
FIBONACCI NUMBERS IN COIN TOSSING SEQUENCES

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The Fibonacci numbers and their generating function appear in a natural way in the problem of computing the expected number [2] of tosses of a fair coin until two consecutive heads appear. The problem of finding the expected number of tosses of a p-coin until k consecutive heads appear leads to classical generalizations of the Fibonacci numbers.

First consider tossing a fair coin and waiting for two consecutive heads. Let $O_n$ be the set of all sequences of $H$ and $T$ of length $n$ which terminate in $HH$ and have no other occurrence of two consecutive heads. Let $S_n$ be the number of sequences in $O_n$. Any sequence in $O_n$ either begins with $T$, followed by a sequence in $O_{n-1}$, or begins with $HT$ followed by a sequence in $O_{n-2}$. Thus,

\begin{equation}
S_n = S_{n-1} + S_{n-2}, \quad S_1 = 0, \quad S_2 = 1.
\end{equation}

Consequently, $S_{n-2} = F_n$, the $n$th Fibonacci number. The probability of termination in $n$ trials is $S_n/2^n$. Letting

\begin{equation}
g(x) = \sum_{n=1}^{\infty} S_n x^n,
\end{equation}

and using the generating function $(1 - x - x^2)^{-1}$ for the Fibonacci numbers, yields $g(x) = x^2/(1 - x - x^2)$. Hence, the expected number of trials is

\[
\sum_{n=1}^{\infty} nS_n/2^n = (1/2)g'(1/2) = 6.
\]

We generalize this result to the following

**Theorem:** Consider tossing a p-coin, $Pr(H) = p$, repeatedly until $k$ consecutive heads appear. If $P_n$ is the probability of terminating in exactly $n$ trials (tosses), then the generating function

\begin{equation}
G(x) = \sum_{n=1}^{\infty} P_n x^n
\end{equation}

is given by $G(x) = \frac{(px)^k (1 - px)}{1 - x + (1 - p)(px)^{k+1}}$.

The expected number of trials, $G'(1)$ is

\begin{equation}
1/p + 1/p^2 + \cdots + 1/p^k = \frac{1}{1 - p} \left[ \frac{1}{p^k} - 1 \right].
\end{equation}

**Proof:** Let $O_n$ be the set of all sequences of $H$ and $T$ of length $n$ which terminate in $k$ heads and have no other occurrence of $k$ consecutive heads. Let $S_n$ be the number of sequences in $O_n$ and $P_n = Pr(O_n)$ be the probability of the event $O_n$. One possibility is that a sequence in $O_n$ begins with a $T$, followed by a sequence in $O_{n-1}$; the probability of this is
The next possibility to consider is that a sequence in $O_n$ begins with HT, followed by a sequence in $O_{n-2}$; this has probability

$$\Pr(HT)\Pr(O_{n-2}) = qP_{n-2}. $$

Continuing in this way, the last possibility to be considered is that a sequence in $O_n$ begins with $k-1$ H's followed by a T and then by a sequence in $O_{n-k}$, the probability of which is $qP_{n-k-1}$. Hence, the recursion:

$$P_n = qP_{n-1} + qpP_{n-2} + \cdots + qp^{k-1}P_{n-k},$$

$$P_1 = P_2 = \cdots = P_{k-1} = 0, P_k = p^k.$$  

(Note that the probability of achieving $k$ heads with $k$ tosses is $p^k$, while with less than $k$ tosses it is impossible.) The technique to find the generating function for the Fibonacci numbers applies to finding

$$G(x) = \sum_{k=0}^{\infty} P_n x^n.$$  

Consider

$$H(x) = \sum_{n=k}^{\infty} P_{n+1} x^n;$$

then

$$xH(x) = \sum_{n=k}^{\infty} P_{n+1} x^n + 1 = \sum_{n=k}^{\infty} P_n x^n - P_k x^k = G(x) - (px)^k.$$  

Hence,

$$H(x) = [G(x) - (px)^k]/x.$$  

On the other hand,

$$H(x) = \sum_{k=0}^{\infty} P_{n+1} x^n = \sum_{k=0}^{\infty} (qP_n + qpP_{n-1} + \cdots + qp^{k-1}P_{n-k+1})x^n$$

$$= q \sum_{k=0}^{\infty} P_n x^n + qp \sum_{k=0}^{\infty} P_{n-1} x^{n-1} + \cdots + qp^{k-1} \sum_{k=0}^{\infty} P_{n-k+1} x^{n-k+1},$$

and recalling that $P_j = 0$ for $j < k$,

$$= q \sum_{k=0}^{\infty} P_n x^n + qp \sum_{k=0}^{\infty} P_{n-1} x^{n-1} + \cdots + qp^{k-1} \sum_{k=0}^{\infty} P_{n-k+1} x^{n-k+1},$$

$$= qG[1 + px + \cdots + (px)^{k-1}] = qG \left[ \frac{1 - (px)^k}{1 - px} \right].$$

Solving for $G$ yields (2).

In the case $p = 1/2$, the combinatorial numbers $S_n = 2^n P_n$ satisfy the recursion $S_n = S_{n-1} + S_{n-2} + \cdots + S_{n-k}$. For these numbers, the generating function $(1 - x - x^2 - \cdots - x^k)^{-1}$ was found by V. Schlegel in 1894. See [1, Chap. XVII] for this and other classical references.

An alternate solution to the problem can be obtained as follows. Consider a sequence of experiments: Toss a $p$-coin $X_1$ times, until a sequence of $k-1$ heads occurs. Then toss the $p$-coin once more and if it comes up heads, set $Y = 1$. If not, toss the $p$-coin $X_2$ times until a sequence of $k-1$ heads occurs again, and then toss the $p$-coin once more and if it comes up heads, set $Y = 2$. If not, continue on in this fashion until finally the value of $Y$ is set. At this time, we have observed a sequence of $k$ heads in a row for the first time, and we have tossed the coin $Y + X_1 + X_2 + \cdots + X_7$ times. The $X_i$ are independent, identically distributed random variables and $Y$ is independent
of all of the $X_i$. Let $E_k = \text{the expected number of tosses to observe } k \text{ heads in a row.}$
\[ E_k = E(Y + 2) = E(2) = E(Y) + E(2) = E(Y) + \sum_{n=1}^{\infty} E(2|Y = n)Pr(Y = n) = E(Y) + \sum_{n=1}^{\infty} nE(X_1)Pr(Y = n) = E(Y) + E(X_1)E(Y). \]

But $E(Y) =$ the expected number of tosses to observe a head $= 1/p$, and $E(Y) = E_{k-1}$. Thus $E_k = 1/p + (1/p)E_{k-1}$, which yields (3).

**REFERENCE**


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**STRONG DIVISIBILITY SEQUENCES WITH NONZERO INITIAL TERM**

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In 1936, Marshall Hall [1] introduced the notion of a $k$th order linear divisibility sequence as a sequence of rational integers $u_0, u_1, \ldots, u_n, \ldots$ satisfying a linear recurrence relation

\[ u_{n+k} = a_1u_{n+k-1} + \cdots + a_ku_n, \]

where $a_1, a_2, \ldots, a_k$ are rational integers and $u_m|u_n$ whenever $m|n$. Some divisibility sequences satisfy a stronger divisibility property, expressible in terms of greatest common divisors as follows:

\[ (u_m, u_n) = u_{(m,n)} \]

for all positive integers $m$ and $n$. We call such a sequence a strong divisibility sequence. An example is the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \ldots$.

It is well known that for any positive integer $m$, a linear recurrence sequence $\{u_n\}$ is periodic modulo $m$. That is, there exists a positive integer $M$ depending on $m$ and $a_1, a_2, \ldots, a_k$ such that

\[ u_{n+M} \equiv u_n \pmod{m} \]

for all $n \geq n_0[m, a_1, a_2, \ldots, a_k]$; in particular, $n_0 = 0$ if $(a_k, m) = 1$.

Hall [1] proved that a linear divisibility sequence $\{u_n\}$ with $u_0 \neq 0$ is degenerate in the sense that the totality of primes dividing the terms of $\{u_n\}$ is finite. One should expect a stronger conclusion for a linear strong divisibility sequence having $u_0 \neq 0$. The purpose of this note is to prove that such a sequence must be, in the strictest sense, periodic. That is, there must exist a positive integer $M$ depending on $a_1, a_2, \ldots, a_k$ such that

\[ u_{n+M} = u_n, \quad n = 0, 1, \ldots. \]