acceptance rule maximizes her profit taking into account the impact of its choices on the agents updating and the future offers she can expect to follow as a result.

With public offers, deviations from equilibrium price offers are observed by future buyers such that different continuation games can be played, while with private offers, out-of-equilibrium price offers do not change the continuation game. Consequently, the out-of-equilibrium behavior of buyers can be more elaborate with public offers than with private offers. We do not require, however, any out-of-equilibrium belief assumptions because the payoffs after the deadline are independent of the beliefs held by buyers. Hence, the game

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10 Note that in equilibrium, both buyers in any period $t$ must have the same beliefs.

11 If the offer is public the updating is also conditional on the offered amount.

12 We discuss the case where $v(1) > 1$ in Section 2.6.
can be solved by backward induction. This fact and intra-temporal competition guarantee
that continuation games off equilibrium remain tractable as we show in Lemma 10.

**Preliminaries**

Before turning to the main analysis, we show in this Section that in equilibrium, sellers play
equilibrium acceptance rules given by cutoff-strategies in response to price offers. Moreover,
every cutoff is only a best response to a single highest price. Hence, the seller’s equilibrium
strategy translates into a one-to-one mapping from cutoffs to prices which simplifies our
analysis significantly.

As in other dynamic games, in equilibrium, sellers with valuations above a cutoff \( k_t(p) \)
reject a price offer \( p \) in period \( t \) while sellers with valuations less than \( k_t(p) \) accept it. In the
bargaining literature, it is the better types that accept first and this property is known as
the skimming property. Since here it is the worse types that trade first, we call it reverse-
skimming instead.

**Lemma 9. (Reverse-skimming property)** In any equilibrium with either type of information
structure, for any period \( t \) and for any price \( p \), there exists a cutoff type \( k_t(p) \) so that a seller
of type \( c \) accepts the offer \( p \) in period \( t \) if \( c < k_t(p) \) and rejects \( p \) in period \( t \) if \( c > k_t(p) \).

This lemma holds independently of the information structure in place (although the cutoff
may differ). The intuition for the lemma is straight forward. If a type-\( c \) seller is willing to
accept a price that, if rejected, would induce a given future price path, then all lower-type
sellers would also be willing to accept that price rather than wait for a price on that path.
While with publicly observable prices, the belief about the remaining seller types in the
market is given by a single cutoff \( k_t \), with private offers, buyers can have beliefs with a richer
support if previous buyers played a mixed pricing strategy. From now on, we denote the cdf
of the cutoff \( k_t(p_t) \) by \( K_t : [0, 1] \to [0, 1] \), where \( p_t \) is the random variable representing the
highest offer in period \( t \), i.e. with public offers \( p_t = \max_{j=1,2} \rho_t^{B_j} \) and with private offers
\( p_t = \max_{j=1,2} \sigma_t^{B_j} \), respectively.

Thanks to the reverse-skimming property, we can, write a buyer’s expected profit conditional on receiving the good if trade takes place as

\[
\Pi_t(p; f_t) = \int_0^{k_t(p)} (v(c) - p) f_t(c) dc,
\]  

(2.1)

if the belief about the remaining types of the sellers is distributed according to a probability
density function \( f_t \). Note, that the actual expected profit of the buyer is the probability that
he receives the good times \( \Pi_t(p; f_t) \).

Furthermore, in equilibrium, buyers’ beliefs about the remaining types are given by
mixtures of truncations of the prior uniform distribution. With public offers, all past prices
are observable, i.e.

\[ f_t(c) = \frac{1}{1 - k_{t-1}(p)}1_{[k_{t-1}(p),1]}(c) \]

where \( p \) is the highest price chosen by buyers up to period \( t - 1 \). In contrast, with private offers, it is given by the belief of period \( t \) buyers about the strategies of buyers in previous periods. Nevertheless, also in this case, the reverse-skimming property guarantees that equilibrium beliefs about the remaining types at the beginning of period \( t \) have a relatively simple structure. Given (possibly mixed) equilibrium strategies of previous buyers \( \sigma^B_1, \ldots, \sigma^B_{t-1} \), the random variable \( k_{t-1} \equiv \max\{k_1(\max_i \sigma^B_i), \ldots, k_{t-1}(\max_i \sigma^B_i)\} \) represents the cutoff at the beginning of period \( t \). Let us denote its cumulative distribution function by \( K_{t-1} \). In equilibrium, it must hold that

\[ f_t(c) = \int_0^c \frac{1}{1 - k}1_{[k,1]}(c) dK_{t-1}(k). \]

With private offers, the price history does not affect future buyers’ beliefs and hence it does not affect their strategies. Hence, the mapping from prices to cutoffs in each period is one-to-one, i.e. in a given period, there exists a unique price \( p_t(k) \) that results in a cutoff \( k \). With public offers, the history of prices affects future buyers’ strategies, so an out of equilibrium deviation offer of a buyer can change the whole continuation game. Nevertheless, in every pure strategy equilibrium the price-cutoff mapping is one-to-one in each period. Moreover, for every pure strategy equilibrium outcome, there exists a PBE in which all equilibrium cutoffs are weakly monotone in time. This is summarized in the following lemma.

**Lemma 10. (Inverse supply)**

(i) (Private offers) With private offers, on equilibrium path, at any point in time \( t \) and for any seller type \( k \) there exists a unique price \( p_t(k) \) that results in a cutoff \( k \). \( p_t(\cdot) = k^{-1}_t(\cdot) \) is increasing and continuous.

(ii) (Public offers) With public offers, given any cutoff-belief \( k_{t-\Delta} \) at time \( t \), there is a unique continuation equilibrium. For any \( k \in [k_{t-\Delta},1] \) there exists a unique price \( p_t(k) \) for which seller \( k \) is the highest seller type accepting the price (given prices are consistent with the continuation game). Let \( \kappa_{t+\Delta}(k) \) be the unique period \( t + \Delta \) cutoff of the continuation equilibrium given at the beginning of the period the cutoff is believed to be \( k \). Then, \( p_t(k) \) is increasing in \( k \) and

\[ p_t(k) = \delta p_{t+\Delta}(\kappa_{t+\Delta}(k)) + (1 - \delta)k. \]  

We call the function \( p_t(\cdot) \) inverse supply function since \( p_t(k_t) \) is the minimal price needed to attract all types up to \( k_t \). With private offers, the existence of an inverse supply function is straightforward because even after an out-of-equilibrium price offer the same continuation game will be played. With public offers, however, the continuation game changes depending on the price offer. Nevertheless, even for general distributions, the cutoff \( \kappa_{t+\Delta}(k_t) \) is always
increasing in yesterday’s cutoff. As a result, $k_t(p)$ is always increasing and an inverse supply function exists. The formal proof can be found in the appendix. Abusing notation slightly, we represent beliefs about past cutoffs and actual cutoffs expected in future periods as part of the continuation game both by cdfs denoted by $K_t$.

An immediate implication of Lemma 10 is that for private offers we can think of the buyers to face supply curves given by the inverse supply function $p_t(k_t) = \delta \left( \int_{k_t}^{1} p_{t+\Delta}(k_{t+\Delta}) dK_{t+\Delta}(k_{t+\Delta}) \right) + K_{t+\Delta}(k_t) p_{t+\Delta}(k_t) \right)$

in periods $t < 1$, as derived in the proof of Lemma 9, and

$$p_t(k_{1-\Delta}) = (1 - \delta)k_{1-\Delta} + \delta v(k_{1-\Delta}). \quad (2.4)$$

With public offers, tomorrow’s cutoff $k_{t+\Delta}$ is a function of today’s cutoff $k_t$, while with private offers, the distribution of tomorrow’s cutoffs $K_{t+\Delta}$ is independent of today’s cutoff. $(1 - \delta)k_t$ represents the payoff of a type $k_t$ buyer if he held on to the good for exactly one more period, $\int_{k_t}^{1} p_{t+\Delta}(k_{t+\Delta}) dK_{t+\Delta}(k_{t+\Delta})$ is the expected price the seller can get if she sells the asset in period $t + \Delta$ and $p_{t+\Delta}(k_t)$ is the expected payoff that the seller can expect if he does not sell tomorrow either. This expected benefit from waiting must correspond to the payoff from selling today given by the price $p_t(k)$.

One drawback of the representation of a buyer’s expected profit given by (2.1) is that the buyer’s belief $f_t$ is defined on the space of cutoffs, whereas buyers choose prices in $[0, v(1)]$. The existence of a unique $p_t(k)$ allows us, on the equilibrium path, to think of buyers essentially choosing cutoffs instead of prices given the seller’s optimal cutoff strategy $k_t(\cdot)$. More precisely, we can write a buyer’s expected profit conditional on winning the bid on the equilibrium path, if he bids a price $p = k_t^{-1}(k)$ and given belief $K_{t-\Delta}^m$ about the cutoff distribution at the beginning of period $t$, by

$$\pi_t(k; K_{t-\Delta}^m) = \int_{0}^{c} \int_{0}^{1} \frac{1}{1-k} dK_{t-\Delta}^m(\tilde{k}) (v(c) - p_t(k)) \, dc. \quad (2.5)$$

If $K_{t-\Delta}^m$ has a one-point support at $l$ (which is always the case with public offers), then we write $\pi_t(k; l)$ instead of $\pi_t(k; K_{t-\Delta}^m)$, abusing notation slightly.

Finally, it is worth noting that thanks to the reverse-skimming property, the only efficiency relevant outcome of the game is the distribution of the cutoff-types in each period. Hence, if we can determine for any PBE the distribution of cutoffs, we are able to calculate the total surplus and to compare the welfare with private and public offers.

\[13\] For general distributions and valuation functions $v(\cdot)$, not all cutoffs can necessarily be attained.
2.3 Transparency and Frequency of Trade

In this Section we establish our first main result. We show that the length of the period (or discount factor) plays an important role for the structure of equilibria with private offers. When trade becomes sufficiently frequent the equilibrium with private offers no longer coincides with the equilibrium with public offers. Despite there being trade in every period with both information structures with public offers there is a unique equilibrium in pure strategies; instead, with private offers there are no pure strategy equilibria. We do not provide a general existence proof for arbitrary number of trading opportunities but as we show by construction in Section 4 there exist multiple mixed strategy equilibria when there are two trading opportunities. We first present our formal result. Then, in Section 3.1, we provide the analysis for the public offers case and in Section 3.2 the analysis for the private offers case. Some of the proofs we defer to the appendix.

Theorem 1. With linear valuations and uniformly distributed costs, the following hold:

(i) With public offers, equilibria must be in pure strategies. Equilibrium prices and cutoffs are unique.

(ii) With private offers, there exists a \( \Delta^* > 0 \) such that the following holds:

1. For all \( \Delta > \Delta^* \) all equilibria are pure strategy equilibria. The equilibrium prices and cutoffs coincide with the equilibrium prices and cutoffs with public offers.

2. For all \( \Delta < \Delta^* \) no pure strategy equilibria exist.

(iii) With both information structures, there is a positive probability of trade in each period.

**Equilibrium with Public Offers**

With public offers, buyers in period \( t \) observe past prices and hence, their belief about the cutoff seller type is given by a single cutoff \( k_{t-\Delta} \). Moreover, today’s cutoff \( k_t \) affects tomorrow’s cutoff \( \kappa_{t+\Delta}(k_t) \) which results from a continuation equilibrium. As shown in the proof of Lemma 10, it follows by backward induction that \( \kappa_{t+\Delta}(k_t) \) is unique and that hence, there is a unique equilibrium which must be in pure-strategies. In the following we recapitulate the main arguments in order to show some additional properties of the equilibrium.

First, equilibrium prices \( p_t(k_t) \) must satisfy (2.2) and (2.4), and a buyer’s expected profit conditional on receiving the good in period \( t \) if he choses a price that results in cutoff \( k_t \) is given by

\[
\pi_t(k_t; k_{t-\Delta}) = \frac{k_t - k_{t-\Delta}}{1 - k_{t-\Delta}} \left[ \frac{A}{2} k_t + \frac{A}{2} k_{t-\Delta} + B - p_t(k_t) \right].
\]  

(2.6)
\[ p_{1-\Delta}(k_{1-\Delta}) \text{ is a linear function and hence, } \pi_{1-\Delta} \text{ is quadratic and continuous in } k_{1-\Delta}. \] The highest seller-type is never willing to trade before the deadline since \( v(1) = 1 \). Consequently, buyers make zero profits in this continuation game and \( \kappa_{1-\Delta}(k_{1-2\Delta}) \) is uniquely given by the largest \( k_{1-\Delta} \) such that \( \pi_{1-\Delta}(k_{1-\Delta}; k_{1-2\Delta}) = 0 \). Thus, \( p_{1-2\Delta}(k_{1-2\Delta}) = \delta p_{1-\Delta}(\kappa_{1-\Delta}(k_{1-2\Delta})) + (1 - \delta)k_{1-2\Delta} \) is linear. If \( p_t(k_t) \) is linear, then \( \pi_t(k_t; k_{t-\Delta}) \) is quadratic. Moreover, it is never profitable for buyers in period \( t \) to set a price that attracts extremely high seller types. Hence, \( \kappa_t(k_{t-\Delta}) \) is uniquely given by the largest \( k_t \) satisfying \( \pi_t(k_t; k_{t-\Delta}) = 0 \) and

\[
p_{t-\Delta}(k_{t-\Delta}) = \delta p_t(\kappa_t(k_{t-\Delta})) + (1 - \delta)k_{t-\Delta}
\]
is linear. The following lemma follows immediately.

**Lemma 11.** In a pure strategy equilibrium resulting in cutoffs \( (k^*_0, \ldots, k^*_{1-\Delta}) \) the following must be satisfied:

(i) Buyers must make zero profits, i.e.,

\[
p_t(k^*_t) = \mathbb{E}^F[v(c) | [k^*_{t-\Delta}, k^*_t]] \tag{2.7}
\]

(ii) Prices are increasing over time, i.e. \( p_t(k^*_t) < p_{t+\Delta}(k^*_{t+\Delta}) \) for all \( t \).

(i) follows immediately from the fact that \( \pi_t(\cdot; k_{t-\Delta}) \) is continuous as argued before. (ii) is on the one hand driven by the fact that by the reverse-skimming property, there are higher types remaining in the market as time goes on, so the willingness to pay of buyers increases over time. On the other hand, the seller’s reservation increases over time since she can get a surplus of \( v(c) - c \) at time 1. In particular as \( \Delta \to 0 \), one can show that

\[
k_t = 1 - e^{-rt} \quad \text{and} \quad p_t = v(k_t).
\]

Figure 2.1 shows the prices at which different seller types trade for \( v(c) = 0.5 + 0.5c, r = 0.5 \), and \( \Delta \in \{\frac{1}{4}, \frac{1}{8}\} \), as well as \( \Delta \to 0 \). \( p_t \) denotes the price and \( k_t \) denotes the equilibrium cutoff in period \( t \). Interestingly, the amounts of trade are not monotone over time.

Finally, there must be a positive probability of trade in every period in a pure-strategy equilibrium. This is in stark contrast to the public offers equilibria in HV that have no trade after the first period except for in the final trade if the opportunity of trade disappears after a deadline. The proof can be found in the appendix and it partially relies on the fact that \( v(\cdot) \) is linear and \( c \) is uniformly distributed. The main results of the paper (in particular Theorem 3) do not rely on this property.  

**Equilibria with Private Offers**

With private offers, the buyers today cannot affect tomorrow’s price. It follows immediately by backward induction that there is trade with positive probability in each period. At time

\[ 14 \text{In Section 2.5 we show that when } \alpha < 1 \text{ there can be some quiet periods at the end.} \]
1 – Δ profits (2.5) are positive for cutoffs k close to the lowest cutoff in the support of $K_{m-\Delta}^r$. This follows from $v(c) – \delta v(k) – (1 – \delta)k > 0$ for c close to k because $v(k) – k > 0$ for all $k < 1$. In earlier periods, there must be trade because buyers today can always imitate future buyers who trade with positive probability without changing the continuation game.  

In addition, a buyer’s profit $\pi_t(k_t; k_{t-\Delta}^*)$ is continuous in $k_t$ and the highest seller type never trades before the deadline. Hence, in a pure-strategy equilibrium the zero-profit condition (2.7) must be satisfied. Moreover, buyers must have correct beliefs about the past cutoff. Consequently, any pure strategy equilibrium with private offers must also be an equilibrium with public offers. 

Applying the chain rule to (2.6) yields that the net marginal benefit (NMB) of a buyer of deviating to a higher price is 

$$\frac{\partial}{\partial k_t} \pi_t(k_t; k_{t-\Delta}^*)|_{k_t=k_t^*} = \left(\frac{k_t^* - k_{t-\Delta}^*}{1 - k_{t-\Delta}^*}\right) \left[\frac{A}{2} - (1 - e^{-r\Delta})\right].$$ 

If $e^{-r\Delta} < 1 - \frac{A}{2}$, then buyers do not have an incentive to increase the price marginally because that would decrease profits. In contrast, for $e^{-r\Delta} > 1 - \frac{A}{2}$, it is profitable for a buyer to increase the price slightly attracting higher seller types. Consequently, there is no pure strategy equilibrium with private offers if $\delta = e^{-r\Delta} > \delta^* \equiv 1 - \frac{A}{2}$, i.e., 

$$\Delta < \frac{1}{r} \log \left(1 - \frac{A}{2}\right)^{-1} \equiv \Delta^*.$$

\footnote{Note that this must not hold true if there is no gap at the bottom, i.e., if $v(0) = 0$. We discuss this in Section 2.6.
The intuition for the difference in the two information structures is that, with public offers, the seller has a stronger incentive to reject high price offers than if the offer had been made privately: Suppose one of the buyers made an off-equilibrium high offer. With public offers the seller gains additional reputation of her type being high by rejecting this offer, the strength of her signal being endogenously determined by the amount of money she left on the table. Consequently, her continuation value increases upon a rejection of the higher price. Instead, with private offers, she cannot use the off-equilibrium higher offer as a signal, so her continuation value remains constant, whereby she is more tempted to take it. Figure 2.2 illustrates the inverse supply function in the first period for $v(c) = 1 + c^2$, $r = 0.8$ and $\Delta = \frac{1}{2}$, such that $\delta \approx 67032 < \delta^*$. With private offers, we calculate the inverse supply function assuming buyers believe that in the next period, buyers bid the pure strategy equilibrium price $p_\Delta(k_\Delta^*)$. With public offers, buyers need to pay higher prices in order to attract better seller types than in equilibrium.

![Figure 2.2: Inverse supply function at $t = 0$](image)

What is the role of the frequency of trade in this context? Intuitively, an increase in the frequency (i.e., a decrease in $\Delta$) has two implications: First, the seller’s value of signaling to future generations rises because the next period starts sooner (effect 2). This effect increases the gap between public and private offers. Second, the trading volume per period becomes lower, because the seller’s cost of delay is lower (effect 1). This affects the marginal benefit of deviation in both information structures equally. However, it scales down the first effect. In particular, as $\Delta \to 0$, the marginal benefit of deviating converges to zero. All in all, the absolute gap between the net marginal benefit of deviation is non-monotonic, but the sign of the difference between private and public offers is driven by the second effect.

The NMB for a buyer to deviate from a given PBE with public offers as a function of $\delta = e^{-r\Delta}$ is illustrated in panel (a) of Figure 2.3. One can see that the NMB becomes vanishingly small as $\delta \to 1$ due to effect 1, but the sign of it remains positive once it passes
\( \delta^* \) due to effect 2. Figure 2.3 (b) shows the expected profits of a buyer conditional on receiving the good in the first period as a function of the cutoff \( k_0^* \) corresponding to the buyer’s price offer \( p_0 \) for the same example. With public offers, Bertrand competition pushes the equilibrium cutoff up to \( k_0^*(\delta) \) because as long as expected profits are positive, buyers want to set higher prices in order to outbid the competitor. However, given the competitor is offering \( p_1 (k_0^*(\delta)) \) deviations to higher prices would lead to losing money and deviations to lower prices would never be accepted. For low \( \delta \) this is still the case with private offers and hence the public offers equilibrium is also an equilibrium with private offers. Instead, when \( \delta \) is sufficiently high (e.g. for \( \delta = 0.8 \)) it becomes profitable for a buyer to increase the price given the other buyer is offering \( p (k_0^*(\delta)) \). Hence, the pure strategy equilibrium with public offers collapses with private offers.

The intuition of these results can be extended to general valuations and distributions as we show in Section 2.6.

In fact, if \( \Delta > \Delta^* \) these pure strategy equilibrium prices are the unique equilibrium outcome with private offers. In other words, no mixed-strategy equilibrium exists for large enough periods. In order to see this, we first show that if \( \Delta > \Delta^* \), then buyers at most mix between countably many prices. Moreover, the inverse supply function \( p_t(\cdot) \) is convex because sellers with higher valuation benefit more from waiting longer. In particular, in a pure strategy equilibrium as illustrated in figure 2.2, \( p_t \) has kinks at future equilibrium cutoffs.

**Lemma 12.** (Convex prices) With private offers, \( p_t(k) \) is differentiable almost everywhere
and differentiable from the right everywhere. The derivative
\[
\frac{\partial}{\partial k_t} p_t(k_t) = 1 - \sum_{s=1}^{N-n-1} \delta^s \left( \prod_{u=t+\Delta}^{t+(s-1)\Delta} K_u(k_t) \right) (1 - K_{t+s\Delta}(k_t)) - \delta^{N-n} (1 - A) \left( \prod_{u=t+\Delta}^{1-\Delta} K_u(k_t) \right) > 0
\]
is nondecreasing for \( t = n\Delta \).

As a result, if \( t \) was the smallest period in which buyers use a mixed strategy, then buyers’ expected profit in period \( t \) is given by
\[
\pi_t(k_t; k_{t-\Delta}) = \frac{k_t - k_{t-\Delta}}{1} \cdot \left( \frac{A}{2} (k_t + k_{t-\Delta}) + B - p_t(k_t) \right)
\]
and it is piece-wise quadratic where by Lemma 12 the coefficient in front of the quadratic part of \( \pi_t(k_t) \) is always smaller than \( \frac{A}{2} - 1 + \delta \).\(^{16}\) Hence, buyers in period \( t \) must play a pure strategy in equilibrium. Consequently, there cannot be a mixed equilibrium if \( \Delta > \Delta^* \).

If \( \Delta < \Delta^* \), then buyers also only mix between at most countably many prices in all periods but the first period. However, multiple mixed-strategy equilibria exist even with only two periods, such that mixed equilibria with more than two periods can become very complicated to construct. In Section 3.3 we characterize all mixed-strategy equilibria for two periods, i.e., \( \Delta = \frac{1}{2} \).

The nonexistence of pure strategy equilibria with private offers might be surprising in light of Kaya and Liu 2012 who show that pure strategy equilibria exist with private offers, but they differ from the ones with public offers equilibria. In particular, independently of the discount factor, trade takes place sooner so that the Coasean force is stronger with private offers. In contrast, we show that the effect of the information structure on a dynamic lemons market is only apparent for sufficiently frequent opportunities to trade.

### 2.4 Transparency and Welfare

For the sake of tractability, we focus our welfare analysis on a situation with two trading opportunities before the deadline (i.e., \( \Delta = \frac{1}{2} \)). We believe our results likely generalize beyond this case but our inability to obtain a tractable characterization of the whole set of private offers equilibria prevents us from formally establishing such results. The main reason we think the result extends is that the economic force behind the results does not depend on the model being linear or there only being two opportunities of trade. The driver

\(^{16}\)This is the case because \( \sum_{s=1}^{N-n-1} \delta^s \left( \prod_{u=t+\Delta}^{t+(s-1)\Delta} K_u(k_t) \right) \cdot (1 - K_{t+s\Delta}(k_t)) - \delta^{1-t-\Delta} \cdot (1 - A) \cdot \left( \prod_{u=t+\Delta}^{1-\Delta} K_u(k_t) \right) \) is smaller than a convex combination of \( \delta, \delta^2, \ldots, \delta^{N-n} \).
for our result is that private offers lack the signaling value of public offers and hence sellers are always more willing to accept a private offer. Since the inefficiency arises because there is too little trade, eliminating the possibility to signal for the seller helps to generate more trade, which in turn increases efficiency.

Equilibria with two trading opportunities

Public offers

A buyer’s expected profit conditional on trading in the last period is given by

\[ \pi_\Delta(k_\Delta; k_0) = \frac{k_\Delta - k_0}{1 - k_0} \left( \frac{A}{2} (k_\Delta + k_0) + B - \frac{(1 - \delta)k_\Delta + \delta(Ak_\Delta + B)}{= p_\Delta(k_\Delta)} \right). \]

Hence, using the zero-profit condition (2.7), the equilibrium cutoff of the continuation game is given by

\[ \kappa_\Delta(k_0) = \frac{(1 - \delta)B + \frac{A}{2}k_0}{1 - \delta - \frac{A}{2} + \delta A}. \]  

(2.8)

Hence, in order to attract a cutoff-type \( k_0 \) in the first period buyers need to bid at least \( p_0 = (1 - \delta)k_0 + \delta p_\Delta(\kappa_\Delta(k_0)) \). Using the zero expected profit condition (2.7), one can solve for the unique equilibrium cutoff in the first period

\[ k_0^* = \frac{2B \cdot (2(1 - \delta)(1 - \delta) + A\delta(1 - \delta) - A(1 - \delta))}{2(1 - \delta)(1 - A)(A\delta - 2\delta + 2) + A^2}. \]  

(2.9)

and by plugging this into (2.8) we can derive the unique cutoff in the last period

\[ k_\Delta^* = \frac{2B \cdot (2(1 - \delta)(1 - \delta) + A\delta(1 - \delta))}{2(1 - \delta)(1 - A)(A\delta - 2\delta + 2) + A^2}. \]  

(2.10)

Private offers

With private offers, the unique equilibrium coincides with (2.9) and (2.10) whenever \( \delta < 1 - \frac{4}{2} \). The following proposition characterizes all equilibria for \( \delta > 1 - \frac{4}{2} \).

**Proposition 4.** In any mixed strategy equilibrium with private offers, the following two must hold:

(i) In the last period, buyers mix between exactly two prices that result in the two cutoffs given by

\[ k_\Delta = \frac{B(1 - \delta)}{A\delta - \delta + 1 - \frac{A}{2}}, \quad \bar{k}_\Delta = \frac{B(1 - \delta^2)}{A\delta^2 - \delta^2 + 1 - \frac{A}{2}}. \]
where \( k_\Delta \) is chosen with probability \( q_\Delta \equiv \frac{\frac{4}{\delta} - (1 - \delta)}{\delta(A\delta+1-\delta)} \).

(ii) In the first period, buyers mix between 0 and cutoffs that lie in \((k_\Delta, \bar{k}_\Delta)\) where 0 is chosen with positive probability.

Intuitively, there must be a non-offer at time 0 with some probability in order to make high price offers at time \( \Delta \) not too profitable. However, buyers at time 0 must also make a relatively high price offers with some probability such that high price offers are just profitable enough. Since prices at \( \Delta \) are relatively high, buyers at time 0 cannot make any positive profit with any price offer even if they do not compete with the other buyer such that non-offers can occur in equilibrium. The value of \( k_\Delta \) can be derived from \( \pi_\Delta(k_\Delta; K_0) = 0 \) and \( \pi_0(\bar{k}_\Delta; 0) = 0 \).

Even though the equilibrium strategy in the first period is not unique, all equilibrium strategies have some properties in common. In particular, the expected cutoff type is constant across equilibria.

**Proposition 5. (Constant Expected Cutoff)** The expected cutoffs in the first period are constant across all mixed-strategy equilibria with private offers and equal to

\[
\int_0^{k_\Delta} kdK_0(k) = \frac{(1 - \bar{k}_\Delta) \left( 1 - \frac{\delta}{1 + \delta} \bar{k}_\Delta \right)}{1 - \bar{k}_\Delta} \frac{1 + 2 \delta \bar{k}_\Delta}{1 + \delta \bar{k}_\Delta} - 1. \tag{2.11}
\]

Moreover, the following must hold

\[
\int_0^{k_\Delta} \frac{1}{1-k} dK_0(k) = \frac{1}{1 - \bar{k}_\Delta} \cdot \frac{(1 + \delta)(1 - \delta + A\delta) - \frac{4}{\delta}}{(1 + \delta)(1 - \delta + A\delta)}. \tag{2.12}
\]

(2.12) follows from \( \pi_\Delta(k; K_0) \leq 0 \) for all \( k \geq \bar{k}_\Delta \). Using this result and \( \pi_\Delta(\bar{k}_\Delta; K_0) = 0 \), (2.11) can be derived. Note that any PBE with private offers must additionally satisfy \( \pi_\Delta(k_\Delta; K_0) = 0 \). Next, we construct an equilibrium in which the buyers in the first period mix between exactly two cutoffs.

Figure 2.4 illustrates for \( v(c) = \frac{1+c}{2} \) the expected profit functions \( \pi_0 \) and \( \pi_\Delta \) in the equilibrium that we have constructed in Section B.2 of the appendix. It highlights how cutoffs in the first period must correspond to kinks of \( \pi_\Delta \) and cutoffs in the second period must correspond to kinks of \( \pi_0 \). Other equilibria, in which buyers in the first period mix between \( \{0\} \) and several prices in \((k_\Delta, \bar{k}_\Delta)\), can coexist.

**Welfare analysis**

Using the properties of equilibria, we next show that on average trade takes place earlier with public offers. In a second step, we show that this results in all seller types making higher expected profits with private offers. In other words, if \( \delta > \delta^* \), all equilibria with
private offers welfare-dominate equilibria with public offers because all surplus goes to the seller by the zero-profit condition.

**Theorem 2.** (i) After each period, expected trade is always greater with private offers than with public offers.

(ii) The unique equilibrium with public offers is ex-ante strictly Pareto-dominated by any equilibrium with private offers whenever \( \delta = e^{-r^*} < \delta^* \). In other words, for each seller type expected welfare is higher with private offers than with public offers.

In the first period, sellers are more inclined to accept a price offers with private offers than with public offers because they cannot signal to future buyers by rejection of a high price. Profits of buyers in the first period non-positive only if price offers in the next period are high. As a result higher types trade in the last period. This in turn implies that more seller types trade.

For the welfare result, it is sufficient to show that all seller-types are ex-ante better off with the private information structure than with the public information structure. Therefore, first note that the expected price in the last period With private offers, the expected price in the last period is given by

\[
\int_0^1 p_\Delta(k) dK_\Delta(k) = (1 - \delta + \delta A) \int_0^1 k dK_\Delta(k) + \delta B
\]

is greater than the price in the last period with public offers \( p_\Delta(k^*_\Delta) = (1 - \delta + \delta A)k^*_\Delta + \delta B \) by Theorem 2 (i). Hence, all seller types \( c > k^*_\Delta \) who trade after the first period with the public information structure, are better off with private offers. The expected price with private
CHAPTER 2. TRANSPARENCY AND DISTRESSED SALES UNDER ASYMMETRIC INFORMATION

offers in the first period is given by

\[ \int_0^1 p_0(k) dK_0(k) = (1 - \delta) \int_0^1 kdK_0(k) + \delta \int_0^1 p\Delta(k) dK\Delta(k) > p_0(k_0^*). \]

Thus, with private offers, all seller types \( c < k_0^* \) receive at least an expected profit of \( \int_0^1 p_0(k) dK_0(k) - c \) which is greater than the profit with public offers \( p_0(k_0^*) - c \). Consequently, all seller types make higher ex-ante profits with public than with private offers.

This theorem is a very clear-cut statement. No matter which equilibrium agents end up playing in the private offer environment, the equilibrium outcome will lead to higher efficiency than in the unique equilibrium of the public offer environment. Hence, at least with linear valuations and uniformly distributed types, the opaque environment strictly dominates the transparent environment. While all seller types ex ante benefit form private offers, the gains are not monotonic in the seller type. Figure 2.5 (b) illustrate the ratio of surplus with private offers and the surplus with public offers for each seller type, if with private offers buyers mix between exactly two prices in each period for \( \Delta = \frac{1}{2}, v(c) = \frac{1+c}{2}, \) and \( e^{-0.5\Delta} = \delta > \delta^* \).

![Figure 2.5: Welfare ratio (private/public) when buyers mix between exactly two prices in both periods](image)

(a) Average welfare ratio by discount rate \( r \)  
(b) Welfare ratio by seller type for \( r = 0.5 \) (i.e., \( \delta > \delta^* \))

While it is not surprising that the welfare ratio is generally non-monotonic in \( r \), it is interesting to note that in figure 2.5 (a) at \( r^* = -\log \left[ 1 - \frac{A}{2} \right] \), the welfare gain with private offers and with buyers mixing between exactly two prices at time 0 increases discontinuously even though the expected cutoffs and prices are constant across mixed-strategy equilibria. This is because total expected welfare is not constant across equilibria.\(^{17}\) As \( \delta \to 1 - \frac{A}{2} \), one

\(^{17}\)There can be another sequence of mixed-strategy equilibria for which the welfare ratio converges to one as \( r \to r^* \).
can easily show (simply by calculation) that the expected cutoff at time 0 with private offers
\[ \int_{0}^{1} kdK_0(k) \] is equal to \( k_\Delta \) which is greater than the public offer cutoff \( k^*_0 \) at time 0:
\[
\lim_{\delta \to 1 - \frac{A}{2}} k_\Delta = \lim_{\delta \to 1 - \frac{A}{2}} \mathbb{E}[k_0] > \lim_{\delta \to 1 - \frac{A}{2}} k^*_0.
\]

Moreover, with private offers, buyers do not mix in the second period in the limit, but choose
\( k_\Delta \) almost surely, and \( k_\Delta \) is equal to the public offer cutoff \( k^*_\Delta \):
\[
\lim_{\delta \to 1 - \frac{A}{2}} \mathbb{k}_2 = \lim_{\delta \to 1 - \frac{A}{2}} k^*_2 \text{ and } \lim_{\delta \to 1 - \frac{A}{2}} q = 0.
\]

Hence, in the limit, there is strictly less expected trade at time 0 with public offers while
at \( t = \Delta \) both information structures coincide. In the limit, period 1 expected profits with
private offers are zero for all cutoffs smaller than \( k_2 \), i.e. period 1 buyers are indifferent
between all prices lower than \( p_1(k_2) \). Hence, if buyers mix between exactly two prices at
time 0, then the limiting surplus as \( \delta \searrow 1 - \frac{A}{2} \) is strictly higher with private offers than with
public offers.

2.5 The Role of Distress

Distress plays an important role in our analysis. On the one hand, the amount of distress
determines how much gains from trade prevail between the seller and the buyer. On the
other hand, the surplus generated after the deadline is reached can be interpreted as another
source of distress. In the following, we discuss the implications of our model if the seller
only receives a surplus of \( \alpha(v(c) - c) \) at time 1. For \( \alpha = 0 \), this incorporates the scenario in
which there is no opportunity of trade after time 1.

Most importantly, Theorem 1 (i) and (ii) are still valid. However, for small \( \alpha \), i.e., with
more distress, trade does not necessarily occur with positive probability in each period.
The intuition is that with low \( \alpha \), there will be a lot of high types willing to sell at \( 1 - \Delta \). As
this happens, the equilibrium price must jump up as well since buyers must break even. Of
course, if prices increase drastically between \( 1 - 2\Delta \) and \( 1 - \Delta \) nobody would want to trade
at \( 1 - 2\Delta \). In this way, there can be one or more periods with no trade when the deadline
effect is severe enough. In figure 2.6 we have plotted pure strategy equilibria with public
offers using the same example as before, i.e., \( v(c) = \frac{c + 1}{2} \) for \( \Delta \in \{ \frac{1}{4}, \frac{1}{8} \} \), \( r = 0.5 \), and \( \alpha = 0.8 \).

This can also be nicely seen in the limiting case \( \Delta \to 0 \). In the limit, there is a mass of
trade at time 1 and some "quiet periods" in which no trade takes place. In particular, in
the last period it must hold that
\[
p_1 = (1 - \alpha)k_1 + \alpha v(k_1) = \mathbb{E}[v(c)[k_{1-}, k_1]]
\]
where at time 1 the mass of seller types \([k_{1-}, k_1]\) trades (where \( k_{1-} \) is the limiting cutoff as
time approaches 1 from the left). Moreover, before the quiet period, for \( A = B = 0.5 \), it
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(a) Prices as a function of seller types for $\Delta = \frac{1}{4}$

(b) Prices and cutoffs over time for $\Delta \in \{\frac{1}{4}, \frac{1}{8}\}$

Figure 2.6: Pure strategy equilibria with $r = 0.5$, $v(c) = \frac{c+1}{2}$, $\alpha = 0.8$

there must be continuous trading over time:

$$k_t = 1 - e^{-rt}.$$  

Finally, the condition that seller $k_{1\Delta}$ must be indifferent between buying just before the quiet period starts and waiting until time 1 pins down the evolution of cutoffs over time.

We can also characterize equilibria for $\Delta = \frac{1}{2}$ for the welfare analysis. The characterization of equilibria can be found in appendix B.2. Given the intuition above, it is not surprising that, with public offers, the equilibrium cutoff in the first period $k_\Delta^*$ is increasing in $\alpha$, while the equilibrium cutoff in the second period $k_2^*$ is decreasing in $\alpha$. It is interesting to note that positive trade occurs for all $\alpha \in (0, 1]$ as long as $\delta < \delta^*$. Hence, for $\delta < \delta^*$ the unique pure strategy equilibrium with public offers is also the unique equilibrium with private offers. Furthermore, if $\delta > \delta^*$, analogously Theorem 1, it follows that there are only mixed-strategy equilibria with private offers. All equilibrium cutoffs $k_\Delta, k_2, k_\Delta$ are decreasing in $\alpha$. The probability of choosing 0 in period 1 is decreasing in $\alpha$. However, buyers choose the lower cutoff $k_2$ with higher probability as $\alpha$ increases.

Hence, there are two forces in place. More distress (smaller $\alpha$) results in more trade in the last period in both information structures. With public offers, this yields to less trade in the period before. However, with private offers, this effect is smaller since the cutoffs in the first period are larger. Nevertheless, there is no trade with higher probability. The intuition for this is that high cutoffs and prices in the last period can only be sustained if there is a similarly large amount of trade in the first period with some probability. As a result, private offers become more efficient compared to public offers as illustrated in figure 2.7.
2.6 Robustness and Generalizations

General Valuations and Distributions

In this Section, we present a generalization of Theorem 1 to arbitrary differentiable valuation functions $v(c)$ and costs $c$ that are distributed according to an arbitrary differentiable cdf $F$ on $[0, 1]$. Even though in this general setup, inverse supply functions do not need to exist for all equilibria, there always exists an equilibrium in which an inverse supply function exists in every period. The main intuition is analogous to the one described in Section 2.3, but the formal analysis is more elaborate.

**Theorem 3.** (i) With public offers, there exists a pure strategy equilibrium for all $0 < \delta < 1$. (ii) Equilibrium cutoffs (and prices) in any pure strategy equilibrium with private offers correspond to equilibrium cutoffs (and prices) in a pure strategy equilibrium with public offers. (iii) There exists a $\Delta^* < 1$ such that if $\Delta < \Delta^*$ there is no pure strategy equilibrium with private offers.

The existence of a pure strategy equilibrium with public offers can be shown by construction. We show that, if all buyers choose pricing strategies that result in a cutoff seller $\kappa_t^*(k_{t-\Delta})$ (defined below) given they believe the current cutoff is $k_{t-\Delta}$, this constitutes an equilibrium. To this end, define $\kappa_t^*(\cdot)$ inductively for $t = 1, \ldots, T - 1$ as follows. First, using $p_{1-\Delta}(k) = \delta v(k) + (1 - \delta)k$, it follows that

$$\kappa_{1-\Delta}(k_{1-2\Delta}) = \sup \left \{ k \in [k_{1-2\Delta}, 1] \left | \frac{1}{1 - F(k_{1-2\Delta})} \int_{k_{1-2\Delta}}^{k} (v(c) - p_{1-\Delta}(k)) f(c) dc > 0 \right \} \right.$$  \hspace{1cm} (2.13)

is left-continuous and $\sup \theta = k_{1-2\Delta}$. Then,

$$p_{1-2\Delta}(k) = \delta p_{1-\Delta}(c_{1-\Delta}(k_{1-2\Delta})) + (1 - \delta)k$$
is left-continuous. In the appendix we show that given left-continuous $\kappa_{t+\Delta}(k)$, it follows that for $t < T - \Delta$ and $p_t(k) = \delta p_{t+\Delta}(\kappa_{t+\Delta}(k)) + (1 - \delta)k$,

$$
\kappa_t(k_{t-\Delta}) = \sup\left\{ k \in [k_{T-2\Delta}, 1] \bigg| \frac{1}{1 - F(k_{T-2\Delta})} \int_{k_{T-2\Delta}}^{k} (v(c) - p_t(k)) f(c) dc > 0 \right\} \quad (2.14)
$$

(with $\sup \emptyset = k_{t-\Delta}$) is left-continuous. Hence, buyers do not make negative expected profits because

$$
k \mapsto \frac{1}{1 - F(k_{t-\Delta})} \int_{k_{t-\Delta}}^{k} (v(c) - (\delta p_{t+1}(\kappa_{t+\Delta}(k)) + (1 - \delta)k)) f(c) dc
$$

is left-continuous. The equilibrium cutoffs $(k_0^*, \ldots, k_{1-\Delta}^*)$ are then, given by $k_0^* = \kappa_0(0)$, $\ldots$, $k_{1-\Delta}^* = \kappa_{1-\Delta}(\kappa_{1-2\Delta}(\ldots \kappa_0(0)))$. None of the buyers has an incentive to deviate form this equilibrium, since by increasing the price offer, buyers will either make zero or negative expected profits by definition of $\kappa_t(\cdot)$ and by decreasing the price they will not receive the good and make zero expected profits. Note that there are generally multiple equilibria because there can be several prices that result in zero expected profits for the buyers.

With private offers, an inverse supply function must always exist. Moreover, there must be trade with positive probability in each period because buyers can always mimic the strategies of tomorrow’s buyers as we have already discussed in Section 2.3. Hence, in any pure strategy equilibrium with cutoffs $(k_0^*, \ldots, k_{T-\Delta}^*)$, the zero-profit condition

$$
\mathbb{E}^F[v(c) | [k_{t-\Delta}^*, k_t^*]] = p_t(k_t^*) \quad (2.15)
$$

with $\mathbb{E}[v(c) | [k_{t-\Delta}, k_t]] = \frac{\int_{k_{t-\Delta}}^{k_t^*} v(c) f(c) dc}{F(k_t^*) - F(k_{t-\Delta}^*)}$ must be satisfied for all $t$. Moreover, a buyer’s expected profit conditional on buying the good is given by

$$
\pi_t(k; k_{t-\Delta}^*) = \frac{F(k) - F(k_{t-\Delta}^*)}{1 - F(k_{t-\Delta}^*)} \left[ \mathbb{E}^F[v(c) | [k_{t-\Delta}^*, k_t]] - p_t(k) \right]. \quad (2.16)
$$

and hence, the marginal profit of buyers at time $t < 1 - \Delta$ is

$$
\frac{\partial}{\partial k_t} \pi_t(k_t; k_{t-\Delta}^*) \bigg|_{k_t = k_t^*} = \frac{f(k_t^*)}{1 - F(k_{t-\Delta}^*)} (v(k_t^*) - p_t(k_t^*)) - \frac{\partial p_t}{\partial k_t} (k_t^*) \frac{F(k_t^*) - F(k_{t-\Delta}^*)}{1 - F(k_{t-\Delta}^*)}. \quad (2.17)
$$

With public offers, given the same pricing strategies of the buyers on equilibrium path, buyers also make zero profits because on equilibrium path beliefs with private offers are correct and thus, correspond to beliefs with public offers. If buyers deviate from the private offer equilibrium price, their profits are of the same form as in (2.16), but with a different
CHAPTER 2. TRANSPARENCY AND DISTRESSED SALES UNDER ASYMMETRIC INFORMATION

inverse supply function $p_t(k)$. In particular, for all $k > k_t^*$, the price with public offers is greater than the price with public offers:

$$\left(1 - \delta\right)k + p_{t+\Delta}(k_t^*) > \left(1 - \delta\right)k + p_{t+\Delta}(k^*_t),$$

because $p_{t+\Delta}$ and $\kappa_{t+\Delta}(\cdot)$ is increasing as we show in the appendix in the proof of Lemma 10. Hence, any pure strategy equilibrium with private offers must be an equilibrium with public offers because deviations to higher prices are even less profitable.

Finally, pure strategy equilibria cease existing for large $\delta$ (or small $\Delta$) since (2.17) becomes positive. The formal proof is shown in the appendix. The main complications in the general setup compared to the linear setup discussed in Theorem 1 is that there can be possibly many pure strategy equilibria with public offers and it is not straightforward that there is a uniform bound on $\Delta$.

**Gap at the top $v(1) > 1$**

Throughout the paper, we have assumed that $v(1) = 1$. This assumption together with continuity and monotonicity of $v(c)$ guarantees that in any equilibrium, a positive mass of high type sellers do not trade before the deadline. The reason is that the expected value of buyers is always smaller than 1, so that the highest type $c = 1$ never trades before information is revealed. Hence, we did not have to worry about off-equilibrium beliefs of buyers if they observe see a seller rejecting even though on equilibrium path all sellers should have traded. The freedom in choice of off-equilibrium beliefs can lead to additional multiplicities of equilibria.

Nevertheless, Theorems 1, 2, and 3 can easily be generalized to settings with $v(1) > 1$ if we assume that the lemons problem is severe enough such that trade does not end before the deadline or if we make some out-of-equilibrium belief assumptions. For example, if buyers’ beliefs remain unchanged (or become more pessimistic) after the last period of trade, the game can still be solved by backward induction and the same arguments can be applied as in the proofs of the theorems. For example figure 2.5 allows for $A + B > 1$.

**No Gap at the bottom ($v(0) = 0$)**

We made the assumption that $v(0) > 0$ to make sure that there is always some trade before time 1. If we have no gap at the bottom then it is possible for trade to completely unravel in all periods before information is revealed if the lemons condition is satisfied. For $v(c) = Ac + B$, the lemons condition is satisfied iff $\frac{A+B}{2} < 1$.

**Proposition 6.** If $B = 0$ and $A < 2$, there always exists a pure-strategy equilibrium with private and public offers. In that equilibrium, there is no trade before time 1.  

\[18\] This proposition can easily be generalized to convex $v(c)$ with $v(0) = 0$. 

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\[18\] This proposition can easily be generalized to convex $v(c)$ with $v(0) = 0$. 

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Two types

Several of the recent papers that look at dynamic adverse selection consider only two possible types (e.g., Daley and Green 2012, Camargo and Lester 2011, Swinkels 1999, Nöldeke and Van Damme 1990) or use the two type case as examples (e.g., HV and Deneckere and Liang 2006). Hence it is interesting to explore what happens if we restricted attention to two types. In order to be closer to the HV setup we assume there is no opportunity of trade after the last period (i.e., $\alpha = 0$).

Consider a situation with $v_H \geq c_H$ and $v_L > c_L = 0$ where the seller’s valuation is $v_L$ with probability $\phi$. Let $\phi$ satisfy the static lemon’s condition (LC)

$$\phi v_L + (1 - \phi)v_H < c_H.$$  

Then, we show that equilibria in both information structures coincide.

**Proposition 7.** With two seller types and two opportunities to trade, equilibria with private and public offers coincide.

This points out that in a dynamic setup it can be important to have a rich enough type space. The differences in results are driven by the inability of sellers to have a rich signal space because Bertrand competition only allows for two prices that make one of the two seller types indifferent between buying and waiting. The formal analysis can be found in appendix B.3.

2.7 Conclusion

We have highlighted in this paper that there are welfare losses from transparency in a setting with both intra and inter period competition. Thus, it might be worth considering if it might be possible to obtain some of the other benefits from price transparency while limiting the negative effects we have highlighted in the paper. When thinking about policy and how to best resolve these tradeoffs the details of the market structure is likely to play an import role. As we have shown the equilibrium dynamics with inter and intra period competition can be very different from other market settings such as HV who allow only for inter period competition. Recall that they find that, with public offers, there is only trade in the first period. In a bilateral bargaining relationship Deneckere and Liang 2006 show that in the limit when offers are very frequent there are quiet periods intersected by bursts of trade. Somewhat similar dynamics are also present in Daley and Green 2012 which allow for the arrival of information as the game unfolds and show trade dries up for intermediate beliefs. Thus, although we believe this paper has helped us understand better these markets there are still many open questions left for future work.
Chapter 3

Mentoring and the Dynamics of Affirmative Action

3.1 Introduction

For decades, affirmative action has remained a topic of heated debate and wildly opposing opinions. While some view it as an outdated form of discrimination against historically favored social classes, others see in it the only way of confronting the reality that race and gender still matter for educational achievement and employment outcomes (see e.g. Schuette v. Coalition to Defend Affirmative Action. 572 U.S. 2014). Part of the political discourse focuses on arguments of justice and righting historical wrongs, topics which cannot be addressed through economic research. However, the issue of productivity is at least as relevant for policy decisions.

There is extensive empirical evidence that race and gender affect an individual’s career prospects irrespective of his or her innate ability Ellison and Swanson 2009; Milkman, Akinola, and Chugh 2014. Differences in hiring rates may arise through preference bias, where decision makers innately favor members of a certain groups, or it may come through statistical discrimination, where rational utility-maximizers infer imperfectly observable information on productivity from correlated, but utility-irrelevant characteristics such as race or gender Phelps 1972, Arrow 1973, Coate and Loury 1993, Moro and Norman 2004, Fang and Moro 2011; Mailath, Samuelson, and Shaked 2000. In particular, certain groups may be stuck in an equilibrium with little skill investment and poor employment prospects, while more fortunate types owe their high returns to education to favorable equilibrium beliefs.

We here focus on a third channel that comes from firms’ inability to harness the full potential of all its workers, resulting in ex-post productivity differences across types. Mentorship by senior professionals for instance is an important determinant of productivity, yet several papers show that mentoring relationships are stronger and more common between members of the same demographic group Ibarra 1992, Dreher and Cox Jr. 1996. In addition to direct mentor interaction, the availability of similar role models also affects the
performance of individuals Carrell, Page, and West 2010. In such a world, an uneven composition of the senior workforce may result in tangible productivity differences among junior employees that affect hiring decisions.

This paper considers a model that incorporates the trade-off between the benefits from mentoring complementarities that arise with homogeneous work forces and the optimal use of pre-mentoring abilities of young workers. For illustrative purposes, we adopt a gender-based model with a minority of female employees, but the insights can be reinterpreted in the context of race or other demographic characteristics. Specifically, we consider an overlapping generations model where the old generation mentors the young cohort by providing them with a productivity boost that adds to their (identically distributed) innate ability. This mentoring boost differs across genders and is increasing in the share of mentors of one’s own type. This may be because mentoring relationships are commonly formed between people of similar demographics and/or because mentors who are exposed to a diverse environment are more adept at crossing cultural barriers.

As such, the basic intuition behind our employment imbalance is as follows: If the senior workforce is predominantly male, then young male employees can be expected to receive a higher mentoring boost than their female colleagues and thereby exhibit higher ex-post productivity. A firm that seeks to maximize its productive output will thus rationally hire more men, including some that have lower ex-ante ability than the threshold required from minority candidates. We assume that a single firm is either too small or myopic in order to internalize the long-run mentorship dynamics, and thus cares only about short-term productivity without regard to the fact that today’s young hires will constitute tomorrow’s mentors.

The properties of the mentoring boost function determine the dynamics of this system, including steady state properties and per-generation productivity. We show that both a completely homogeneous and a balanced work force can be stable steady states of the economy. A homogeneous work force makes maximal use of the mentoring complementarities, while a balanced base of mentors optimally promotes the innate ability of all workers and increases the total labor force by boosting more workers above the industry productivity standards.

The goal of the paper is to develop conditions under which regulations can be used to increase total productivity of the economy and to compare the effectiveness of different policies. In particular, we are interested in the following questions: Under which conditions should we regulate at all? How radical should quotas or subsidies be? Is it better to have a very radical policy initially and then terminate the regulation? Who are the losers and winners of different policies?

Our model highlights that affirmative action is only useful if the following conditions are satisfied: First, the marginal benefit of more similar mentors should be significantly larger for minorities than for majorities. For example, an increase in minority mentors should have a relatively strong impact on the productivity of female students, but a negligible impact on male students. Furthermore, unless a policy reaches a critical fraction of minority mentors,
the economy will eventually fall back to the undesirable steady state. This might explain why different empirical studies can reach opposing conclusions depending on whether a certain affirmative action policy has been “strong enough” to push the economy above this threshold.

Moreover, we show that affirmative action policies can increase the productivity through two channels: On the one hand, the average productivity of workers becomes higher when firms optimally harness the potential of both genders. On the other hand, more minority workers enter the labor force without necessarily crowding out majority workers. This latter effect is particularly large if the policy maker focuses on subsidies for minority employment, in industries with a general under-supply of workers, as is for example in STEM fields as pointed out for example in Arcidiacono, Aucejo, and Hotz 2013. In such fields, subsidies for minority employees do not harm the prospects of majority workers and can thus be a powerful tool to meet the demand for qualified workers in the long-run.

We also compare different policies such as quota, subsidies, and laws against differential treatment of ex-ante equal workers based on their gender or race. We show that quotas are the only tool that maximize productivity, but that quotas will necessarily lead to crowding-out of majority workers. This can be an issue when selling the policy to an electorate. Instead, subsidies do not crowd out majority workers in the short term, and when the mentoring boost for the majority remains constant, their job prospects are unharmed even in the long term.

Finally, we show that the benefit of a policy is very sensitive to the duration of time that they are in place. In particular, we show that if the policy maker is sufficiently forward-looking and the hiring cost per worker is not too high (or, equivalently, ex-ante worker ability is not too low), then the optimal policy is in place for at most one period, which is the life span of a mentoring generation. We also show some simulations using a continuous time version of the model to highlight the trade-offs between stricter quotas versus longer timespan and evaluate effects of different policies.

**Related Literature.** Our analysis provides a rationale for effective temporary affirmative action, which is in contrast to some previous theoretical research. Our opposing predictions stem from differing assumptions on the source of the hiring imbalance. Indeed, when taste-based discrimination is at work, affirmative action may well be a zero-sum game where the benefit to the minority is offset by a direct utility loss of the majority. Under statistical discrimination, larger minority participation need not translate into updated beliefs. Quite to the contrary: Under certain parameter values, employment quotas may actually reinforce negative stereotypes against certain groups Coate and Loury 1993. The intuition is simple: When minority employment is mandated by law, firms may have to hire minority members even if they are unskilled. This in turn may actually reduce their returns to education and thereby further lower equilibrium skill investment. A similar conclusion is reached when agents infer their personal success probability from their own type’s employment history as in Chung 2000. In an unregulated market, observing successful people with similar
characteristics sheds a positive light on one's own prospects in the labor market and hence encourages investment. However, these positive inferences disappear under temporary hiring restrictions, and agents will not be any more optimistic once the employment constraints are lifted. It is important to emphasize that these arguments rely purely on informational inferences and assume no direct productivity benefit from relatable role models.

Together, these models seem to suggest that affirmative action is futile at best, or downright harmful at worst. If individuals however receive a direct productivity boost from mentoring, we show that a more positive view is warranted. The group complementarities at the heart of our model have been observed in the empirical literature. Dreher and Cox Jr. 1996 find that not only were female MBA graduates and students of color less likely to form mentoring partnerships with white men, but these missing relationships also had a tangible impact on later compensation. Indeed, students mentored by white men earned on average $16,840 more annually than those with mentors of other demographic profiles. In a similar vein, Ibarra 1992 analyzes the professional network within an advertising firm and finds that differential patterns of network connectivity helped men reap greater network returns than a woman in the same position. Bettinger and Long 2005 find that an increased share of female faculty positively influences course selection and major choice for female undergraduates in some (though not all) disciplines where women are historically underrepresented. Observations by Carrell, Page, and West 2010 also seem to support the hypothesis of a measurable productivity boost from relatable role models. They show that assigning a female professor to mandatory introductory science and math classes significantly increases course grades for female students without hurting male performance, to the point of eradicating the gender gap in grades and STEM majors.

The structure of our model is very similar to that of Athey, Avery, and Zemsky 2000, as they study the intertemporal promotion decision of long-lived firms. As in our model, they assume that senior workers offer an additive mentoring boost to junior employees, and that the size of this boost is an increasing function in the share of senior mentors of one's own type. Since their firms are forward looking, they essentially share the same objective function as the social planner in our case with fixed labor force (see section 3.5). We believe that our two papers are complementary: They offer additional theoretical insight into the family of boost functions that admit certain steady states, while we mainly restrict our attention to a specific mentoring function and contrast ways in which a social planner can guide myopic firms to a more productive equilibrium. In particular, we show that the optimal policy requires only temporary “corrective bias” in the form of affirmative action. By studying also the case of an unsaturated labor market (which seems particularly relevant for high-skill sectors), we show how quotas affect both the intensive and the extensive margin of productivity.
3.2 Model

In each period students from two ex ante identical groups of mass 1, labeled as female and male, arrive. Their ex ante ability is distributed according to a cdf $F$ on a subset of $\mathbb{R}$. Students receive a mentoring boost that is an increasing and weakly concave function $m$ of the fraction of mentors from the same group. We normalize $m(0) = 0$ without loss of generality. Denote the fraction of male mentors by $\phi$. To simplify notation, we will use $m_i(\phi)$ to denote the mentoring boost for type $i \in \{\phi, \mu\}$ given the fraction of females is $\phi$, where $m_\phi(\phi) \equiv m(\phi)$ and $m_\mu(\phi) \equiv m(1 - \phi)$. Then, the productivity of a worker of type $i$ with ex ante ability $a$ who was born in period $t$ is

$$a + m_i(\phi_t)$$

if $\phi_t$ is the fraction of female mentors in period $t$. Each worker only works for one period and acts as a mentor in that period.

We assume that the labor market has unlimited positions, but each firm incurs a fixed cost $c$ of hiring a worker. Each firm captures all the surplus. Firms are myopic, their best strategy is to hire anybody with positive ex-post productivity, such that wages are normalized to zero. Then, a worker of type $i$ with ex ante productivity $a_{t-1}$ finds a job in period $t$ if and only if

$$a_{t-1} + m_i(\phi_{t-1}) \geq c.$$

Hence, in equilibrium, the labor force in a period of state $\phi$ is given by

$$L(\phi) = 1 - F(c - m(\phi)) + 1 - F(c - m(1 - \phi)) \equiv \kappa_{\phi}(\phi) + \kappa_{\mu}(\phi)$$

where $\kappa_i$ denotes the ex-ante ability required by workers of group $i$. Depending on the role model composition, this value may differ between women ($i = \phi$) and men ($i = \mu$).

Then, if the fraction of mentors today is $\phi$, the fraction of female mentors in the next period is given by

$$\phi^+(\phi) = \frac{1 - F(\kappa_{\phi}(\phi))}{L(\phi)}.$$

The transition function $\phi^+$ not only gives us the limiting behavior of the economy, but also sheds light on the dynamics of the system. As such, we call a steady state $\bar{\phi}$ stable whenever a small perturbation in the number of role models does not affect the long-term convergence. Stability is useful, because in practice there is only hope to sustain a desirable steady state without having to regulate the market forever if the steady state is stable.

\footnote{We discuss the situation in which the number of jobs is fixed in section 3.5.}
Definition 1. (i) A state $\phi$ is a steady state, if the gender ratio remains constant once $\phi$ is reached, i.e., if $\phi = \phi^+(\phi)$. 
(ii) A steady state $\phi$ is stable if there exists $\epsilon > 0$ such that $\lim_{t \to \infty} (\phi^+(\phi) - \overline{\phi}) = 0$ for all $\phi \in (1 - \epsilon, 1 + \epsilon) \cap [0, 1]$.

3.3 Dynamics and Welfare

Steady States

There are a few general statements that we can make about the dynamics of such an economy. Due to symmetry, it is immediate that $\phi = \frac{1}{2}$ is always a steady state. However, it does not have to be stable and there can exist other extreme steady states. We show that if the mentoring boost is not very sensitive to changes in $\phi$ around $\frac{1}{2}$, then $\frac{1}{2}$ is a stable steady state and quotas can be a useful tool to sustain this steady state. Finally, if the most able students are not productive than $c$ without a mentoring boost, $\phi \in \{0, 1\}$ are steady states. This is summarized in the following lemma.

Proposition 8. (i) $\phi = \frac{1}{2}$ is always a steady state.

(ii) If $m$ is differentiable at $\frac{1}{2}$, $\phi = \frac{1}{2}$ is stable if and only if

$$m'\left(\frac{1}{2}\right) < 2 \cdot \frac{1 - F\left(c - m\left(\frac{1}{2}\right)\right)}{f(c - m(\frac{1}{2})).}$$

(iii) If $F$ has a finite support $[a, \overline{a}]$, then $\phi \in \{0, 1\}$ are steady states if and only if $\overline{a} < c$.

Intuitively, since $m$ is assumed to be concave, (3.1) is satisfied whenever the mentoring boost is not too sensitive to changes in the number of similar mentors for $\phi > \frac{1}{2}$, but more sensitive for small levels of $\phi$. This was for example observed in Carrell, Page, and West 2010: The gender of teachers did not affect males students, but had a “powerful effect on female students’ performance in math and science classes”. For this reason we use the following example in most of our analysis in this paper. The formal proof of the proposition is deferred the appendix.

Example. If $m'\left(\frac{1}{2}\right) = 0$, then (3.1) is trivially satisfied. Hence, to make things as simple as possible, we focus our analysis in most of the paper on the mentoring function

$$m(\phi) = \begin{cases} \frac{M}{s} \phi & \text{if } \phi \leq s \\ \frac{M}{\overline{s}} & \text{if } \phi \geq s \end{cases}.$$  

$m$ is linear for low values and achieves its maximum for some threshold $0 < s \leq 0.5$, as depicted in (3.1). $s$ can be interpreted as the critical fraction of similar mentors necessary in order to benefit from the full mentoring boost. $M$ is the maximal mentoring boost. ♦
Figure 3.1: Example for mentoring function.

(3.1) is also more easily satisfied if the workforce is large if the mentor gender is balanced, i.e., $\phi = \frac{1}{2}$. If the number of jobs was fixed to mass 1, the right hand side of (3.1) would be equal to $\frac{1}{2}$ as we show in section 3.5.

Finally, the assumption in Proposition 8 (iii) is useful for technical reasons, since it simplifies the structure of the problem. For this reason we work in most of this paper with uniformly distributed abilities on $[0, 1]$. With full support, e.g., if $F$ is normally distributed, extreme steady states close to $\phi \in \{0, 1\}$ still exist under some assumptions, but they are never equal to 0, 1. We believe that the qualitative results of our paper do not change if $F$ has full support, but small tails (as we can show for example for the normal distribution).

All in all, affirmative action policies in favor of minorities only make sense only if the following two conditions are satisfied. First, there must be more than one stable steady states. Second, the heterogeneous steady state must be welfare dominating the other steady state. Moreover, if there are two stable steady state, there must exist an intermediate steady state in between. It is crucial that the policy maker pushes the economy past this critical point in order to prevent the economy bouncing back to the undesirable steady state, once the policy has been abolished.

Example. Let the ex ante ability be distributed according to an uniform distribution for both genders, $a \sim U[0, 1]$, and that the mentoring boost function $m$ be given by (3.2). We assume $M \geq c - 1$ to avoid the trivial situation where even the most able individuals under perfect mentoring are unattractive to employers. We also assume $c > 1$. In the illustrating figures, we chose $M \in \{1, 2\}$, $s = 0.3$ and $c = 1.4$. The two values for $M$ contrast situations where full employment is not possible ($M < c$) against those where it is ($M \geq c$).

The dynamics in this case are simple to solve. The number of female mentors is $l_\phi = \min\{1, M - c + 1\}$ as long as $\phi \geq s$. Since $c \geq 1$, there exists a minimal mentoring level that is necessary for the employment even of the most able women, implying that $l_\phi = 0$. 
whenever

$$\phi \leq \hat{\phi} = \frac{s}{M} (c - 1).$$

For intermediate values, mentoring leads to a linear shift in women’s ex-post productivity. Under the uniform distribution, this linearity is preserved in the female labor force participation $l_\phi = \min\{1, \alpha(\phi - \hat{\phi})\}$, where $\alpha = \frac{M-c+1}{s(1-\frac{c}{M})}$. There are $l_\mu$ male mentors in every period for $\phi \leq 1 - s$. Everything else follows by symmetry. For the transition function, this implies

$$\phi^+(\phi) = \begin{cases} 
0 & \text{if } \phi \leq \hat{\phi} := \frac{s}{M} (c - 1) \\
1 - \frac{s - \hat{\phi}}{\phi + s - 2\hat{\phi}} & \text{if } \hat{\phi} \leq \phi \leq s' \\
\frac{1}{2} & \text{if } s' \leq \phi \leq 1 - s' \\
\frac{s - \hat{\phi}}{1 - \phi + s - 2\phi} & \text{if } 1 - s' \leq \phi \leq 1 - \hat{\phi} \\
1 & \text{if } 1 - \hat{\phi} \leq \phi,
\end{cases}$$

where $s' = \min\{s, \frac{1}{\alpha} + \hat{\phi}\}$ is the lowest state that reaches maximal female employment. The transition function admits three stable $\{0, 0.5, 1\}$ and two unstable steady states $\{\tilde{\phi}, 1 - \tilde{\phi}\}$ with $\tilde{\phi} \in [\hat{\phi}, s]$ as depicted in figure 3.2.

2As long as full employment is at most barely feasible ($M \leq c$), we obtain $s' = s$. 
Welfare

In order to determine the desirability of steady states, it is useful to compare the total productivity in different steady states. We use total productivity and welfare interchangeably, but we discuss losers and winners given different policies in section 3.4. In a period with fraction $\phi$ of female mentors, welfare is given by

$$W(\phi) = \int_{\kappa(\phi)}^{1} (x + m(\phi) - c)f(x)dx + \int_{\kappa(\phi)}^{1} (x + m(1 - \phi) - c)f(x)dx.$$ 

**Proposition 9.** Let $m$ be twice differentiable at $\phi = \frac{1}{2}$. Then the following hold:

(i) $\phi = \frac{1}{2}$ is always a local maximum or minimum of $W(\phi)$.

(ii) $\phi = \frac{1}{2}$ is a local maximum if and only if

$$m'(\frac{1}{2}) < \sqrt{-m''(\frac{1}{2}) \cdot \frac{1 - F(c - m(\frac{1}{2}))}{f(c - m(\frac{1}{2}))}}$$

or $m'(\frac{1}{2}) = 0$.

(iii) Whenever

$$-m''(\frac{1}{2}) < \frac{4(1 - F(c - m(\frac{1}{2})))}{f(c - m(\frac{1}{2}))}, \tag{3.3}$$

then $\phi = \frac{1}{2}$ is stable only if it is locally maximizing welfare. If 3.3 holds with equality, $\phi = \frac{1}{2}$ is stable if and only if it is locally welfare maximizing. Otherwise, $\phi = \frac{1}{2}$ is locally maximizing welfare only if it is stable.

Intuitively, (i) and (ii) suggest a link between stability and welfare of the mixed steady state $\frac{1}{2}$. Essentially, the mentoring boost function should not be too steep at $\frac{1}{2}$, if it is the welfare optimal outcome. Nevertheless, the link is not clear-cut. We can show in (iii), however, that whenever (3.3) is satisfied, stability of $\phi = \frac{1}{2}$ implies that it is at least locally welfare optimal. In other words, policies geared towards the heterogeneous steady state cannot be too harmful, since the mixed steady state can only be sustained if it also maximizing total productivity. (3.3) is satisfied whenever, the mentoring boost function is not “too concave” at $\frac{1}{2}$. The mentoring boost function 3.2 that we use throughout the paper satisfies the condition with equality, so that the mixed outcome is always welfare optimal.

**Example.** Using the same example as before it is immediate that welfare is maximized for generations with $\phi \in [s, 1 - s]$. Figure 3.3 shows the welfare generated by a generation with a fraction $\phi$ of female role models. ♦

Similarly to (3.1), (3.3) is more easily satisfied if the labor force is large. Consequently, there are two driving forces that make a heterogeneous work force more attractive. On
the one hand, in the heterogeneous steady state, ex ante abilities of workers are used more optimally since the best students from both groups work. On the other hand, an increase in the fraction of minority mentors boosts the labor force for the minority significantly without decreasing the male labor force much if the mentoring boost function is flat for $\phi > \frac{1}{2}$. These two effects can be illustrated using our leading example.

**Example.** Productivity conditional on employment is distributed according to $U[0, \ell_i]$ with mean $\frac{1}{2}\ell_i$ and productivity conditional on employment is distributed according to $U[0, \ell_i]$ with mean $\frac{1}{2}\ell_i$, as long as we have non-complete employment of type $i \in \{\phi, \mu\}$. When everybody of a given type works, this conditional productivity is distributed according to $U[\mu_i(\phi) - c, \mu_i(\phi) - c + 1]$ with mean $\mu_i(\phi) - c + 0.5$. Figure 3.4 depicts mean productivity of employed female workers both when full employment can and cannot be reached. One can see that the average productivity of workers is maximized for $\phi \geq s$. Figure 3.5 depicts female labor force participation graphically. It is also maximized for intermediate $\phi \geq s$. These are the driving forces for welfare being greatest for generations with $\phi \in [s, 1 - s]$.
3.4 Affirmative Action Policies

In order to optimize long-term welfare, the social planner may set restrictions on the firm’s hiring policies. This section considers and contrasts several different approaches. Throughout the section we assume that \( F \) is the uniform distribution on \([0, 1]\) and that the mentoring boost is of the form (3.2) as in the examples discussed above. We additionally assume that the least able people are unproductive even under full monitoring \( (m(1) \leq c) \).

Fractional hiring quotas

The most straightforward implementation of a gender quota is to fix a target fraction \( \phi \) of the female workforce. This leaves firms to choose the overall size of the labor force and can be implemented in a decentralized economy.

Suppose that the current female representation is given by \( \phi < 0.5 \) (the other case follows by symmetry). Assume that the government sets a target level \( \overline{\phi} \in [0, 0.5] \), representing the minimal fraction of positions that each firm has to assign to female employees. Low quotas with \( \overline{\phi} \leq \phi^+(\phi) \) are without impact as even an unregulated market would move to the desired representation levels. Hence, we focus on the case \( \overline{\phi} > \phi^+(\phi) \). We show that the total labor force is increasing if a quota is set. However, there are always a few male workers whose post-mentoring productivity is higher than \( c \), but who do not find employment due to the quota. The increase in female labor force offsets this effect. Hence, firms optimally respond to the quota both by hiring some women with negative productivity and foregoing some men with positive productivity.

**Proposition 10.** Let the current state be \( \phi \). If the social planner sets a quota of \( \phi^+(\phi) < \)
\( \overline{\phi} < \frac{1}{2} \), the equilibrium labor force is given by
\[
\ell^* (\overline{\phi} \mid \phi) = \min \left\{ \frac{1}{1 - \overline{\phi}}, \max \left\{ 0, \frac{1 - c + \overline{\phi} m(\phi) + (1 - \overline{\phi}) m(1 - \phi)}{1 - 2 \overline{\phi} (1 - \overline{\phi})} \right\} > \ell^* (\phi^+(\phi) \mid \phi).
\]

Hence, the quota increases the overall labor force relative to the unregulated market. The male equilibrium labor force is given by
\[
\ell^*_\mu (\overline{\phi} \mid \phi) = (1 - \phi) \ell^* (\overline{\phi} \mid \phi) < \ell^*_\mu = (1 - \phi^+(\phi)) \ell^*(\phi^+(\phi) \mid \phi)
\]
while the female equilibrium labor force is given by
\[
\ell^*_\phi (\overline{\phi} \mid \phi) = \overline{\phi} \ell^* (\overline{\phi} \mid \phi) > \ell^*_\phi = \phi^+(\phi) \ell^*(\phi^+(\phi) \mid \phi).
\]

As a result, setting a quota reduces the male and increases the female labor force relative to the unregulated market.

The equilibrium labor force can be determined by maximizing the total productivity in the economy given the current state is \( \phi \), but under the constraint that the fraction of female workers must be \( \overline{\phi} \):
\[
\begin{align*}
\max_{\ell} & \quad \overline{\phi} \ell \left[ 1 - \frac{\ell}{2} - c + m(\phi) \right] + (1 - \overline{\phi}) \ell \left[ 1 - \frac{(1 - \overline{\phi}) \ell}{2} - c + m(1 - \phi) \right] \\
\text{s.t.} & \quad 0 \leq \ell \cdot (1 - \overline{\phi}) \leq 1
\end{align*}
\]

The equilibrium labor force must maximize total productivity since no firm can benefit from hiring additional workers if the best \( \ell^*\overline{\phi} \) and the best \( \ell^*(1 - \overline{\phi}) \) are employed, if \( \ell^* \) solves (3.4). Intuitively, some of the male labor force must be sacrificed because the cost of firing the worst men with post-mentoring productivity 0 is smaller than the cost of hiring a woman with strictly negative productivity. The formal proof can be found in the appendix.

Finally, it is worth noting that the optimal labor size is not monotone in the quota \( \overline{\phi} \) or in the current representation \( \phi \). In other words, total labor force is not maximized for extremely strict quota close to \( \frac{1}{2} \).

In order to determine the optimal quota, we need to analyze the entire dynamics of the problem. We show some simulations in section 3.4 in a continuous time version of the model.

**Fixed hiring targets**

A more invasive approach consists in dictating gender-specific, minimal ex-ante ability cutoffs \( \kappa_i \) (or, equivalently, ex-post productivity thresholds \( \kappa_i - c + m(\frac{1}{1 - \kappa_i + 1 - \kappa_\mu}) \)) above which all agents need to be hired. Practically, this would correspond to a situation where firm’s only choice is to participate in the market or not, but the hiring policy of all participating firms is fully dictated by the social planner. In other words, here the government chooses both a target fraction \( \overline{\phi} \) of minority employees and explicitly sets the labor force to \( (1 - \kappa_\phi) + (1 - \kappa_\mu) \).
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This example is included not primarily for realism, but rather as a benchmark scenario of what would be possible if the social planner had more instruments at hand. Indeed, as it turns out, the decentralized choice of labor size is equal to the first best, since mentoring is driven only by the relative representation of the two genders, and not the absolute size of the labor force. As a result, both the (long-lived) government and the (myopic) firms will agree on the one-period productivity-maximizing labor size $\ell^*$. 

**Proposition 11.** Let $W(\kappa_\phi, \kappa_\mu | \phi) = \sum_{i \in \{\phi, \mu\}} (1 - \kappa_i) \left[ 1 - \frac{1 - \kappa_i}{1 - \kappa_\phi - \kappa_\mu} - c + m_i(\phi) \right]$ denote the one-period welfare generated by fixed ability cutoffs $(\kappa_\phi, \kappa_\mu) \in [0, 1]^2$.

A social planner maximizing long-term welfare

$$V^*(\phi) = \max_{\kappa_\phi, \kappa_\mu} W(\kappa_\phi, \kappa_\mu | \phi) + \delta V^* \left( \frac{1 - \kappa_\phi}{1 - \kappa_\phi + 1 - \kappa_\mu} \right)$$

sets $(\kappa^*_\phi, \kappa^*_\mu)$ such that labor force satisfies

$$2 - \kappa^*_\phi - \kappa^*_\mu = \ell^* \left( \frac{1 - \kappa^*_\phi}{2 - \kappa^*_\phi - \kappa^*_\mu} \bigg| \phi \right),$$

where $\ell^*|\phi)$ is given by (3.4) above.

Given this result, it is without loss of generality to restrict our attention to fractional quotas as outlined in section 3.4. If there is, however, a stock of mentors that lives longer than the firm’s horizon, this equivalence will not hold true any more. We consider such a model in section 3.4 when we look at a continuous time version of this overlapping generations model.

**Preventing discriminatory hiring practices**

A third conceivable policy concerns setting an upper bound $\pi$ on $|\kappa_\phi - \kappa_\mu|$, the threshold levels of ex-ante ability required for employment. Such a policy makes sense if the ability boost from mentoring is realized only once workers are active in the work force but ex-ante ability can be observed before. In such a situation, a regulatory body may work as a “watch dog” ensuring that individuals of similar ability have the same chances to enter an industry independent of their gender, class or race.

Again, when determining the optimal value of $\bar{\pi}$, the social planner has to take into account firm’s strategic response in choosing the final labor composition $\phi$ and size $\ell$. Certainly, the social outcome cannot exceed that of the previous intervention, where both values of interest are directly imposed. Thus, the real question is rather how close to this first best solution that government can get using this less invasive strategy.

As a first step, we show that monitoring hiring thresholds does not affect the endogenously chosen labor size $\ell$. It thus affects the long-run optimum only through its impact on representation $\phi$. 


Proposition 12. Imposing a cap $|\kappa_\phi - \kappa_\mu| \leq \bar{\kappa}$ has no effect on overall labor force participation.

This proposition can be shown by noting that in equilibrium firms maximize

$$
\begin{align*}
\max_{\ell, \phi'} & \quad \phi' \ell \cdot \left[ 1 - \frac{\phi' \ell}{2} - c + m(\phi) \right] + (1 - \phi') \ell \cdot \left[ 1 - \frac{(1 - \phi') \ell}{2} - c + m(1 - \phi) \right] \\
\text{s.t.} & \quad |(1 - \phi') - \phi'| \cdot \ell \leq \bar{\kappa} \quad 0 \leq \ell \cdot (1 - \phi') \leq 1.
\end{align*}
$$

The solution $\ell^*(\bar{\kappa} \mid \phi)$ is independent of $\bar{\kappa}$. The complete proof is deferred to the appendix.

The intuition for the result is as follows: Suppose that the equilibrium ex-post cutoffs don’t sum to zero. In particular, assume $(\kappa_\phi - c + m(\phi)) + (\kappa_\mu - c + m(1 - \phi)) > 0$, implying that the least productive male employee more than compensates for the negative productivity of the threshold female worker. In that case, it is strictly profitable to employ some additional workers of each gender since their joint productivity is positive. Conversely, when the threshold productivities are jointly negative, the firm would be better off reducing its labor force. However, when the sum is exactly zero, overall labor force is exactly the same size as in an unregulated market, the only difference being that some male workers may have been substituted by the same measure of female workers.

Apart from the closed form expression for labor, the previous result also implies that this type of policy intervention will be less efficient than setting fractional hiring quotas, at least within the framework modeled here. Indeed, the first-best approach from section 3.4 was shown to be equivalent to fractional hiring quotas seen in section 3.4, and (10) revealed that the optimum induces changes in labor size. As a result, a social planner with an eye only on differential hiring thresholds will reach the welfare implemented by fractional quotas.

Direct subsidy to the minority

Lastly, a social planner could intervene by offering a direct subsidy to firms for each minority employee. Within academia, endowments for a minority professorship take on this flavor. Formally, firms will incorporate this as a direct productivity boost in addition to $m(\phi)$. At the optimum, they will hire any worker with positive productivity, and so this policy is the only one that will not affect the current male work force.

While this approach is computationally appealing, since the productivity impact in a given generation is fully captured by its effect on one gender, it is less efficient at optimizing long-term welfare. This is because, again, we have shown above that the first-best approach agrees with fractional quotas (see (11)), and those induce changes to the male workforce (see (10)). If the subsidy is accompanied by a tax for the majority, one can, however, obtain the same outcome as with a quota. For simplicity we focus our analysis on the case $M \leq c$. 

Proposition 13. There exists a $c^* > 0$ such that the following holds:

(i) For $c < c^*$, there exists a $\delta^* < 1$ such that for all $\delta \in [\delta^*, 1]$, the optimal strategy consisting of subsidies only entails a subsidy in the first period and no subsidy in any other period.

(ii) For $c > c^*$, the optimal subsidy is implemented over several periods independently of the discount factor.

First, it is straight-forward that it can only be optimal to subsidize in the first period only if the discount factor is not too small because the cost of substitute is paid today, while the benefit from it is collected in the future by being closer to the optimal steady state. The marginal gain from subsidizing more today and not subsidizing tomorrow is that already tomorrow more minority workers will work. However, with a high hiring cost this benefit becomes vanishingly small. On the contrary, in order to achieve a share $\bar{\phi}$ of female workers tomorrow, the seller has to subsidize more today as $c$ increases. Thus, for large hiring cost $c$, the cost of implementing the quota today is too high to offset the benefit from having a higher female work force tomorrow already.

Simulations in Continuous Time

In order to understand the dynamic implications of different kinds of policies, it is useful to consider a continuous time version of the model studied above and to simulate the evolution of the economy. There is a continuous inflow of students who receive a mentoring boost and enter the labor market immediately. Mentors leave the market at a rate of 1.

The model we are analyzing is the limiting model of the following discrete time model with period length $\Delta$. In each period a mass $[t, t+\Delta]$ of students from each group is available to enter the market and to receive a mentoring boost. Let $L(t)$ denote the labor force at time $t$. Then,

$$L(t + \Delta) = \Delta \left( 1 - F(\kappa_\phi(\phi(t))) \right) + 1 - F(\kappa_\mu(\phi(t))) + (1 - \Delta) L(t).$$

Hence, as $\Delta \to 0$, the labor force evolves according to

$$\dot{L}(t) = \ell(\phi(t)) - L(t).$$

The number of mentors must satisfy

$$\dot{\phi}(t + \Delta) L(t + \Delta) = (1 - \Delta) \phi(t) L(t) + \Delta \frac{1 - F(\kappa_\phi(\phi(t)))}{\ell(\phi(t))} \ell(\phi(t)).$$

In the limit this results in the differential equation

$$\dot{\phi}(t) = \frac{1 - F(\kappa_\phi(\phi(t)))}{L(t)} - \phi(t).$$
In a steady state it must hold that $\dot{L}(t) = 0$ and $\dot{\phi}(t) = 0$. Hence, $\phi$ is a steady state if and only if
$$
\phi = \frac{1 - F(\kappa_{\phi}(\phi))}{\ell(\phi)}
$$
and the labor force is equal to $L(t) = \ell(\phi)$. Consequently, all steady states are as analyzed in the discrete model.

We focus our analysis on policies that enforce cutoffs $\pi_{\phi}, \pi_{\mu}$ for time $T$. Let
$$
\mathcal{L} = 1 - F(\pi_{\phi}) + 1 - F(\pi_{\mu})
$$
be the resulting mass of workers hired at each period $t \in [0, T]$ while the policy is in place. Then, the dynamics of the labor force $L$ and fraction of female workers $\phi$ is given by
$$
\begin{align*}
\dot{L}(t) &= \mathcal{L} - L(t) \\
\dot{\phi}(t) &= \frac{1 - F(\pi_{\phi})}{L(t)} - \phi(t).
\end{align*}
$$

**Proposition 14.** With a policy enforcing cutoffs $\pi_{\phi}, \pi_{\mu}$ for a time interval $[0,T]$, the dynamics of the labor force $L$ and fraction of female workers $\phi$ is given by
$$
\begin{align*}
L(t) &= (1 - e^{-t})(\mathcal{L} - L(0)) + L(0) \\
\phi(t) &= \int_0^t e^{-(t-u)} \left( \frac{1 - F(\pi_{\phi})}{L(u)} - \phi(0) \right) du + \phi(0).
\end{align*}
$$

The differential equations follow from appendix C.2. Using this result we can simulate the effect of different policies. Figure 3.6 illustrates the evolution of the labor force and the female labor share if the economy is in the extreme steady state with $\phi(0) = 0$ and $L(0) = 2 - c$ at time zero and different policies $(\kappa_{\phi}, \kappa_{\mu})$ are in place afterwards.

First, the labor force is increasing at a faster rate the smaller the sum of cutoffs $\pi_{\phi} + \pi_{\mu}$. In particular, the evolution of labor force is the same for $(\pi_{\phi}, \pi_{\mu}) \in \{(0.3, 0.3), (0.4, 0.2)\}$. The female labor force share is increasing in the number of female mentors $1 - \pi_{\phi}$. Consequently, holding the labor force $1 - \pi_{\phi} + 1 - \pi_{\mu}$ fixed, the policy maker can always increase the speed of convergence to the favorable steady state rising $\phi$ with a policy such that $\frac{1 - \pi_{\phi}}{1 - \pi_{\phi} + 1 - \pi_{\mu}} = \bar{\phi} = \bar{\phi}$.

However, implementing a large $\bar{\phi}$ is not necessarily optimal because the average productivity of female workers is lower than that of their male counterparts as long as $\phi(t) < s \left( < \frac{1}{2} \right)$. Given the uniform distribution, total discounted welfare up to time $t$ can be written as
$$
\mathcal{W}(t) = \int_0^t e^{-ru} \left[ \int_{\pi_{\phi}(\phi(u))}^{1} (a + m(\phi(u)) - c) da + \int_{\pi_{\mu}(\phi(u))}^{1} (a + m(1 - \phi(u)) - c) da \right] du.
$$
Hence, $W(t)$ represents the discounted total productivity up to time $t$ disregarding what happens after time $t$. In particular, we can solve for the break-even horizon $t^*$ in $W(t^*) = 0$, corresponding to the point in time when accrued welfare improvements amortize the cost of implementation of a quota early on.

Figure 3.7 shows the total surplus generated at a point in time $t$. Naturally, there is a loss in welfare compared to the benchmark in early periods, because some minority workers who did not receive much of a mentoring boost get hired. However after about a generation the productivity outperforms the benchmark without a policy in place. Obviously, the discount rate of the policy maker $r$ plays an important role when determining the break-even horizon $t^*$ and when evaluating optimal policies. This is illustrated in figure 3.8.
Figure 3.8 displays $W(t)$ if the policy maker cares a lot about the future ($r = 0.1$) and if he is rather myopic ($r = 3$). One can think of the lifetime of a generation of mentors to be 1 because we normalized the death rate of mentors to 1. With $r = 0.1$, the discount factor for the average lifetime of mentors is $\approx 0.9$, with $r = 3$, it is $\approx 0.05$. The thin black line depicts the benchmark welfare if the government does not regulate and stays in the extreme steady state with only male workers. With $r = 0.1$, even the weakest policy with $\kappa \phi = 0$ is amortized in less than a generation. With a myopic government, however, any policy is generally too costly. The discount rate of a policy maker could vary depending on the political and economic stability in a country. Affirmative action policies only make sense at all in a relatively stable economic situation.

Figure 3.9 considers a discount rate of $r = 1$ which is equivalent to a discount factor of $\approx 0.37$. The break-even time is very close and less than a generation for policies with $\pi \in [0, 0.4]$. If half as many women as men are hired ($\pi \phi = 0.6$), then the break-even time increases by almost 50 per cent. It is also interesting to note that the extreme policy where all women are hired, which maximizes the drift of labor force $L$ and fraction of female labor force $\phi$, is strictly dominated by a policy in which the same number of men and women are hired. Consequently, it can be very costly to only hire minority workers for an entire period.

In order to solve for the optimal long-run policy, we would also need to take into account how the economy would evolve after the policy has been abolished. This makes the problem even more complicated. In general, this analysis shows that the effectiveness of affirmative action can be very sensitive to the time that the policy has been in place and that the discount rate of the social planner is crucial when evaluating the costs and benefits of different policies.

### 3.5 Fixed labor demand

In the above analysis, the labor force played an important role. An alternative benchmark could be where the number of jobs is fixed. What role do mentoring complementarities play in such a situation?
Model, steady states, and welfare

We use the same model as in section 3.2, but without a cost of hiring. The cutoff ex-post ability $\xi(\phi)$ is now determined by the constraint that only mass 1 of workers can be hired. As a consequence, if more workers from one group are hired, fewer workers from the other group will be hired. For simplicity, we again use the mentoring function given by (3.2) and we assume that the ex-ante ability of workers is distributed uniformly on $[0, 1]$. Then, the cutoff types for females and males are given by

$$\kappa_{\phi}(\phi) = \max\{\xi(\phi) - m(\phi), 0\} \quad \kappa_{\mu}(\phi) = \max\{\xi - m(1 - \phi), 0\}$$

where $\xi(\phi)$ solves

$$1 - \kappa_{\phi}(\phi) + 1 - \kappa_{\mu}(\phi) = 1. \quad (3.5)$$

Hence, we can write

$$\xi(\phi) = \begin{cases} 
M & \text{if } \phi \leq \frac{M-1}{M}s \\
\frac{1+M}{2} + \frac{M}{2s}\phi & \text{if } \frac{M-1}{M}s < \phi \leq s \\
\frac{1}{2} + M & \text{if } s < \phi \leq 1 - s \\
\frac{1+M}{2} + \frac{M}{2s}(1 - \phi) & \text{if } 1 - s < \phi \leq 1 - \frac{M-1}{M}s \\
M & \text{if } 1 - \frac{M-1}{M}s < \phi
\end{cases}.$$
The transition function for the fraction of female workers is given by

\[
\phi^+(\phi) = \begin{cases} 
0 & \text{if } \phi \leq \frac{M-1}{M}s \\
1 - \frac{1+M}{2} + \frac{M}{2s} \phi & \text{if } \frac{M-1}{M}s < \phi \leq s \\
\frac{1}{2} & \text{if } s < \phi \leq 1 - s \\
\frac{1+M}{2} - \frac{M}{2s}(1 - \phi) & \text{if } 1 - s < \phi \leq 1 - \frac{M-1}{M}s \\
1 & \text{if } 1 - \frac{M-1}{M}s < \phi
\end{cases}
\]

The transition function is piecewise linear in that case and as in the baseline setup, \( \phi = \frac{1}{2} \) is a stable steady state. Moreover, for all \( M > 1 \), \( \phi \in \{0, 1\} \) are also stable steady states.

![Figure 3.10: Transition function \( \phi^+(\phi) \)]

Finally, per period welfare with fraction \( \phi \) of female workers is given by

\[
W(\phi) := \int_{\kappa_{\phi}(\phi)}^{1} a + m(\phi)da + \int_{\kappa_{1}(\phi)}^{1} a + m(1 - \phi)da.
\]

In this example, welfare at \( \phi = \frac{1}{2} \) is always greater than with \( \phi \in \{0, 1\} \):

\[
W\left(\frac{1}{2}\right) = \frac{3}{4} + M > \frac{1}{2} + M = W(\phi).
\]
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Policy Implications

First, if there is only a fixed number of jobs, then all affirmative action quotas discussed in section 3.4 become equivalent. All of these policies consist of a quota $\phi$ that result in a cutoff $\kappa_\phi = 1 - \phi$ for female workers and in a cutoff $\kappa_\mu = \phi$. Moreover, note that only quotas greater or equal than $\bar{\phi} = M - 1\over s - 2$ make sense since only with $\phi > \bar{\phi}$ the economy eventually converges to $\phi = 1/2$. With fixed labor demand, we can get the following clear-cut result.

**Proposition 15.** There exists a $\delta^*$ such that for all $\delta > \delta^*$, the optimal policy consists of a quota $\bar{\phi} < \phi$ in period one and no quota thereafter.

The intuition for this result is similar to the intuition of (13). The result is more clear-cut here because $\phi^+$ is linear in the relevant region. If $m$ was strictly concave in $[0, s]$, $\phi^+$ would also be strictly concave and the result would not hold unconditionally. Hence, the policy maker might have an incentive to rather implement quota step by step. Nevertheless, he should always make sure the set quota greater than $\bar{\phi}$.

All in all, with fixed labor supply, a similar intuition holds as with an unlimited number of jobs, but a cost of hiring new workers. However, with fixed demand for labor, the majority will always be strictly worse off after such a policy. Moreover, for more general mentoring boost functions, the policy maker will have a stronger incentive to regulate in markets with a shortage of workers because there is a benefit from increasing the labor force when more minority workers enter the market.

3.6 Discussion and conclusion

In this paper we provide a framework that allows us to derive conditions under which affirmative action is necessary if it is motivated by mentoring complementarities. If a balanced workforce brings about other benefits on top of mentoring complementarities, affirmative action is even more likely to be an effective tool to increase the average productivity in the society. However, in markets, in which the demand for labor is limited, some majority workers will always be crowded out which can make it hard for the policy maker to sell such a regulation. In future work we plan to extend our results to asymmetric situations with a larger and a smaller group. It is not straight-forward how our results would generalize in such a setup. Finally, it would be interesting to analyze the consequences of firms having less bargaining power, so that they do not extract the entire surplus. This would only affect the setup with unlimited jobs and it would result in firms only partly internalizing the marginal productivity of workers. The effect on affirmative action policies, is however, ambiguous.
Bibliography


BIBLIOGRAPHY


Appendix A

Appendix to Chapter 1

Proof. (Lemma 3)

(i) Without commitment power, in an equilibrium with cycle length $N$, the seller must never have an incentive to have a temporary sale before $N$-th low valuation buyer has arrived given the buyer’s strategies and beliefs. Hence, it must hold for any $n < N$ that

$$p_H(N) \cdot \frac{\lambda \pi}{r_s} + (1 + R_s)^{-(N-l)} \cdot \left( v_L N - C_A + \Pi(0; N) - p_H(N) \frac{\lambda \pi}{r_s} \right) \geq v_N n - C_A + \Pi(0; N).$$

This inequality can be simplified to

$$C_A \geq \left( N - (N - n) \cdot \frac{1 - (1 + R_s)^{-N}}{1 - (1 + R_s)^{-(N-n)}} \right) v_L.$$

Note that all expressions are independent of $p_H(N)$, because beliefs are fixed and hence, the only trade-off for the seller is between clearing the market more frequently and paying the cost $C_A$ more frequently. The function $x \mapsto \frac{x}{1 - (1 + R_s)^{-x}}$ is increasing and at 1 equal to $\frac{\lambda(1 - \pi) + r_s}{r_s}$. Hence, the first inequality holds for all $n \leq N$ if it holds for $n = N - 1$, that is the seller has no incentive to deviate from $N$ by having an earlier sale if and only if

$$C_A \geq \left( N - \frac{\lambda(1 - \pi) + r_s}{r_s} \cdot (1 - (1 + R_s)^{-N}) \right) v_L.$$

(ii) A cutoff-demand $N$ can be supported by an equilibrium only if after the arrival of the $N$-th low valuation buyer, the seller prefers an advertised sale to waiting for more $v_L$-buyer to arrive. Hence, for any $n \geq N$ that is consistent with an equilibrium of a continuation game, it must hold that

$$v_L n - C_A + \Pi(0; N) \geq p_H(N) \frac{\lambda \pi}{r_s} + (1 + R_s)^{-(n-N)} \left( v_L n - C_A + \Pi(0; N) - p_H(N) \frac{\lambda \pi}{r_s} \right).$$
In particular it must hold for \( n = N + 1 \). The inequality can be simplified to

\[
\left(N - \frac{1 - (1 + R_s)^{-N}}{1 - (1 + R_s)^{-N}}(1 + R_s)^{-(n-N)} \cdot (n - N)\right) v_L \geq C_A
\]

The function \( x \mapsto \frac{(1+R_s)^{-x} x}{1-(1+R_s)^{-x}} \) is decreasing in \( x \) and at 1 equal to \( \frac{\lambda(1-\pi)}{r_s} \). Hence, the seller does not have an incentive to accumulate more than \( N \) buyers if and only if

\[
\left(N - \frac{1 - (1 + R_s)^{-N}}{1 - (1 + R_s)^{-N}}(1 + R_s)^{-(n-N)} \cdot (n - N)\right) v_L \geq C_A.
\]

(iii) I show that \( N - N = -1 \) for all parameter values. From the necessary conditions (1.3) and (1.4), the following equality for \( N \) and \( N \) follows:

\[
N - \left(1 - (1 + R_s)^{-N}\right) \cdot \left(1 + \frac{\lambda(1-\pi)}{r_s}\right) = N - \left(1 - (1 + R_s)^{-N}\right) \cdot \frac{\lambda(1-\pi)}{r_s}.
\]

This equality is equivalent to

\[
(1 + R_s)^{N-N+1} = 1 + (N - N + 1) \cdot (1 + R_s)^N \cdot R_s.
\]

One can immediately see that for \( N - N = -1 \) the equality is satisfied. Moreover, since \( C(N) > C(N) \) everywhere and \( C, C \) are both increasing functions, for each \( N \) there should only be one solution of this equality for \( N - N + 1 \). Hence, there is only a single integer number \( N^*(C_A) \in [N, N] \).

**Proof.** (Proposition 1)

Let (1.5) be satisfied. In order to show that an EDLP equilibrium can always be sustained, it is sufficient to show that the seller does not have a profitable deviation. Note that by the definition of equilibrium, buyers’ beliefs remain unchanged after observing a deviation. Hence, the seller can never benefit from deviating to a price smaller than \( v_L \). Similarly, I have already discussed in Section 1.2 why a deviation to a price that is rejected by both buyer types is never profitable. As discussed in the main text, the following continuation equilibrium can be supported: high-valuation buyers only accept prices \( p \leq v_L + c \) and the seller returns to an EDLP after the first deviation. It is immediate that buyers do not have a profitable deviation. Moreover, after a buyer has rejected a price \( p \leq v_L + c \) and hence, \( n = 1 \), the seller cannot make higher profits by accumulating \( N - 1 \) more low-valuation buyers than with EDLP because by (1.5) for all \( N \)

\[
(1 - (1 + R_s)^{-(N-1)}) \frac{\lambda(1-\pi)}{r_s}(v_L + c) + (1 + R_s)^{-(N-1)}(Nv_L - C_A + \Pi_L) < \Pi_L.
\]
Given this continuation equilibrium, a deviation is not profitable for the seller because (1.5) implies that \( \pi(v_L + c) < v_L \).

Next, I show that if \( \Pi(0, N^*) > \Pi_L \), a cutoff-demand \( N^*(C_A) \) and regular price \( p_H(N^*(C_A)) \) can be supported in equilibrium. This is because a deviation to a price smaller than \( p_H(N^*(C_A)) \) but greater than \( v_L \) decreases profits by the no off-equilibrium signaling assumption and deviating to a regular price of \( v_L \) is not profitable because \( \Pi(0, N^*) > \Pi_L \).

\[ \text{Proof. (Corollary 1)} \]

(i) If \( C_A < \overline{C}(1) \), then in an advertising equilibrium, the seller drops the price immediately after she has observed a buyer not buying and \( N^* = 1 \). Hence, \( \Pi(0; 1) = \Pi_L - \frac{(1+R_s)^{-1}}{1-(1+R_s)^{-1}}C_A \) and by Proposition 1 there is no advertising equilibrium.

(ii) Because \( C(N) = C(N+1) \) is increasing in \( N \) and \( C_A \in [\underline{C}(N^*(C_A)), \overline{C}(N^*(C_A))] \), an increase in \( C_A \) implies that \( N^*(C_A) \) must be greater.

(iii) If \( C_A \to \infty \), then \( N^* \to \infty \) in an advertising equilibrium, \( \Pi(0; N^*) \to \Pi_H \). Hence, if \( v_H \pi \geq v_L \), then there exists an advertising equilibrium that sustains the static monopoly profit \( \Pi_H \). If \( v_H \pi < v_L \), then there is no advertising equilibrium, but the static monopoly profit \( \Pi_L \) can be sustained in an EDLP equilibrium.

\[ \text{Proof. (Corollary 2)} \]

By uniqueness of the equilibrium, it suffices to do comparative statics using one of \( \overline{C}, \underline{C} \). It is easy to check that \( C \) is increasing in \( R_s \) and increasing in \( N \).

(i) Since \( \overline{C}, \underline{C} \) are decreasing in \( R_s \), \( N^*(C_A) \) is decreasing in \( C_A \). As \( R_s \to \infty \), \( \overline{C}(N) = \underline{C}(N+1) \) converges to \( N \cdot v_L \). Hence, \( N^*(C_A) \to \frac{C_A}{v_L} \). As \( R_s \to 0 \), \( \overline{C}(N) = \underline{C}(N+1) \) converges to 0 because by l'Hôpital

\[
\lim_{R_s \to 0} \left( 1 - (1 + R_s)^{-N} \right) R_s^{-1} = \lim_{R_s \to 0} \frac{(1 + R_s)^N - 1}{(1 + R_s)^N R_s} = \lim_{R_s \to 0} \frac{N(1 + R_s)^{N-1}}{N(1 + R_s)^N R_s (1 + R_s)^{-1}} = N.
\]

Hence, \( N^*(C_A) \to \infty \) since \( \overline{C}(N), \underline{C}(N) \) are increasing in \( R_s \).

(ii) Hence, an increase in \( \lambda \) implies an decrease in \( R_s \), i.e. the equilibrium cutoff-demand \( N^*(C_A) \) increases by (i). In order to show that the increase is less than proportional, I
analyze the effect of a proportional increase in $N$ and $\lambda$. In particular, one can calculate

$$
\frac{\partial}{\partial \gamma} \left( N \cdot \gamma - \left( 1 - \left( 1 + \frac{R_s}{\gamma} \right)^{-N} \right) \cdot \frac{\gamma}{R_s} \right) \cdot v_L|_{\gamma=1} = 
$$

$$
N - \frac{1}{R_s} \left( 1 - \left( 1 + R_s \right)^{-N} \right) + N \cdot \left( 1 + R_s \right)^{-N-1} + \frac{1}{R_s} \cdot \left( 1 + R_s \right)^{-N} \cdot \log \left( 1 + R_s \right)^{-N} >
$$

$$
N - \frac{1}{R_s} \left( 1 - \left( 1 + R_s \right)^{-N} \right) + N \cdot \left( 1 + R_s \right)^{-N-1} + \frac{1}{R_s} \cdot \left( 1 + R_s \right)^{-N} - 1 =
$$

$$
N - \frac{2}{R_s} \left( 1 - \left( 1 + R_s \right)^{-N} \right) + N \cdot \left( 1 + R_s \right)^{-N-1} > 0
$$

for all $N, R_s > 0$. Since the average length of time between two sales if given by $\frac{N}{\lambda}$, an increase in $\lambda$ leads to shorter time intervals between two sales in equilibrium.

(iii) One can calculate

$$
\frac{\partial}{\partial \gamma} \left( N \cdot \gamma - \left( 1 - \left( 1 + \frac{R_s}{\gamma} \right)^{-N} \right) \cdot \frac{\gamma}{R_s} \right) \cdot v_L|_{\gamma=1} =
$$

$$
\left( N \cdot \left( 1 + R_s \right)^{-N-1} + \frac{1}{R_s} \cdot \left( 1 + R_s \right)^{-N} \cdot \log \left( 1 + R_s \right)^{-N} \right) v_L < 0
$$

for all $N, R_s > 0$. Since the average length of time between two sales if given by $\frac{N}{\lambda}$, an increase in $\lambda$ with a proportional decrease of $v_L$ leads to longer time intervals between two sales in equilibrium.

\[\square\]

**Proof.** (Lemma 4)

(i) Note that for all $x > 0$,

$$
x \log(x) \geq x - 1 \quad (A.1)
$$

where it holds with equality if only if $x = 1$. It follows that

$$
\frac{\partial}{\partial N} p_H(N) = -(v_H - v_L) \cdot \left( 1 + \frac{\lambda(1 - \pi)}{r_b} \right) \cdot 
$$

$$
\left( 1 + \frac{r_a}{\lambda(1 - \pi)} \right)^{-N} - 1 - \left( 1 + \frac{r_a}{\lambda(1 - \pi)} \right)^{-N} \cdot \left( \log \left( 1 + \frac{r_a}{\lambda(1 - \pi)} \right)^{-N} \right) \frac{N^2}{}\right) > 0
$$
and
\[
\frac{\partial^2}{(\partial N)^2} p_H(N) = -(v_H - v_L) \cdot \left(1 + \frac{\lambda(1 - \pi)}{r_b}\right) \cdot \frac{1}{N^3} \cdot \left[\left(1 + \frac{r_b}{\lambda(1 - \pi)}\right)^{-N} \cdot \left(\log \left(1 + \frac{r_b}{\lambda(1 - \pi)}\right)^{-N}\right)^2 + 2 \left(1 + \frac{r_b}{\lambda(1 - \pi)}\right)^{-N} \cdot \log \left(1 + \frac{r_b}{\lambda(1 - \pi)}\right)^{-N} + 2 \left(1 - \left(1 + \frac{r_b}{\lambda(1 - \pi)}\right)^{-N}\right)^2\right] < 0.
\]

It is immediate that
\[
\frac{\partial}{\partial N} \frac{-(1 + R_s)^{-N}}{1 - (1 + R_s)^{-N}} = \frac{(1 + R_s)^N \cdot \log (1 + R_s)}{(1 + R_s)^N - 1} > 0
\]
and
\[
\frac{\partial^2}{(\partial N)^2} \frac{-(1 + R_s)^{-N}}{1 - (1 + R_s)^{-N}} = \frac{-(1 + R_s)^N \cdot \log (1 + R_s)^2}{(1 + R_s)^N - 1} < 0.
\]

(ii) By inequality (A.1) it follows that
\[
\frac{\partial}{\partial N} \frac{(1 + R_s)^{-N}}{1 - (1 + R_s)^{-N}} \cdot N = \frac{(1 + R_s)^N - 1 - (1 + R_s)^N \log (1 + R_s)^N}{(1 + R_s)^N - 1} < 0
\]
and
\[
\frac{\partial^2}{(\partial N)^2} \frac{(1 + R_s)^{-N}}{1 - (1 + R_s)^{-N}} \cdot N = \frac{(1 + R_s)^N \cdot \log (1 + R_s)}{(1 + R_s)^N - 1} \cdot \left(-2 \cdot (1 + R_s)^N + (1 + R_s)^N \cdot \log (1 + R_s)^N + \log (1 + R_s)^N + 2\right) > 0
\]
as can be easily checked.\(^1\)

\[\square\]

**Proof. (Lemma 5)**

\(^1\)This can be seen by noting \(1 \cdot \log 1 + \log 1 + 2 - 2 \cdot 1 = 0\) and \(\frac{d}{dy} \left(y \cdot \log y + \log y + 2 - 2 \cdot y = \log y - 1 + \frac{1}{y}\right) \geq 0\) by inequality (A.1).
Let \( r_s = r_b = r \). Then, I can write

\[
\Pi(N) = v_H \frac{\lambda \pi}{r} - \frac{\lambda \pi}{r} \cdot \left(1 + \frac{\lambda(1 - \pi)}{r}\right) \cdot \frac{1 - \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}}{N} \cdot (v_H - v_L) + \frac{(Nv_L - C_A) \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}}{1 - \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}}
\]

which is less than \( \frac{\lambda \pi}{r} v_H \) for all \( N \) if and only if

\[
\left(\frac{\lambda \pi}{r}\right) r + \frac{\lambda(1 - \pi)}{r} \cdot (v_H - v_L) \geq \frac{N^2 \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}}{\left(1 - \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}\right)^2} \left(v_L - \frac{C_A}{N}\right)
\]

(A.2)

holds for all \( N \). Using that \( h : x \mapsto \frac{x^a - x}{(1 - a)^2} \) is decreasing for all \( a > 0 \) and for all \( x \geq 0 \) and

\[
\lim_{N \to 1} \frac{N^2 \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}}{\left(1 - \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}\right)^2} = \frac{\lambda(1 - \pi)}{r} \cdot \frac{\lambda(1 - \pi) + r}{r}^N, \text{ the above inequality is satisfied if}
\]

\[
\pi \cdot (v_H - v_L) \geq (1 - \pi) \cdot v_L,
\]

(A.3)

that is if \( v_H \pi \geq v_L \). Next I show that that if \( v_H \pi < v_L \), then \( \Pi(0; N) < \frac{\lambda}{r} v_L \) for all \( N \).

\[
\Pi(N) < \frac{\lambda v_L}{r} \left(1 - \frac{1 - \left(1 + \frac{r_b}{\lambda(1 - \pi)}\right)^{-N}}{N} \cdot \left(1 + \frac{\lambda(1 - \pi)}{r_b}\right)\right) + \frac{\lambda \pi v_L}{r} \cdot \frac{1 - \left(1 + \frac{r_b}{\lambda(1 - \pi)}\right)^{-N}}{N} \cdot \left(1 + \frac{\lambda(1 - \pi)}{r_b}\right) + \frac{N \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}}{1 - \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}} v_L
\]

\[
= \frac{v_L \lambda}{r} \left(1 - \left(1 - \pi\right) \frac{1 - \left(1 + \frac{r_b}{\lambda(1 - \pi)}\right)^{-N}}{N} \cdot \left(1 + \frac{\lambda(1 - \pi)}{r_b}\right)\right) + \frac{N \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}}{1 - \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}} v_L
\]

Since \( x \mapsto \frac{x^a - x}{(1 - a)^2} \) is decreasing for all \( a > 1, x \geq 0 \) and \( \lim_{N \to 1} \frac{N \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}}{1 - \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}} = \frac{\lambda(1 - \pi)}{r} \) and

\( x \mapsto \frac{1 - a^x}{x} \) is decreasing for all \( a > 1, x \geq 0 \) and \( \lim_{N \to 1} \frac{1 - \left(1 + \frac{r}{\lambda(1 - \pi)}\right)^{-N}}{N} = \frac{r}{r + \lambda(1 - \pi)} \), I can further conclude

\[
\Pi(N) < v_L \frac{\lambda}{r} \pi + \frac{\lambda(1 - \pi)}{r} v_L = v_L \frac{\lambda}{r}.
\]
Proof. (Lemma 6)

(i) follows by noting that \( \lim_{N \to \infty} \Pi^{FC}(N) = \Pi_H \) and

\[
\Pi_H - \Pi^{FC}(N) > \frac{\lambda \pi (v_H - v_L)}{r_s} \cdot \left( 1 + \frac{\lambda (1 - \pi)}{r_b} \right) \cdot \frac{1 - \left( 1 + \frac{r_s}{\lambda (1 - \pi)} \right)^{-N}}{N} \to 0, \quad \text{as } N \to \infty
\]

\[
> 0, \quad \text{as } N \to \infty, \quad \frac{N \cdot (1 + R_b)^{-N}}{1 - (1 + R_b)^{-N} \cdot v_L}
\]

and

\[
\frac{N \cdot (1 + R_b)^{-N}}{1 - (1 + R_b)^{-N} \cdot v_L} = o \left( \frac{1 - \left( 1 + \frac{r_s}{\lambda (1 - \pi)} \right)^{-N}}{N} \right), \quad N \to \infty.
\]

(ii) First, note that \( r_b \) affects the seller’s profit only through \( p_H(N) \) or the expected discount to a sale \( E \left[ e^{-r_s \tau(p_H(N))} \right] = \frac{1}{N} \sum_{n=0}^{N-1} \left( 1 + \frac{r_s}{\lambda (1 - \pi)} \right)^{-n} \). This is decreasing in \( r_b \), that is from (1.2) and (1.7) it follows that \( p_H(N) \) and \( \Pi(N) \) are increasing in \( r_b \). Since \( \Pi(N) \) is increasing in \( N \) for all \( N \), \( \max_N \Pi(N) \) is also increasing in \( r_b \).

(iii) is straightforward, since profit are only affected negatively by \( C_A \).

(iv) First, note that by (1.2), given all parameters, for any \( N \) and all \( \epsilon > 0 \), there exists a \( \tau_b(N, \epsilon) \) such that for all \( r_b \geq \tau_b(N, \epsilon) \), \( p_H(N) > v_H - \epsilon \). Let

\[
\hat{N} = \arg \max_N \left( N v_L - C_A \right).
\]

Hence, by (1.7),

\[
\Pi^{FC}(\hat{N}) > \frac{\lambda \pi (v_H - \epsilon)}{r_s} \cdot \left( 1 + \frac{\lambda (1 - \pi)}{r_b} \right) \cdot \left( \hat{N} v_L - C_A \right).
\]

Hence, let us choose

\[
\epsilon = \frac{r_s}{\lambda \pi} \cdot \left( \frac{1}{\left( 1 + \frac{r_s}{\lambda (1 - \pi)} \right)^{\hat{N}}} \cdot (\hat{N} v_L - C_A) - \max \left\{ \frac{\lambda (v_L - v_H \pi)}{r_s}, 0 \right\} \right).
\]

Then, \( \Pi^{FC}(\hat{N}) \geq \max \left\{ \frac{\lambda \pi v_H}{r_s}, \frac{\lambda}{r_s} \right\} \) for all \( r_b \geq \tau_b(\hat{N}, \epsilon) \).

If \( v_L - \pi v_H \geq \frac{r_s}{\lambda} \max_N \frac{1}{(1 + \frac{r_s}{\lambda (1 - \pi)})^N - 1} (N v_L - C_A) \), then

\[
\Pi(0; N) < \frac{\lambda \pi}{r} v_H + \max_N \frac{1}{(1 + \frac{r_s}{\lambda (1 - \pi)})^N - 1} (N v_L - C_A) \leq \frac{\lambda}{r} v_L.
\]
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Proof. (Lemma 7)

(i) First, note that

\[ \frac{\partial}{\partial N} \Pi^{FC}(N) = \frac{\lambda \pi}{r_s} \cdot (\frac{\partial}{\partial N} p_H(N)) - v_L \cdot \left( \frac{\partial}{\partial N} \frac{(1 + \frac{r_b}{\lambda(1-\pi)})^{-N}}{1 - \frac{r_b}{\lambda(1-\pi)}} \right) + C_A \left( \frac{\partial}{\partial N} \frac{-\frac{r_b}{\lambda(1-\pi)}^{-N}}{1 - \frac{r_b}{\lambda(1-\pi)}} \right). \]

(A.4)

Using the expression of \( \frac{\partial}{\partial N} p_H(N) \) from the proof of Lemma 4 (i), one can derive

\[ \frac{\partial^2}{\partial N \partial r_b} \Pi^{FC}(N) = \frac{\lambda \pi}{r_s} \cdot (v_H - v_L) \cdot \frac{1}{N^2} \cdot \frac{\lambda(1-\pi)}{r_b^2} \cdot \left[ \left( 1 + \frac{r_b}{\lambda(1-\pi)} \right)^{-N} - N r_b \cdot \left( 1 + \frac{r_b}{\lambda(1-\pi)} \right)^{-N} \cdot \left( \log \left( 1 + \frac{r_b}{\lambda(1-\pi)} \right)^{-N} \right) \right]. \]

Figure A.1: Illustration of the proof of Proposition 7

Step 1: For any fixed \( N \) there exists a \( \tilde{r}_b(N) \), such that \( \frac{\partial^2}{\partial N \partial r_b} \Pi^{FC}(N) > 0 \) if and only if \( r_b < \tilde{r}_b(N) \) and \( \frac{\partial^2}{\partial N \partial r_b} \Pi^{FC}(N) < 0 \) if and only if \( r_b > \tilde{r}_b(N) \)

\[ \frac{d}{dy} \left( y^{-N} - 1 - (1 + N(y-1)) y^{-N} \left( \log y^{-N} \right) \right) = N y^{-N-1} (y-1) \left( N \log y^{-N} - \log y^{-N} + N \right) \]
APPENDIX A. APPENDIX TO CHAPTER 1

has only one null for $y > 1$, namely $y = e^{-\frac{1}{y^N}}$ and is positive at $y + \epsilon$ for small $\epsilon > 0$. Moreover, the following holds

- $\lim_{y \to 1} \left( y^{-N} - 1 - (1 + N(y - 1)) \cdot y^{-N} \cdot (\log y^{-N}) \right) = 0$,
- $\lim_{y \to \infty} \left( y^{-N} - 1 - (1 + N(y - 1)) \cdot y^{-N} \cdot (\log y^{-N}) \right) = -1 < 0$,
- $\lim_{y \to +1} N \cdot y^{-N-1} \cdot (y - 1) \cdot (N \log y^{-N} - \log y^{-N} + N) = 0^+$.

All together with continuous differentiability of all functions this shows the existence of a unique $\tilde{r}_b(N) > 0$ for every $N$ and $\tilde{r}_b(\cdot)$ is a continuous function. The solid line in Figure A.1 illustrates $\tilde{r}_b$.

Step 2: Any local maximum cutoff-demand $\tilde{N}^z(r_b)$ is increasing for small $r_b$ and decreasing for other $r_b$.

By the Implicit Function Theorem, any local maximum $\tilde{N}^z(r_b)$ of $\Pi$ is locally differentiable in $r_b$. $\tilde{N}^z$ is illustrated by the dotted blue line in Figure A.1. Moreover, it follows that $\frac{\partial^2 \tilde{N}^z}{\partial N \partial r}(\tilde{N}^z(r_b), r_b)$ and $\tilde{N}^z(r_b)$ must have the same sign. Hence, I can show that $\tilde{N}^z(r_b)$ can only change from positive to negative and not the other way around by contradiction. Assume that there exists an $\overline{r}_b$ such that for some $\epsilon_1, \epsilon_2 > 0$, $\tilde{N}^z(r) < 0$ for all $r \in (r_b - \epsilon_1, \overline{r}_b)$ and $\tilde{N}^z(r) > 0$ for all $r \in (\overline{r}_b, \tilde{r}_b + \epsilon_2)$ (and hence $\tilde{N}^z(\overline{r}_b) = 0$). Then, there are $r_1 < r_2$ with $\tilde{N}^z(r_1) = \tilde{N}^z(\overline{r}_b) = N$ such that $\tilde{N}^z(r_1) < 0 < \tilde{N}^z(r_2)$. Hence, such that $\frac{\partial^2 \tilde{N}^z}{\partial N \partial r}(N, r_1) < 0 < \frac{\partial^2 \tilde{N}^z}{\partial N \partial r}(N, r_2)$. This is, however, a contradiction to Step 1 which concludes the proof.

(ii) follows immediately from Lemma 4.

\[ \square \]

Proof. (Lemma 8)

(i) follows immediately from (1.9).

(ii) Let $\delta_b = \left( 1 + \frac{r_b}{1 - \pi} \right)^{-1}$, $\delta_s = \left( 1 + \frac{r_s}{1 - \pi} \right)^{-1}$. Then, one can write

\[
\Pi(N) - \frac{\lambda}{r_s} v_L = \frac{\lambda (1 - \pi) v_L}{r_s} \left[ \frac{\pi v_L - \pi}{1 - \pi} \cdot \left( 1 - \frac{1 - \delta_b^N}{N \cdot (1 - \delta_b)} \right) - (1 - \delta_s^{N-1}) \right].
\]

First, note that $\Pi(N)$ is increasing in $N$ for large enough $N$. Hence, $\Pi(N) - \frac{\lambda}{r_s} v_L$ has at most one maximum and at most one minimum, if and only if it is either first increasing, then
decreasing and then increasing or first decreasing and then increasing. Therefore, consider the first derivative of $\Pi(N) - \frac{\lambda}{r_s} v_L$ which is given by

$$
\frac{\partial}{\partial N} \left( \Pi(N) - \frac{\lambda}{r_s} v_L \right) = \frac{\lambda (1 - \pi) v_L}{r_s \cdot N^2} \cdot \left[ C \cdot \frac{1 - \delta_b^N}{1 - \delta_b} + C \cdot \frac{N \delta_b^N \log(\delta_b)}{1 - \delta_b} + N^2 \delta_s^{N-1} \log(\delta_s) \right].
$$

I need to show that $H(N)$ is either first positive, then negative and then positive or negative and then positive. Therefore consider

$$
\frac{d}{dN} H(N) = N \cdot (\log(\delta_s))^2 \cdot \left[ C \cdot \frac{\delta_b}{1 - \delta_b} \cdot \left( \frac{\delta_b}{\delta_s} \right)^{N-1} \left( \frac{\log(\delta_b)}{\log(\delta_s)} \right)^2 + \frac{2}{\log(\delta_s)} + N \right].
$$

$\frac{d}{dN} H(N)$ is changing signs at most twice because $J(N)$ it is quasi-convex. If it changes signs twice, it is first positive, then negative and then positive again. In that case, since $H(0) = 0$, $H$ is first positive, then negative and then positive again and the claim follows immediately. If $J$ changes signs at most once, then $H$ changes signs at most twice and $\Pi(N) - \Pi_L$ changes signs at most twice. Since $\Pi(1) - \Pi_L = 0$, the claim follows immediately.

(iii) The derivative of $\Pi$ at $N = 1$ is given by

$$
\Pi'(N)|_{N=1} = \frac{\lambda (1 - \pi)}{r_s} v_L \left( C + C \cdot \frac{\delta_b \log \delta_b}{1 - \delta_b} + \log \delta_s \right)
$$

is decreasing in $\delta_b$, i.e. increasing in $r_b$. For $r_b \to \infty$, $\delta_b \to 0$ and $\lim \Pi'(N)|_{N=1} < 0$. □

Proof. (Proposition 2)

For advertising costs $C_A \in \mathcal{C}$, an advertising equilibrium exists if and only if

$$
\Pi(C^{-1}(C_A)) \geq \Pi_L.
$$

Hence, the proof follows immediately from Lemma 8, since $\Pi$ is either first decreasing and then increasing or, if $r_b$ is large enough, first increasing, then decreasing and then increasing again. Since $\Pi$ is increasing in $r_b$, $T_1$ is decreasing and $T_0$ increasing in $r_b$. □

Proof. (Proposition 3)

The proof follows from the fact that $\Pi$ has a single maximum and from Proposition 7 (i). □
Appendix B

Appendix to Chapter 2

B.1 Proofs

Model and Preliminaries

Proof. (Lemma 9; Reverse-skimming property)

In both information structures, a seller rejects a price $p_t$ in period $t$ if and only if $p_t - c$ is less than accepting a price at any (possibly random) time $\tau$ in the future $\mathbb{E}[e^{-r\tau}(p_{\tau} - c)]$. $p_{\tau}$ is independent of $c$ except for if $\tau = 1$ in which case it is equal to $v(c)$ which is increasing in $c$. Hence, if a seller type $c$ accepts the price offer $p_t$, then a seller type $c' < c$ must accept the price offer. (Note that it is irrelevant that with public offers, $p_{\tau}$ is a function of $p_t$.) \hfill $\Box$

Proof. (Lemma 10; Inverse supply)

(i) (Private offers) With private offers, beliefs of buyers are independent of price histories. Hence, the continuation game in an equilibrium is unaffected by past offers. We argue by backward induction. Consider

Base of induction: In period $1 - \Delta$, a seller of type $c$ accepts an offer $p$ if and only if $p \geq \delta v(c) + (1 - \delta)c \equiv p_{1-\Delta}(c)$. $p_{1-\Delta}$ is increasing and continuous. Consequently, $p_{1-\Delta}(k) = k_{1-\Delta}^{-1}(k)$ is the unique price that results in a cutoff $k$ in period $1 - \Delta$.

Induction hypothesis: Let us now assume, there exists an increasing and continuous inverse supply function $p_s(\cdot) = k_s^{-1}(\cdot)$ for all $s \in \{t + \Delta, \ldots, 1 - \Delta\}$ for a $t = n\Delta$. Denote the cdfs of cutoffs that result from the pricing strategies in the continuation game after period $t$ by $K_s$, $s = t + \Delta, \ldots, 1 - \Delta$. 
APPENDIX B. APPENDIX TO CHAPTER 2

Induction step: In period $t$, the continuation payoff of a seller $c$ who rejects an offer in period $t$ is given by

$$W(c) = \sum_{m=1}^{N-n-1} \left( \delta^m \left( \prod_{h=n+1}^{m+n-1} K_h(c) \right) \int_1^c \left( p_{(n+m)\Delta} \left( k_{(n+m)\Delta} \right) - c \right) dK_{(n+m)\Delta}(k_{(n+m)\Delta}) \right) + \delta^{N-n}(v(c) - c) \left( \prod_{h=n+1}^{N-1} K_h(c) \right) + c$$

$$= \delta \left[ \left( \int_1^c p_{t+\Delta}(k_{t+\Delta})dK_{t+\Delta}(k_{t+\Delta}) \right) + K_{t+\Delta}(c)p_{t+\Delta}(c) \right] + (1-\delta)c.$$

Since $p_{t+\Delta}$ is increasing, $W(c)$ is increasing and continuous. $p_t(c) = W(c)$ defines an inverse demand function. Note that $p_t(k)$ is the unique price that results in a cutoff $k$ and hence $p_t = k_t^{-1}$.

(ii) (Public offers) With public offers, linear valuations, and uniformly distributed costs, the unique equilibrium can be constructed by backward induction. Because of uniqueness of any continuation equilibrium, there exists an inverse supply function $p_t(k)$ at any point in time $t$ for any cutoff $k$.

Base of induction: In the last period, independently of the belief of buyers, there is a unique price $p_{1-\Delta}(k) = \delta v(k) + (1-\delta)k$ that results in a cutoff $k$. $p_{1-\Delta}(k)$ is increasing and linear. Given a cutoff belief $k_{1-2\Delta}$, buyers’ profits conditional on purchasing the asset are

$$\frac{k_{1-\Delta} - k_{1-2\Delta}}{1 - k_{1-2\Delta}} \cdot \left[ \frac{A}{2} (k_{1-\Delta} + k_{1-2\Delta}) + B - p_{1-\Delta}(k_{1-\Delta}) \right]$$

which is quadratic in $k_{1-\Delta}$. Hence, there is a unique $k_{1-\Delta}(k_{1-2\Delta})$ such that for no price greater than $p_{1-\Delta}(k_{1-\Delta}(k_{1-2\Delta}))$ buyer make non-positive profits. As a result $k_{1-\Delta}(k_{1-2\Delta})$ is the unique cutoff in period $1 - \Delta$ given buyers believe the current cutoff is $k_{1-2\Delta}$.

Induction hypothesis: In period $t$, given any cutoff belief $k_{t-\Delta}$, for any $k \geq k_{t-\Delta}$ there is a unique price $p_t(k)$ that results in a cutoff seller $k$. $p_t(k)$ is linear and increasing in $k$. Moreover, there is a unique continuation equilibrium starting in period $t$, given buyers believe the current cutoff is $k_{t-\Delta}$. The unique equilibrium period $t$ cutoff $\kappa_t(k_{t-\Delta})$ is linear and increasing $k_{t-\Delta}$.

Induction step: Given buyers believe the current cutoff is $k_{t-2\Delta}$, if buyers in period $t - \Delta$ charge a price $p$ then, the cutoff seller type $k$ satisfies

$$p = \delta p_t(\kappa_t(k)) + (1-\delta)k.$$

Since $\kappa(k)$ is unique, there is a unique price $p_{t-\Delta}(k) = \delta p_t(\kappa_t(k)) + (1-\delta)k$ that results in cutoff $k$. $p_{t-\Delta}$ is increasing and linear and a buyer’s expected profit conditional on buying the good at price $p_{t-\Delta}(k_{t-\Delta})$ is given by

$$\frac{k_{t-\Delta} - k_{1-2\Delta}}{1 - k_{1-2\Delta}} \cdot \left[ \frac{A}{2} (k_{t-\Delta} + k_{1-2\Delta}) + B - p_{t-\Delta}(k_{t-\Delta}) \right]$$
which is quadratic in $k_{t-\Delta}$ and results in a unique highest price $p_{t-\Delta}(k_{t-\Delta})$ such that there is no higher price at which buyers make positive profits. The resulting cutoff $\kappa_{t-\Delta}(k_{t-2\Delta})$ is linear in $k_{1-2\Delta}$ because the profit function is quadratic. Hence, $p_{t-\Delta}(k_{t-\Delta})$ is linear.

For general valuation functions and distributions, there can be multiple continuation equilibria. As a result, $\kappa_t(k_{t-\Delta})$ is not unique. Nevertheless, for $k_{t-\Delta} > k'_{t-\Delta}$, $\kappa_t(k_{t-\Delta}) \geq \kappa_t(k'_{t-\Delta})$ for all $\kappa_t(k_{t-\Delta})$, $\kappa_t(k'_{t-\Delta})$ that can occur in a continuation equilibrium. The reason is that with continuous $v$, $p_{t-\Delta}$ exists and is increasing so $\pi_{1-\Delta}(\cdot; k_{t-2\Delta})$ is well defined. In any continuation equilibrium, buyers should not be able to receive positive profits by increasing the price and cutoff. If $p_t$ is well-defined and increasing, then $\pi_t(\cdot; k_{t-\Delta})$ is uniformly increasing in $k_{t-2\Delta}$. Hence, if buyers could not get positive profits by increasing $k_t$ given $k_{t-\Delta}$, then they cannot get positive profits by increasing the price from $k_t$ given $k'_{t-\Delta} < k_{t-\Delta}$. Hence, $\kappa_t(k_{t-\Delta}) > \kappa_t(k'_{t-\Delta})$ and

$$p_{t-\Delta}(k) = (1 - \delta)k + (1 - \delta)p_t(\kappa_t(k_{t-\Delta}))$$

is increasing and well defined. The only caveat is that $p_t$ does not have to be defined on the entire $[k_{t-\Delta}, 1]$ if the image of the function $k_t(p)$ from Lemma 9 does not contain $[k_{t-\Delta}, 1]$. □

**Transparency and Frequency of Trade**

**Equilibrium with Public Offers**

*Proof. (Lemma 11)*

The proof of the first part is in the text. For (ii) note first that $k_0^* \leq k_{2\Delta}^* \leq \cdots \leq k_{t-\Delta}^*$. We argue by backward induction. First, note that $p_{1-\Delta}(k_{1-\Delta}) < v(k_{1-\Delta}^*)$. Moreover, if $p_{t+\Delta}(k_{t+\Delta}) < p_{t+2\Delta}(k_{t+2\Delta})$, then

$$p_t(k_t^*) = (1 - \delta)k_t^* + \delta p_{t+\Delta}(k_{t+\Delta}) < (1 - \delta)k_{t+\Delta}^* + \delta p_{t+2\Delta}(k_{t+2\Delta}) = p_{t+\Delta}(k_{t+\Delta}).$$

□

*Proof. (Theorem 1 (iii))* We here present the proof for public offers. For private offers, the proof can be found in the text. First, note that by (2.6), $\kappa_t(k_{t-\Delta}) = k_{t-\Delta}$ for some $k_{t-\Delta}$ if and only if

$$\frac{Ak_{t-\Delta} + B - p_t(k_{t-\Delta})}{v(k_{t-\Delta})} \leq 0.$$

However, we show that for all $t$ and $k$, $v(k) - p_t(k) > 0$. Hence, $\kappa_t(k_{t-\Delta})$ is always greater than $k_{t-\Delta}$. In the last period, we have for all $k \in [0, 1]$

$$v(k) - p_{1-\Delta}(k) = (1 - \delta)(v(k) - k) > 0.$$


Next, we show that if \((1 - \delta)(v(k) - k) \geq v(k) - p_{t+\Delta}(k) > 0\) for all \(k \in [0,1]\), then \((1 - \delta)(v(k) - k) \geq v(k) - p_t(k) > 0\) for all \(k \in [0,1]\). By linearity of \(v(\cdot)\) and because \(c \sim U[0,1]\),

\[
p_{t+\Delta}(k_{t+\Delta}(k)) - v(k) = v(k_{t+\Delta}(k)) - p_{t+\Delta}(k_{t+\Delta}(k)).
\]

Hence,

\[
v(k) - p_t(k) = (1 - \delta)(v(k) - k) - \delta \left( p_{t+\Delta}(k_{t+\Delta}(k)) - v(k) \right) > 0
\]

because \(v(k_{t+\Delta}(k)) - p_{t+\Delta}(k_{t+\Delta}(k)) \leq (1 - \delta)(v(k_{t+\Delta}(k)) - k_{t+\Delta}(k))\) and because \(v(k) - k\) is decreasing in \(k\). It also follows immediately that \(v(k) - p_t(k) \leq (1 - \delta)(v(k) - k)\). \(\square\)

**Equilibria with Private Offers**

**Proof.** (Lemma 12)

We argue by backward induction over \(t\). \(p_{1-\Delta}(k_{1-\Delta}) = \delta (Ak_{1-\Delta} + B) + (1 - \delta)k_{1-\Delta} = (1 - \delta + \delta A)k_{1-\Delta} + \delta B\) is differentiable and \(\frac{\partial}{\partial k_{1-\Delta}} p_{1-\Delta} = 1 - \delta + \delta A\). Given \(p_{t+\Delta}\) is differentiable with a nondecreasing derivative, (2.3) is differentiable and

\[
\frac{\partial}{\partial k_t} p_t(k_t) = \delta K_{t+\Delta}(k_t) \frac{\partial}{\partial k_{t+\Delta}} p_{t+\Delta}(k_t) + (1 - \delta)
\]

is nondecreasing and piecewise constant. \(\square\)

**Proof.** (Theorem 1 (ii) 1.)

The main proof of the result that for \(\delta < \delta^*\) there are no mixed-strategy equilibria can be found in the text. Here, we only show that for \(\delta < \delta^*\) buyers at most mix between countable many prices. We also show that with \(\delta > \delta^*\) there is only countable mixing after the first period which will be useful for the equilibrium construction in Section 3.3.

**Lemma 13.** If \(\delta < 1 - \frac{4}{\pi^2}\), buyers in period 1 mix at most between countably many cutoffs. If \(\delta > 1 - \frac{4}{\pi^2}\) and expected period 1 profit \(\pi_0(k;0) = 0\) for all \(k \in (a,b)\), then any \(k \in (a,b)\) cannot be in the support of \(K_\Delta, \ldots, K_{1-\Delta}\) since it must hold that \(K_\Delta(k) = \frac{\delta - 1 + \frac{4}{\pi^2}}{\delta \pi^2 p_\Delta(k)}\).

**Proof.** In the first period, expected buyers’ profits are given by

\[
\pi_0(k;0) = k \cdot \left[ \frac{A}{2} k + B - p_0(k) \right].
\]

If buyers mix between all cutoffs \(k \in (a,b)\) in the first period, then they must make zero profits for all such cutoffs, i.e., for all \(k \in (a,b)\)

\[
\delta \left( \int_k^1 p_\Delta(k_\Delta) dK_\Delta(k_\Delta) + K_\Delta(k)p_\Delta(k) \right) + k(1 - \delta) = \frac{A}{2} k + B
\]
or equivalently
\[ \delta \left( \int_k^1 p_\Delta(k) dK_\Delta(k) + K_\Delta(k) p_\Delta(k) \right) = \left( \delta - \left( 1 - \frac{A}{2} \right) \right) k + B. \]

Note that the left hand side of the identity must be nondecreasing in \( k \), so if \( \delta < 1 - \frac{A}{2} \), then there cannot be mixing on \((a, b)\) in the first period. If \( \delta \geq 1 - \frac{A}{2} \), then the left hand side is differentiable, so the right hand side must be differentiable, so that
\[ K_\Delta(k) = \frac{\delta - (1 - \frac{A}{2})}{\delta \frac{\partial}{\partial k_\Delta} p_\Delta(k)} \]
on \( k \in (a, b) \). Since \( K_\Delta \) is a cdf, \( \frac{\partial}{\partial k_\Delta} p_\Delta(k) \) cannot be increasing on \((a, b)\), so that by Lemma 12 \( \frac{\partial}{\partial k_\Delta} p_\Delta(k) \) must be constant on \((a, b)\). This implies that the support of \( K_\Delta \) is disjoint from \((a, b)\) and because \( \frac{\partial}{\partial k_\Delta} p_\Delta(k) \) must be constant on \((a, b)\). Finally, the intersection of the supports of \( K_\Delta, \ldots, K_{1\Delta} \) and \((a, b)\) must be empty. \qed

Transparency and Welfare

Equilibria with three periods

Proof. (Proposition 4)

Before turning to the main proof we show the general result that buyers do not mix between a continuum of prices after time 0 and if they mix continuously at time 0, then those cutoffs cannot be cutoffs in future periods.

Lemma 14. (Mixed Strategy Equilibria; Linear Case) With private offers and \( \Delta < \Delta^* \) all mixed strategy equilibria must satisfy the following properties.
(i) Whenever \( t > \Delta \), buyers mix between at most countably many prices.
(ii) If buyers in period \( \Delta \) mix continuously between prices that result in cutoffs in an interval \((a, b)\), then buyers in periods \( t > \Delta \) do not choose any price that results in a cutoff in \((a, b)\) with positive probability.

Proof. Let us argue by induction. Assume there exists and interval \((a, b)\) such that buyers in period \( t \) mix between all cutoffs, i.e., for all \( k_t \in (a, b) \)
\[ \pi_t(k_t; K_{t-\Delta}^m) = \int_0^{k_t} \int_0^c \frac{1}{1 - k} dK_{t-\Delta}(k) (Ac + B - p_t(k_t)) dc = 0. \]

After applying integration by parts and setting \( H(k_t) \equiv \int_0^{k_t} \left( \int_0^c \frac{x}{1 - k} dK_{t-\Delta}(k) dx \right) dc \), one can see that this is equivalent to the ordinal differential equation
\[ AH'(k_t)k_t - AH(k_t) = H'(k_t) (p_t(k_t) - B). \]
Thus, we can conclude that

\[ H(k_t) \equiv \int_0^{k_t} \left( \int_0^c \int_0^x \frac{1}{1-k} dK_{t-\Delta}^m(k) dx \right) dc = \text{const} \cdot \exp \left( \int_0^{k_t} \frac{1}{z - p(z) - B} dz \right) \]

and by Fubini’s Theorem \( H(k_t) = \int_0^{k_t} \frac{k_t - k}{2(1-k)} dK_{t-\Delta}^m(k) \) which is increasing because \( \frac{k_t - k}{2(1-k)} > 0 \) for \( 0 < k < k_t \). Thus, the cdf \( K_{t-\Delta}^m(k) \) must be strictly increasing everywhere on \((a,b)\). Hence, if buyers mix on \((a,b)\) at time \( \Delta \), then they must mix at time 0 which is a contradiction by Lemma 13. Hence, by induction there cannot be mixing on an interval at any time \( t > \Delta \). \( \square \)

The proof of Proposition 4 for \( \Delta = \frac{1}{2} \) follows in three steps. First, we show in step 1 that buyers at time \( \Delta \) mix between exactly two prices and we show the first part of (ii). Step 2 discusses the second part of (ii), i.e. that there must be non-offers with positive probability in period 1. Finally, in step 3 we can pin down the exact values of \( k_\Delta \) and \( \overline{k}_\Delta \).

**Step 1:** At time \( \Delta \), buyers mix between exactly two prices resulting in cutoffs \( k_\Delta, \overline{k}_\Delta \) and cutoffs at time 0 must be in \( \{0\} \cup [k_\Delta, \overline{k}_\Delta] \).

First, note that with two periods, buyers in both periods must mix between at least two cutoffs. The reason is that if buyers at time 0 would play pure strategies, then there is a unique price at which buyers at time \( \Delta \) make zero profits, i.e. the unique Bertrand equilibrium in that period contains only pure strategies of the buyers. If buyers at time \( \Delta \) played pure strategies in equilibrium, then the same argument holds for expected profits at time 0. Since we have already established in Theorem 1 that if \( \delta > 1 - \frac{A}{2} \) there cannot be pure strategy equilibria, there must be mixing in both periods.

Let us first consider the continuation game at time \( \Delta \) given beliefs about the current cutoffs represented by the cdf \( K_0 \). Buyers’ profits are then given by

\[ \pi_\Delta(k_\Delta; K_0) = \int_0^{k_\Delta} \left( \int_0^c \frac{1}{1-k} dK_0(k) \right) (Ac + (1 - \delta)B - k_\Delta(\delta A + 1 - \delta)) dc. \]

\( \pi_\Delta \) is continuous and at the smallest element \( k_0^m < 1 \) in the support of \( K_0 \), for all \( \epsilon \) small enough we have

\[
\frac{\partial \pi_\Delta}{\partial k_\Delta}(k_\Delta; K_0)|_{k_\Delta=k_0^m+\epsilon} = \int_0^{k_0^m+\epsilon} \frac{1}{1-k_0} dK_0(k_0)(1 - \delta) (B + (k_0^m + \epsilon)(A - 1)) - \int_{k_0^m}^{k_0^m+\epsilon} \left( \int_0^c \frac{1}{1-k_0} dK_0(k_0) \right) dc(A\delta + 1 - \delta) > \int_0^{k_0^m+\epsilon} \frac{1}{1-k_0} dK_0(k_0) \left[ (1 - \delta) \left( B + k_0^m (A - 1) \right) - \epsilon(2(A\delta + 1 - \delta) - A) \right] > 0,
\]
so in equilibrium, buyers at time $\Delta$ do not choose prices that result in a cutoff type smaller or equal to $k^m_0$ with positive probability since if they did increasing the price a little bit would be a profitable deviation for any buyer. In particular, in any equilibrium, seller types close to zero trade at time $\Delta$, that is in the continuation game starting at time $\Delta$ it must hold that $K_\Delta(k_\Delta) = 0$ for small $k_\Delta$.

Now, we analyze the behavior of buyers at time 0. By Proposition 14, the support of $K_\Delta$ is discrete and $p_0(\cdot)$ is piecewise linear, continuous and by Lemma 12, it is also weakly convex. Hence, buyers’ expected profit at time 0

$$\pi_0(k_0; 0) = k_0 \cdot \left( \frac{A}{2} k_0 + B - \left( \int_{k_0}^1 p_\Delta(k_\Delta) dK_\Delta(k_\Delta) \right) + K_\Delta(k_0)p_\Delta(k_0) \right) + k_0(1 - \delta) \right) \right).$$

is continuous, piecewise quadratic and at any cutoff in the support of $K_\Delta$ it has a “downward” kink (that is the slope is dropping discontinuously) because of the convexity of $p_0$. Hence, in equilibrium, expected profits at time 0 must qualitatively look like one of the graphs in figure B.1. Note that for small $k$, $p_0(k) = \delta \int_k^1 p_\Delta(k_\Delta) dK_\Delta(k_\Delta) + k(1 - \delta)$ because $K_\Delta(k) = 0$ for small $k$. Hence, the parabola most to the left must be open above because $\frac{A}{2} - (1 - \delta) > 0$. We have already argued that buyers must mix between at least two prices in every period, so we can exclude the possibility of the expected profit function in period 1 having a shape as in figure B.1 (c). Hence, there exist cutoffs $0 < \underline{k}_\Delta \leq \overline{k}_\Delta < 1$ such that buyers at time 0 choose only prices with positive probability that are in $\{0\} \cup [\underline{k}_\Delta, \overline{k}_\Delta]$.

![Figure B.1: Possible shapes of buyers’ profits in period 1](image)

Using these insights about $\pi_0$, we can conclude that $\pi_\Delta$ is piecewise quadratic on $[0, 1] \setminus [\underline{k}_\Delta, \overline{k}_\Delta]$ where the coefficient in front of $(k_\Delta)^2$ is negative as a multiple of $\frac{A}{2} - (1 - \delta) - \delta A < 0$. Hence, all pieces of $\pi_\Delta$ are open below. At every cutoff that is chosen with positive probability in period 1, $\pi_\Delta$ has a kink. Hence, period 2 expected profits are qualitatively as in figure B.2. Note however, that $\pi_\Delta$ does not have to be piecewise quadratic in $[\underline{k}_\Delta, \overline{k}_\Delta]$ as in figure B.2.
Next, we argue that \( \pi_0 \) must look like in figure B.1 (b). Let us first assume that none of the pieces of \( \pi_0 \) is constant and equal to zero as is the case in figure B.1 (a). Then, in a mixed strategy equilibrium, buyers at time 0 mix between exactly two prices that result in cutoff types 0 and \( k_0 = k_\Delta = \overline{k}_\Delta \), respectively. Moreover, \( k_0 \) must be a cutoff type at time \( \Delta \), because it corresponds to a kink of \( \pi_0 \). We can conclude \( \pi_0(k_\Delta) = \pi_\Delta(\overline{k}_\Delta) = 0 \) and \( \pi_\Delta(k) \leq 0 \) for all \( k \geq \overline{k}_\Delta \). In addition, \( \pi_\Delta \) has its only kink at \( \overline{k}_\Delta \), so buyers do not mix between prices in period 2, but choose a price with probability one that results in a cutoff \( \overline{k}_\Delta \). This cannot be an equilibrium as argued before. Hence, there cannot be an equilibrium where none of the pieces of \( \pi_0 \) is constant and equal to zero.

Finally, period 2 buyers must mix between exactly two cutoffs \( \{k_\Delta, \overline{k}_\Delta\} \). This can be seen as follows: One can infer directly from Lemma 14 (ii) that period 2 buyers do not choose prices that result in cutoffs in \( (k_\Delta, \overline{k}_\Delta) \). Moreover, because \( \pi_0(k) = 0 \) on \( \{0\} \cup \{k_\Delta, \overline{k}_\Delta\} \) only, \( \pi_\Delta \) can have kinks in that region only. Hence, \( \pi_\Delta(k_\Delta) = \pi_\Delta(\overline{k}_\Delta) = 0 \), \( \pi_\Delta(k) \leq 0 \) for \( k \geq k_\Delta \) and the fact that \( \pi_\Delta \) is piece-wise quadratic on \( [0, k_\Delta] \cup [\overline{k}_\Delta, 1] \) with parabolas that are open below imply that \( \pi_\Delta(k) > 0 \) for \( k \in (0, k_\Delta) \) and \( \pi_\Delta(k) < 0 \) for \( k \in (\overline{k}_\Delta, 1] \).

Thus, in any equilibrium the support of \( K_0 \) is a subset of \( \{0\} \cup \{k_\Delta, \overline{k}_\Delta\} \) and the support of \( K_\Delta \) is \( \{k_\Delta, \overline{k}_\Delta\} \) for some \( k_\Delta, \overline{k}_\Delta \in (0, 1) \). Let \( K_\Delta(k_\Delta) = q_\Delta \) and \( K_0(0) = q_0 \), noting that we already know from Lemma 13 that \( q_\Delta = \frac{A_1+\frac{\delta}{2}}{\delta(1-\delta+\delta A)} \neq 0 \).

**Step 2:** In any mixed strategy equilibrium, there must be non-offers with positive probability in period 1, i.e. \( q_0 > 0 \).

Let us assume \( q_0 = 0 \) and let us denote the smallest element in the support of \( K_0 \) by \( \underline{k} < 1 \). Note that \( Ak + (1-\delta)B - k(\delta A + (1-\delta)) = (1-\delta)(A-1) + B \geq (1-\delta)B(1-k) \) which is strictly positive for \( B > 0 \) and \( k < 1 \). Hence, there exists an \( \epsilon > 0 \) such that \( Ak + (1-\delta)B - (k+\epsilon)(\delta A + (1-\delta)) > 0 \). Then, \( \pi_\Delta(k+\epsilon) > 0 \) which is a contradiction to \( k_\Delta < \underline{k} \) being in the support of \( K_\Delta \).

**Step 3:** \( k_\Delta = \frac{B(1-\delta)}{A\delta-\delta+1-\frac{2}{2}} \) and \( \overline{k}_\Delta = \frac{B(1-\delta^2)}{A\delta^2-\delta^2+1-\frac{2}{2}} \).

In equilibrium, it must hold that \( \pi_\Delta(k_\Delta) = 0 \), that is

\[
\int_0^{k_\Delta} Ac + (1-\delta)B - k_\Delta(\delta A + 1 - \delta)dc = k_\Delta \cdot \left( \frac{A}{2}k_\Delta + (1 - \delta)B - k_\Delta(\delta A + 1 - \delta) \right) = 0
\]
which is equivalent to $k_\Delta = \frac{B(1-\delta)}{A\delta - \delta + 1 - \frac{\delta}{2}}$. For $k_\Delta$, we use that $\pi_0(k_\Delta) = 0$ since this is equivalent to

$$k_\Delta \cdot \left(\frac{A}{2}k_\Delta + (1 - \delta^2)B - \delta(A - 1)k_\Delta - (1 - \delta)k_\Delta\right) = 0$$

because $K_\Delta(k_\Delta) = 1$. Hence, $k_\Delta = \frac{B(1-\delta^2)}{A\delta^2 - \delta^2 + 1 - \frac{\delta^2}{2}}$.

Proof. (Proposition 5)

First, note that in any equilibrium it must hold that $\pi_\Delta(k_\Delta; K_0) = 0$ and for all $d > k_\Delta$, $\pi_\Delta(k; K_0) \leq 0$, i.e.,

$$\int_0^{k_\Delta} \int_0^c \frac{1}{1-k} dK_0(k)(Ac + B - ((1 - \delta + A\delta)k_\Delta + \delta B))dc = 0$$

$$\int_0^d \int_0^c \frac{1}{1-k} dK_0(k)(Ac + B - ((1 - \delta + A\delta)k + \delta B))dc \leq 0 \ \forall \ d > k_\Delta.$$

Let us first simplify the first equality. By applying Fubini’s Theorem and then, noting that $k_\Delta - k_{\Delta} = 1 + k_\Delta - 1$ and $\frac{k_\Delta - k_{\Delta}}{1-k} = 1 + k_\Delta - 1$, we can deduce

$$\int_0^{k_\Delta} \int_0^c \frac{1}{1-k} dK_0(k)(Ac + B - ((1 - \delta + A\delta)k_\Delta + \delta B))dc$$

$$= A \int_0^{k_\Delta} \frac{k_\Delta^2 - k^2}{1-k} dK_0(k) + ((1 - \delta)B - (1 - \delta + A\delta)k_\Delta) \int_0^{k_\Delta} \frac{k_\Delta - k}{1-k} dK_0(k)$$

$$= A \int_0^{k_\Delta} \frac{k_\Delta^2 - k^2}{1-k} dK_0(k) + \frac{A}{2} \int_0^{k_\Delta} kdK_0(k)$$

$$+ \int_0^{k_\Delta} \frac{1}{1-k} dK_0(k) \left(\frac{1}{1 + \delta} - \frac{A}{2} + (k_\Delta - 1)((1 - \delta)B - (1 - \delta + A\delta)k_\Delta)\right)$$

$$= \frac{A}{2} \cdot \left(1 - \frac{1 + 2\delta}{1 + \delta} \frac{k_\Delta}{1 + \delta} + \int_0^{k_\Delta} kdK_0(k) + (k_\Delta - 1) \left(1 - \delta \frac{k_\Delta}{1 + \delta} \right) \int_0^{k_\Delta} \frac{1}{1-k} dK_0(k)\right).$$

Thus, in equilibrium, the following must hold

$$1 - \frac{1 + 2\delta}{1 + \delta} \frac{k_\Delta}{1 + \delta} + \int_0^{k_\Delta} kdK_0(k) = (1 - k_\Delta) \left(1 - \delta \frac{k_\Delta}{1 + \delta} \right) \int_0^{k_\Delta} \frac{1}{1-k} dK_0(k). \quad (B.1)$$
In order to simplify the second inequality, we can use that \( \pi(k) = 0 \), and see that for \( d > k \)

\[
\int_0^d \int_0^c \frac{1}{1-k} dK_0(k)(Ac + B(1-\delta) - (1-\delta + A\delta)d)dc
\]

\[
= \int_0^\kappa \int_0^c \frac{1}{1-k} dK_0(k)dc(\kappa - d)(1-\delta + A\delta) + \int_\kappa^d \int_0^c \frac{1}{1-k} dK_0(k)(Ac + B(1-\delta) - (1-\delta + A\delta)d)dc
\]

\[
= (d - \kappa) \left[ \int_0^\kappa \frac{1}{1-k} dK_0(k) \left( \frac{A\kappa}{2} + B(1-\delta) - \left(1-\delta + A\delta - \frac{A}{2}\right) \right) \right] > 0
\]

is quadratic in \( d \) and the parabola is open below. The parabola has a zero at \( \kappa \) and we will show in the following that it cannot have another zero. If \( \pi(k') = 0 \) for a \( k' > \kappa \), then \( \pi(k) > 0 \) for \( k \in (\kappa - \epsilon, \kappa) \) which leads to a contradiction. Finally, if there is continuous mixing on some \( (\kappa - \epsilon, \kappa) \), then since the slope from the right of \( \pi(k) \) is negative at \( \pi(k) \), the slope from the left must also be negative because

\[
\frac{\partial}{\partial k} \pi(k) = \frac{\partial}{\partial k} \int_0^k \int_0^c \frac{1}{1-k} dK_0(k)(Ac - (1-\delta + A\delta)k + B(1-\delta))dc
\]

\[
= \int_0^k \frac{1}{1-k} dK_0(k)(1-\delta)(Ak - k + B) - (1-\delta + A\delta) \int_0^k \int_0^c \frac{1}{1-k} dK_0(k)dc
\]

and \( k(A - 1) + B > 0 \). This again cannot hold in equilibrium. As a result, the parabola can only have one zero \( \kappa_2 \) and it follows from by plugging in the value of \( \kappa \) calculated in
Proposition 4 that

\[
\int_0^{\bar{k}_\Delta} \frac{1}{1-k} dK_0(k) \left( \frac{A}{2} \bar{k}_\Delta + B(1-\delta) \right) - \int_0^{\bar{k}_\Delta} \int_0^c \frac{1}{1-k} dK_0(k) dc (1-\delta + A\delta) = \bar{k}_\Delta
\]

\[
\int_0^{\bar{k}_\Delta} \frac{1}{1-k} dK_0(k)(1-\delta + A\delta - \frac{A}{2})
\]

\[
\leftrightarrow \frac{A}{2} \bar{k}_2 + B(1-\delta) - \int_0^{\bar{k}_\Delta} \int_0^c \frac{1}{1-k} dK_0(k) dc (1-\delta + A\delta) = B(1-\delta^2) \frac{(1-\delta - \frac{A}{2} + A\delta)}{(1-\delta^2 - \frac{A}{2} + A\delta^2)}
\]

\[
\leftrightarrow \frac{A}{2} \bar{k}_2 \cdot \frac{1}{1-\delta^2 + A\delta + A\delta^2} = \frac{1}{1} \int_0^{\bar{k}_\Delta} \frac{1}{1-k} dK_0(k) - \int_0^{\bar{k}_\Delta} \frac{1}{1-k} dK_0(k)
\]

\[
\leftrightarrow \frac{A}{2} \bar{k}_2 \cdot \frac{1}{1-\delta^2 + A\delta + A\delta^2} = 1 + (\bar{k}_\Delta - 1) \int_0^{\bar{k}_\Delta} \frac{1}{1-k} dK_0(k)
\]

This proves (2.12). Plugging (2.12) into (B.1), shows (2.11). \(\square\)

Welfare analysis

Proof. (Theorem 2)

(i) In period 1, we can easily calculate the difference between the expected cutoff with private offers and the private offer period 1 cutoff using (2.9), (2.10), Proposition 5 and Proposition 4 and see that it is positive:

\[
\int_0^{\bar{k}_\Delta} \frac{1}{1-k} dK_0(k)(1-\delta + A\delta) + \int_0^{\bar{k}_\Delta} \frac{1}{1-k} dK_0(k) - \int_0^{\bar{k}_\Delta} \frac{1}{1-k} dK_0(k)
\]

\[
= \frac{1}{2} \bar{k}_\Delta \cdot \frac{2A(1-\delta) - (A-1)^2(1-\delta)^2 - 3(1-\delta)^2 + \delta^2 A}{(4-4A + A^2)(1-2\delta + \delta^2) + 2A - 2A\delta^2 + 2A^2\delta^2}
\]

\[
\geq \frac{1}{2} \bar{k}_2 \cdot \frac{(1-\delta)^2 - (A-1)^2(1-\delta)^2 + \delta^2 A}{(4-4A + A^2)(1-2\delta + \delta^2) + 2A - 2A\delta^2 + 2A^2\delta^2}
\]

for \(\delta > 1 - \frac{A}{2}\) and \(A + B = 1\). In period 2, the difference between the cutoffs in the two
information structures is given by
\[
\int_0^1 kd K_\Delta(k) - k_\Delta^* = qk_\Delta + (1-q)\bar{k}_\Delta - \frac{2B \cdot (A\delta - 2\delta + 2) \cdot (1-\delta)}{2(1-\delta)(1-A)(A\delta - 2\delta + 2) + A^2} \\
= \frac{\delta}{2} - (1-\delta) \frac{B(1-\delta)}{A\delta - \delta + 1 - \frac{\delta}{2}} + \left(1 - \frac{\delta}{2} - (1-\delta) \frac{B(1-\delta^2)}{A\delta^2 - \delta^2 + 1 - \frac{\delta}{2}} \right) \\
= - \frac{2B \cdot (A\delta - 2\delta + 2) \cdot (1-\delta)}{2(1-\delta)(1-A)(A\delta - 2\delta + 2) + A^2} \\
= - \frac{A^2 B (-1 + \delta)(-2 + A + 2\delta)}{(1-\delta + A\delta)(2 - A - 2\delta + 2A\delta)((1-\delta)^2(2 - A)^2 + 2A\delta(1-\delta) + A^2\delta^2)}.
\]

Note that this function is zero if and only if the numerator is zero which is a quadratic function in \( \delta \). It is easy to check that the numerator is zero if and only if \( \delta \in \{1 - \frac{A}{2}, 1\} \). Hence, we can conclude that the function is positive in \([1 - \frac{A}{2}, 1]\) because it is positive at \( \delta = 1 - \frac{A}{4} \) and there are no singularities for \( \delta \in [0, 1] \).

(ii) is shown in the main text. \( \square \)

Robustness and Generalizations

General Valuations and Distributions

Proof. (Theorem 3)

(i): It remains to show that the functions \( \kappa_t(\cdot) \) inductively defined for \( t = 0, \ldots, 1 - \Delta \) by (2.13) and (2.14) are left-continuous.

Step 1: If \( \pi_t(k; k_{t-\Delta}) \) is left-continuous in \( k \), then \( \kappa_t \) is increasing
First note, that because of left-continuity of \( \pi_t(\cdot; k_{t-\Delta}) \), we either have \( \pi_t(\kappa_t(k_{t-\Delta}), k_{t-\Delta}) > 0 \) or \( \pi_t(\kappa_t(k_{t-\Delta}), k_{t-\Delta}) = 0 \). Moreover, note that \( \pi_t(k; k_{t-\Delta}) \) is always differentiable in \( k_{t-\Delta} \). Let us consider an arbitrary \( k_{t-\Delta} \) and an infinitesimal increase in \( k_{t-\Delta} \). If \( \pi_t(\kappa_t(k_{t-\Delta}), k_{t-\Delta}) > 0 \), there exists an \( \epsilon > 0 \) so that \( \pi_t(\kappa_t(k_{t-\Delta}), k_{t-\Delta} + \gamma) > 0 \) for all \( \gamma < \epsilon \). Hence, \( \kappa_t(k_{t-\Delta} + \gamma) > \kappa_t(k_{t-\Delta}) \) for all \( \gamma < \epsilon \). On the other hand, if \( \pi_t(\kappa_t(k_{t-\Delta}), k_{t-\Delta}) = 0 \), then
\[
\left. \frac{\partial}{\partial k_{t-\Delta}} \pi_t(k; k_{t-\Delta}) \right|_{k=\kappa_t(k_{t-\Delta})} = \frac{\int_{k_{t-\Delta}}^{\kappa_t(k_{t-\Delta})} f(y) \, dc}{(1 - F(k_{t-\Delta}))^2} \int_{k_{t-\Delta}}^{\kappa_t(k_{t-\Delta})} (v(c) - p_t(\kappa_t(k_{t-\Delta}))) f(c) \, dc - (v(k_{t-\Delta}) - p_t(c_t^0(k_{t-\Delta}))) \\
= - \frac{\int_{k_{t-\Delta}}^{\kappa_t(k_{t-\Delta})} (v(c) - p_t(\kappa_t(k_{t-\Delta}))) f(c) \, dc}{1 - F(k_{t-\Delta})} > 0
\]
because if \( v(k_{t-\Delta}) - p_t(\kappa_t(k_{t-\Delta})) \geq 0 \), then \( \int_{k_{t-\Delta}}^{\kappa_t(k_{t-\Delta})} (v(c) - p_t(\kappa_t(k_{t-\Delta}))) f(c) \, dc > 0 \) (note that \( v \) is increasing) contradicting the zero-profit assumption \( \pi_t(\kappa_t(k_{t-\Delta}), k_{t-\Delta}) = 0 \). Hence, \( \kappa_t(\cdot) \) is increasing at \( k_{t-\Delta} \).
Step 2: \( \kappa_t(\cdot), p_t(\cdot) \) and \( \pi_t(\cdot; k_{t-\Delta}) \) are left-continuous

We argue by backward induction in \( t \). \( p_{1-\Delta}(\cdot) \) is left-continuous because \( v \) is continuous and hence, \( \pi_{1-\Delta}(k_{1-2\Delta}; k) \) is left-continuous in \( k \). (It is even continuous.) Let \( k_{1-2\Delta}^{(n)} \uparrow k_{1-2\Delta} \). Then, \( \kappa_{1-\Delta}^{*}(k_{1-2\Delta}^{(n)}) \leq \kappa_{1-\Delta}(k_{1-2\Delta}) \) for all \( n \) and \( \kappa_{1-\Delta}(k_{1-2\Delta}^{(n)}) \) is an increasing sequence by step 1. Hence, \( \lim_{n \to \infty} \kappa_{1-\Delta}(k_{1-2\Delta}^{(n)}) \) exists. We will show next that \( \lim_{n \to \infty} \kappa_{1-\Delta}(k_{1-2\Delta}^{(n)}) = \kappa_{1-\Delta}(k_{1-2\Delta}) \). Therefore, consider an arbitrary sequence \( k_{1-2\Delta}^{(m)} \uparrow \kappa_{1-\Delta}(k_{1-2\Delta}) \) such that \( \pi_{1-\Delta}(k_{1-2\Delta}^{(m)}; k_{1-2\Delta}) > 0 \) (which must exist by definition of \( \kappa_{1-\Delta} \)). Then, for any \( m \), there exists an \( n(m) \) such that \( \pi_{1-\Delta}(k_{1-2\Delta}^{(m)}; k_{1-2\Delta}^{(n)}) > 0 \) for all \( n \geq n(m) \) because \( \pi_{1-\Delta}(k_{1-2\Delta}^{\cdot}; \cdot) \) is continuous for all \( k \). Hence, \( k_{1-2\Delta}^{(m)} \leq \kappa_{1-\Delta}(k_{1-2\Delta}^{(m)}) \leq \kappa_{1-\Delta}(k_{1-2\Delta}) = \lim_{m \to \infty} k_{1-2\Delta}^{(m)} \). Hence, \( \lim_{n \to \infty} \kappa_{1-\Delta}(k_{1-2\Delta}^{(n)}) = \lim_{m \to \infty} \kappa_{1-\Delta}(k_{1-2\Delta}^{(m)}) = \kappa_{1-\Delta}(k_{1-2\Delta}) \) and thus, \( \kappa_{1-\Delta}(\cdot), p_{1-\Delta}(\cdot) \) and \( \pi_{1-\Delta}(\cdot; k_{1-2\Delta}) \) are left-continuous.

Let now assume that \( c_{t+\Delta}(\cdot), p_{t}(\cdot) \) and \( \pi_{t}(\cdot; k_{t-\Delta}) \) are left-continuous. Hence, \( \kappa_{t}(\cdot) \) is increasing by step 1. The rest of the argument works analogously to above, so that \( \kappa_{t}(\cdot), p_{t}(\cdot) \) and \( \pi_{t}(\cdot; k_{t-2\Delta}) \) are left-continuous for all \( t \).

(ii) is proven in the main part of the paper.

(iii) Let us denote for a given \( \delta \) the set of all possible equilibrium cutoffs of a private offer game by \( PE^{private}(\delta) \) and for a public offer game in which the zero-profit condition is satisfied by \( PE^{public}(\delta) \), respectively. Let us assume that for any \( \delta^* \) there exists a \( \delta > \delta^* \) such that \( (k_{0}(\delta), \ldots, k_{1-\Delta}(\delta)) \in PE^{private}(\delta) \). Hence, there exists a sequence \( \delta_{n} \to \delta_{\infty} \) 1 and a corresponding sequence \( (k_{0}(\delta_{n}), \ldots, k_{1-\Delta}(\delta_{n})) \in PE^{private}(\delta_{n}) \). Let us define a function \( \delta \mapsto (k_{0}(\delta), \ldots, k_{1-\Delta}(\delta)) \) such that \( k_{t}(\delta_{n}) = k_{t}^{*}(\delta_{n}) \) for all \( t \) and infinitely many \( n \) (which is possible because \( k_{t}^{*}(\delta) \to \delta \) as \( t \) increases) continuously differentiable and \( k_{t-\Delta} < k_{t} \) for all \( t \). Hence, \( k_{t}(\cdot) \) is a smoothly interpolated version of \( k_{t}^{*} \). Recall that by (2.17) at any equilibrium \( (k_{0}^{*}, \ldots, k_{1-\Delta}^{*}) \in PE^{private}(\delta) \) the marginal profit of buyers at time \( t < 1 - \Delta \) is given by (2.17), i.e. with private offers

\[
\frac{\partial}{\partial k_{t}} \pi_{t}(k_{t}^{*}; k_{t-\Delta}^{*}) = F(k_{t}^{*}) - F(k_{t-\Delta}^{*}) \left[ f(k_{t}^{*}) \left( \frac{v(k_{t}^{*})}{F(k_{t}^{*})} - \frac{\mathbb{E}[v(c)|[k_{t-\Delta}^{*}, k_{t}^{*}]]}{F(k_{t}^{*}) - F(k_{t-\Delta}^{*})} \right) \right] - (1 - \delta)
\]

by the zero profit condition (2.15). Note that since \( k_{t}(\cdot) \) are continuously differentiable, by
applying l’Hopital’s Lemma twice, we can write
\[
\lim_{\delta \to 1} \frac{v(k_t(\delta))(F(k_t(\delta)) - F(k_{t-\Delta}(\delta))) - f^{k_t(\delta)} v(c) dc}{(F(k_t(\delta)) - F(k_{t-\Delta}(\delta)))^2}
\]
\[
= \lim_{\delta \to 1} \frac{v'(k_t(\delta)) \frac{\partial k_t(\delta)}{\partial \delta}}{2 \left( f(k_t(\delta)) \frac{\partial k_t(\delta)}{\partial \delta} - f(k_{t-\Delta}(\delta)) \frac{\partial k_{t-\Delta}(\delta)}{\partial \delta} \right)}
\]
\[
+ \frac{f(k_{t-\Delta}(\delta)) \frac{\partial k_{t-\Delta}(\delta)}{\partial \delta}}{f(k_t(\delta)) \frac{\partial k_t(\delta)}{\partial \delta} - f(k_{t-\Delta}(\delta)) \frac{\partial k_{t-\Delta}(\delta)}{\partial \delta}} \cdot \frac{v'(k_t(\delta)) - v(k_t(\delta))}{2 \left( f(k_t(\delta)) \frac{\partial k_t(\delta)}{\partial \delta} - f(k_{t-\Delta}(\delta)) \frac{\partial k_{t-\Delta}(\delta)}{\partial \delta} \right)}
\]
\[
= \lim_{\delta \to 1} \frac{v'(0) \partial k_t(1) / \partial \delta}{2 f(0) \left( \frac{\partial k_t(1)}{\partial \delta} - \frac{\partial k_{t-\Delta}(1)}{\partial \delta} \right)} - \frac{f(0) \frac{\partial k_{t-\Delta}(1)}{\partial \delta}}{2 f(0)^2 \left( \frac{\partial k_t(1)}{\partial \delta} - \frac{\partial k_{t-\Delta}(1)}{\partial \delta} \right)^2}
\]
\[
= \frac{v'(0)}{2 f(0)} > 0
\]
if \( f(0) \geq 0 \), because the equilibrium cutoffs converge all to 0 as \( n \to \infty \). We can conclude
\[
\lim_{\delta \to 1} \left[ f(k_t^*) \left( \frac{v(k_t)}{F(k_t) - F(k_{t-\Delta})} - \frac{\mathbb{E}[v(c)|k_{t-\Delta}, k_t]}{F(k_t) - F(k_{t-\Delta})} \right) - (1 - \delta) \right] > 0.
\]
Thus, there exists a \( n^* \) such that for all \( n > n^* \) \( (k_0(\delta_n), \ldots, k_{t-\Delta}(\delta_n)) \notin PE^{Private}(\delta_n) \) which is a contradiction. Hence, there must exist a \( \delta^* < 1 \) such that for all \( \delta > \delta^* \), there does not exist a pure strategy equilibrium with private offers.

Proof. (Proposition 6)

Let \( B = 0 \) and \( A < 2 \). The zero-profit condition implies that the cutoff at time 0 must satisfy
\[
p_0 = v \left( \frac{k_0}{2} \right) = \frac{A}{2} k_0.
\]
Moreover, it must hold that
\[
p_0 \geq (1 - \delta) k_0 + \delta p_1 \geq k_0.
\]
However, this can never hold simultaneously for \( A < 2 \) except if \( k_0 = 0 \). Hence, in the unique pure strategy equilibrium (with private and public offers), there is no trade before the deadline.
B.2 Equilibrium with two prices in both periods for \( \alpha \in [0, 1] \)

With public offers, the equilibrium cutoffs are given by

\[
\begin{align*}
    k_0^* &= \frac{2B \cdot (2(1 - \delta)(1 - \alpha \delta) + Aa\delta(1 - \delta) - A(1 - \alpha \delta))}{2(1 - \delta)(1 - A)(Aa\delta - 2a\delta + 2) + A^2} \\
    k_\Delta^* &= \frac{2B \cdot (2(1 - \delta)(1 - \alpha \delta) + Aa\delta(1 - \delta))}{2(1 - \delta)(1 - A)(Aa\delta - 2a\delta + 2) + A^2}.
\end{align*}
\]

With private offers, buyers in period 2 mix between the cutoffs

\[
\begin{align*}
    k_\Delta &= \frac{B(1 - \alpha \delta)}{Aa\delta - a\delta + 1 - \frac{A}{2}}, \\
    k_\Delta &= \frac{B(1 - \alpha \delta^2)}{Aa\delta^2 - a\delta^2 + 1 - \frac{A}{2}},
\end{align*}
\]

where they choose \( k_\Delta \) with probability \( q_\Delta = \frac{A - (1 - \delta)}{\delta(Aa\delta + 1 - a\delta)} \). In order to illustrate the tradeoffs with general \( \alpha \), we focus on the equilibrium, in which buyers in the first period also mix between exactly two cutoffs and with \( A + B = 1 \). Buyers in the first period choose a cutoff of 0 with probability

\[
q_0 = \frac{(1 - \alpha \delta)^2(-2(-1 + A)\alpha \delta^2 + (-2 + A))}{2(-2(-1 + A)\alpha^3 \delta^5 + (-2 + A) - 2\alpha \delta(-2\delta + A\delta - 1 + A) + \alpha^2 \delta^2(-2\delta^2 + (-4 + 6A)\delta - A))}
\]

and the cutoff

\[
k_0 = -\frac{2B(2(1 - A)\alpha^3 \delta^5 - (2 - A) - 2\alpha \delta(-2 - A)(1 + \delta) + 1) + \alpha^2 \delta^2(-2\delta^2 + (-4 + 6A)\delta - A))}{-4(1 - A)^2 \alpha^3 \delta^5 + (2 - A)^2 - 4(1 - A)\alpha^2 \delta^3(-\delta + 2(-1 + A)) + (2 - A)\alpha\delta(-2(\delta + 1) + A(2\delta + 3))}
\]

with probability \( 1 - q_0 \).

B.3 Equilibrium with two periods and two types

Consider a situation with \( v_H \geq c_H \) and \( v_L > c_L = 0 \) where the seller’s valuation is \( v_L \) with probability \( \phi \). The static lemon’s condition (LC) is satisfied if

\[
\phi v_L + (1 - \phi)v_H < c_H.
\]

Finally, denote the fraction of \( v_L \)-sellers such that the lemons condition is just satisfied by \( \phi^* \), i.e.,

\[
\phi^* v_L + (1 - \phi^*)v_H = c_H.
\]

We solve the situation by backward induction. Given the belief \( \phi_2 \) about the fraction of \( v_L \)-sellers in the market, buyers’ expected period 2 profits are given by

\[
\pi_2(p) = \begin{cases} 
\phi_2 v_L + (1 - \phi_2)v_H - p & \text{if } p \geq c_H \\
v_L - p & \text{if } c_L < p < c_H \\
0 & \text{otherwise}
\end{cases}
\]
if they sell at a price $p$. Since buyers compete in a Bertrand fashion, the equilibrium price is

$$p^*_2(\phi_2) = \begin{cases} 
\phi_2 v_L + (1 - \phi_2) v_H & \text{if } \phi_2 < \phi^* \\
\{\phi_2 v_L + (1 - \phi_2) v_H, v_L\} & \text{if } \phi_2 = \phi^* \\
v_L & \text{if } \phi_2 > \phi^* 
\end{cases}$$

If (LC) is satisfied with $\phi_2$, only low types trade and $p^*_2 = v_L$.

The price in in the continuation equilibrium is as in figure B.3. If (LC) is satisfied, then

<table>
<thead>
<tr>
<th>Figure B.3: Period 2 price</th>
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</thead>
<tbody>
<tr>
<td>$p_2$</td>
</tr>
<tr>
<td>$c_H$</td>
</tr>
<tr>
<td>$v_L$</td>
</tr>
<tr>
<td>$v_H$</td>
</tr>
<tr>
<td>$\phi^*_2$</td>
</tr>
</tbody>
</table>

the period 1 price is always $p_1 = v_L$. Moreover, the following holds:

1. If $\delta \leq \frac{v_L}{v_H}$, then all $v_L$-sellers trade in period 1 and $p_2 = v_H$.
2. If $\frac{v_L}{v_H} < \delta < \frac{v_L}{c_H}$, then in period 1 enough $v_L$-sellers trade such that in period 2

$$\phi_2 v_L + (1 - \phi_2) v_H = \frac{v_L}{\delta}.$$ 

Note that $\phi_2 < \phi$, such that in period 2, $p_2 = \phi_2 v_L + (1 - \phi_2) v_H$.

3. If $\frac{v_L}{c_H} < \delta$, then in period 1 enough $v_L$-sellers trade such that in period 2 $\phi_2 = \phi^*$, such that in period 2, buyers are indifferent between bidding $c_H = \phi^* v_L + (1 - \phi^*) v_H$ and $v_L$. They mix between the two such that

$$\mathbb{E}[p_2] = \frac{v_L}{\delta}.$$ 

These are by construction all equilibria with both private and public offers.
Appendix C

Appendix to Chapter 3

C.1 Proofs

Dynamics and Welfare

Proof. (Proposition 8)

(i) and (iii) follow immediately from the definitions. (ii) can be shown by noting that whenever $m$ is differentiable,

$$
(\phi^+)'(\phi) = \frac{(1 - \phi^+(\phi))f(c - m(\phi))m'(\phi) + \phi^+(\phi)f(c - m(1 - \phi))m'(1 - \phi)}{L(\phi)}.
$$

Proof. (Proposition 9)

(i) follows immediately from

$$
W'(\phi) = (1 - F(\kappa^\phi(\phi))) \cdot m'(\phi) - (1 - F(\kappa^\mu(\phi))) \cdot m'(1 - \phi).
$$

and $W'(\frac{1}{2}) = 0$. (ii) follows from

$$
W''(\phi) = f(\kappa^\phi(\phi))m'(\phi)^2 + f(\kappa^\mu(\phi))m'(1 - \phi)^2
+ (1 - F(\kappa^\phi(\phi)))m''(\phi) + (1 - F(\kappa^\mu(\phi)))m''(1 - \phi).
$$

(iii) follows from (ii) and Proposition 8. \qed
Affirmative Action Policies

Proof. (Proposition 10)

The objective function of (3.4) is strictly concave since
\[ \frac{\partial^2 W}{(\partial l)^2} (l, \bar{\phi} | \phi) = -(1 - \bar{\phi})^2 - \bar{\phi}^2 \]
is negative for all \( \bar{\phi} \in (0, 0.5) \). To complete the proof, it is thus sufficient to show that we can improve on \( W(\cdot, \bar{\phi} | \phi) \) by decreasing the labor force from \( l = \frac{1 - \phi^+(\phi)}{1 - \bar{\phi}} l^*(\phi^+(\phi) | \phi) \) and by increasing it from \( l = l^*(\phi^+(\phi) | \phi) \). Since \( \frac{\phi^+(\phi)}{\bar{\phi}} l^*(\phi^+(\phi) | \phi) \leq 1 - \phi \), this also implies that additional women will be hired.

First, let us assume firms keep hiring all \( 1 - c + m(1 - \phi) = (1 - \phi^+(\phi)) l^*(\phi^+(\phi) | \phi) \) positive productivity men and add the best available women until they reach the desired quota \( \bar{\phi} \). Now, consider the change in the objective function if they reduce the labor force by \( \Delta l > 0 \). This corresponds to letting go of the worst \( (1 - \bar{\phi}) \Delta l \) men with mean productivity \( \frac{1 - \bar{\phi}}{2} \Delta l > 0 \), as well as the worst \( \bar{\phi} \Delta l \) women with mean productivity \( 1 - c + m(\phi) - \bar{\phi} \cdot \ell + \frac{\bar{\phi}}{2} \Delta l < 0 \). As \( \Delta L \to 0 \), only the female labor force has a marginal effect on welfare, implying that
\[ \frac{\partial W}{\partial l} (\ell, \bar{\phi} | \phi) = \bar{\phi} \cdot (1 - c + m(\phi) - \bar{\phi} \cdot \ell) < 0. \]

Similarly, labor force \( \ell \) corresponds to hiring the same overall mass of people as in an unregulated market. As a result, the least productive male employee must have a productivity \( p > 0 \) that exactly compensates that of the lowest female employee \( -p \). This is because an unregulated firm would hire \( p \) more men and, if labor size is to remain unchanged, \( p \) less women. However, this also implies that increasing the labor force by \( \Delta \ell > 0 \) adds \( \bar{\phi} \Delta l \) women with negative mean productivity \( -p \) as well as \( (1 - \bar{\phi}) \Delta \ell \) men with positive mean productivity \( p \). Again, as \( \Delta \ell \to 0 \), we obtain
\[ \frac{\partial W}{\partial \ell} (\ell, \bar{\phi} | \phi) = \bar{\phi} \cdot (-p) + (1 - \bar{\phi}) \cdot p = p \cdot (1 - 2\bar{\phi}) > 0. \]

As a result, we conclude that the optimal labor force is between these two extremes, \( l^*(\bar{\phi} | \phi) \in (\ell, \bar{l}) \), from which the result follows.

Proof. (Proposition 11)

There exists a one-to-one mapping between cutoffs \( (\kappa_\phi, \kappa_\mu) \) and ratio-labor force pairs \( (\bar{\phi}, l) \),
\[ \bar{\phi} = \frac{1 - \kappa_\phi}{2 - \kappa_\phi - \kappa_\mu} \quad \text{and} \quad l = 2 - \kappa_\phi - \kappa_\mu. \]

As a result, one can equivalently perform the optimization on the second pair of variables – and since \( l \) doesn’t affect the outcome in subsequent periods, the solution is given by the one-period optimum of (3.4).
Proof. (Proposition 12)

When the constraint isn’t binding at the optimum, the result follows trivially. And when it does, then any $\pm dl$ change in male employment needs to be compensated by a $\pm dl$ change in female employment. At the optimal $(\phi^*, l^*)$, such a change cannot have any impact on the objective value, yielding the bellman equation

$$
\frac{1}{d\ell} \left[ d\ell \cdot \left(1 - (1 - \phi^*)\ell - c + m(1 - \phi) + O(d\ell)\right) + d\ell \cdot \left(1 - \phi^* \ell - c + m(\phi) + O(d\ell)\right) \right] \to 0,
$$

from which it follows that $\ell^* = 2 - 2c + m(1 - \phi) + m(\phi)$ independent of the fraction $\phi^*$ or the upper bound $\overline{\kappa}$.

Proof. (Proposition 13)

Assume $M \leq c$. Let the current fraction of female workers be $\phi$. $\phi^- : (0, 1) \to (0, 1)$ be the inverse of $\phi^+$ for $\phi \in (0, 1) \setminus \{\frac{1}{2}\}$ and $\phi^- (\frac{1}{2}) = s'$. Hence,

$$
\phi^-(\phi) := \frac{s - \frac{s}{M} (c - 1)}{1 - \phi} - s + 2\frac{s}{M} (c - 1).
$$

First note, that the best quota $\bar{\phi}$ that results in $\phi^+(\bar{\phi}) = \frac{1}{2}$ is the smallest such quota, i.e., $\bar{\phi} = s' < s$, because it minimizes the inefficiencies in the current period. Similarly, a quota $\bar{\phi}$ that results in $\phi^+(\bar{\phi}) \in \{0, 1\}$ in the next period can never be optimal.

Consider a situation in which the policy maker wishes to reach a fraction $\bar{\phi} \leq s$ after two periods. We need to show that it is optimal to set a quota of $\phi^- (\bar{\phi})$ in the first period if $c$ is sufficiently small and $\delta$ is sufficiently large. Let the total welfare if a quota of $\bar{\phi}$ is set in the first period and a quota of $\bar{\phi}$ is set in period two by $V(\phi \to \bar{\phi} \to \bar{\phi})$. Moreover, let the total welfare in a given period with fraction $\phi$ of female workers and no regulation by $W(\phi)$ and let the total continuation welfare be $V(\phi)$. Let us denote the equilibrium female labor force by $\ell_\phi (\phi) = 1 - (c - \frac{M}{s} \phi)$.

We need to show that the solution of

$$
\max_{\bar{\phi} \leq \phi^- (\bar{\phi})} V(\phi \to \bar{\phi} \to \bar{\phi})
$$

is $\phi^- (\bar{\phi})$. This can be done by showing that $V(\phi \to \bar{\phi} \to \bar{\phi})$ is increasing in $\bar{\phi}$. Note that

$$
V(\phi \to \bar{\phi} \to \bar{\phi}) = W(\phi^- (\bar{\phi})) - \ell_\phi (\phi^- (\bar{\phi}))(m(\phi^- (\bar{\phi})) - m(\phi))
+ \delta \left[ V(\phi^- (\bar{\phi})) - \ell_\phi (\phi^- (\bar{\phi}))(m(\phi^- (\bar{\phi})) - m(\bar{\phi})) \right].
$$
We can write
\[ \frac{\partial}{\partial \phi} V(\phi \to \tilde{\phi} \to \bar{\phi}) = \left( \frac{M}{s} \right)^2 (\phi - \phi^-(\bar{\phi})) \frac{s - \frac{\delta M (c - 1)}{(1 - \bar{\phi})^2}}{1 - \phi} + \delta \frac{M}{s} \left( 1 - \frac{c}{s} \phi^-(\bar{\phi}) \right) \]
\[ > \left( \frac{M}{s} \right)^2 \left[ \delta \phi^- (\bar{\phi}) - \frac{s \phi^- (\bar{\phi})}{(1 - \bar{\phi})^2} \right] - \frac{M}{s} (c - 1) \left[ \frac{\delta}{\phi^- (\bar{\phi})} \right] \]

Since \( x - s \cdot \frac{x}{(1-x)^2} > 0 \) for all \( s, x \in (0, 0.5) \), it follows immediately, that there exists a \( c^* > 1 \) such that for all \( c < c^* \) there exists a \( \delta^* \) such that for all \( \delta > \delta^* \), \( \frac{\partial}{\partial \phi} V(\phi \to \tilde{\phi} \to \bar{\phi}) > 0 \). This concludes the proof. \( \square \)

**Saturated markets with fixed labor force**

*Proof. (Proposition 15)*

The proof works analogous to the proof of Proposition 13. More precisely, if in the first period a quota of \( \phi^- \) is set and in the second period a quota of \( \bar{\phi} \) is set, then the total discounted welfare is given by
\[ V(\phi \to \tilde{\phi} \to \bar{\phi}) = W(\phi^- (\phi)) - \bar{\phi} (m(\phi^- (\phi)) - m(\phi)) + \delta \left[ V(\phi^- (\phi)) - \bar{\phi} (m(\phi^- (\phi)) - m(\phi)) \right] \]
with \( \phi^- \) being the inverse of \( \phi^+ \). However, now the policy also affects the number of males hired. Thus, for \( \phi < \bar{\phi} < \phi^- \leq s \),
\[ \frac{\partial}{\partial \phi} V(\phi \to \tilde{\phi} \to \bar{\phi}) = \left( 1 - \phi + m(\phi^- (\phi)) \right) + \bar{\phi} \cdot \frac{M}{s} \cdot \frac{\partial}{\partial \phi} \phi^- (\bar{\phi}) - (\phi + 1) \]
\[ - (m(\phi^- (\phi)) - m(\phi)) - \bar{\phi} \cdot \frac{M}{s} \cdot \frac{\partial}{\partial \phi} \phi^- (\bar{\phi}) + \delta \cdot \frac{M}{s} \cdot \bar{\phi} \]
\[ = \frac{M}{s} \left( \phi + \delta \bar{\phi} \right) - 2\bar{\phi} \]
\[ > \frac{M}{s} \delta \bar{\phi} - 2\phi^- (\bar{\phi}) \]
\[ = \left( \frac{M}{s} \delta - \frac{4s}{M} \right) \bar{\phi} - \frac{2s (M - 1)}{M} \]
\[ > \frac{(M - 1)s}{M} \left( \frac{M}{s} \right)^2 \delta - 4 \left( \frac{M}{s} \right) - 2 \]
using $\bar{\phi} < \phi - (\bar{\phi}) = \frac{2s}{M} \left( \frac{M-1}{2} + \bar{\phi} \right)$ and $\bar{\phi} > \frac{M-1}{M-2}$. This expression is greater than zero for sufficiently large $\delta$ whenever

$$\frac{(M-1)s}{M} \left( \frac{\left( \frac{M}{s} \right)^2 - 4}{\frac{M}{s} - 2} \right) > 0.$$ 

This is always satisfied since $M > 1$.

\begin{flushright}
$\square$
\end{flushright}

### C.2 Solution to differential equation

Let $x(t)$ be given by the differential equation

$$\dot{x}(t) = f(t) - g(t)x(t)$$

with initial value $x(0)$. We use this differential equation frequently throughout the paper. Then, the solution to this equation is given by

$$x(t) = \int_0^t e^{-\int_r^t g(r)dr} \left( f(s) - x(0)g(s) \right) ds + x(0).$$