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ABSTRACT

The perturbation series for a stationary, homogeneous, isotropic turbulent flow mixed by a gaussian random force is investigated starting with a path integral formulation of the theory. For a class of correlation functions of the mixing force we find that perturbation theory is excellent at large wave numbers, while at small wave numbers it is badly divergent. If the correlation function of the mixing force behaves at large wave number as a power in wave number \( k \), or even as \( k^\alpha e^{-k} \), \( \alpha < 3 \) for three dimensions, perturbation theory is accurate. This is called strong mixing. The renormalization group is used to study the small \( k \) behavior of the theory. At three dimensions the energy spectral function behaves at \( k^{-\rho} \) where \( \rho \), a "critical exponent," is \(-1/3\) in lowest order. The renormalization group is then used to evaluate the velocity correlation function for all \( k \), when the fluid is strongly mixed.

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I. INTRODUCTION

The analogy between quantum field theory and the theory of turbulent flow has been realized for many years.\textsuperscript{1,2} This paper further explores this connection and utilizes some techniques from recent developments in quantum field theory, in particular the renormalization group\textsuperscript{3}, to examine the behavior of turbulent flows at very low wave numbers and, in the case of what we denote strong mixing, at very large wave numbers as well.

We study homogeneous, isotropic, stationary turbulence in an incompressible fluid. The flow is maintained by an external random force, $F_j(x,t)$, which we take to be gaussian with correlation function

$$\langle F_j(x,t)F_\ell(y,\tau)\rangle = \frac{\gamma_0^2}{4} \delta(t-\tau) \int \frac{dk}{(2\pi)^D} \Lambda_{j\ell}(k) e^{-i\mathbf{k}\cdot(x-y)} \Gamma(k^2/k_0^2)$$

(1)

where the delta function in time indicates the force correlation is white. The projection operator

$$\Lambda_{j\ell}(k) = \delta_{j\ell} - k_j k_\ell/k^2$$

(2)

arises from the incompressibility of the fluid. The fourier transform, $\Gamma(k^2/k_0^2)$, of the force correlation is dimensionless and contains a scale factor $k_0$ which determines the wave numbers at which the fluid is mixed. All other dimensions reside in $\gamma_0$ which is a measure of the strength of the forcing. The number of space dimensions is taken to be $D$.

The first issue addressed in this paper is the behavior of perturbation theory in the non-linearity of the Navier-Stokes equation for the calculation of, say, the velocity correlation function. We find the answer...
to this depends on the $k$ dependence of $\Gamma(k_0^2/k^2)$. At large $k$, perturbation theory is excellent, if $\Gamma$ falls as a power of $k$ or even as an exponential times some powers. If

$$\Gamma(k) \sim k^\alpha e^{-k} \quad (3)$$

for large $k$, then when $\alpha < 6-D$, perturbation theory is good for large $k$. We denote this as **strong mixing**.

If $\Gamma(k)$ falls faster than an exponential, say as $\exp - k^2/k_0^2$ or even as a step function $\theta(k_0^2-k^2)$, then perturbation theory at large $k$ is very bad. Such a situation will be denoted as **weakly mixed**. In this instance, the mixing that takes place for $k < k_0$ is "forgotten" for large $k$ because the transfer of energy via the Navier-Stokes non-linearity dominates the very weak mixing. In the **strong mixing** case, the fluid never "forgets" about how it is being stirred, and the mixing dominates the non-linearity until wave numbers high enough for dissipation (again linear) to take prominence.

When we have strong mixing, one should not expect, in general, a Kolmogorov $k^{-5/3}$ behavior for the energy spectral function. Basically the inertial range, where the method of mixing the fluid is forgotten, never develops. For weakly mixed fluids, a Kolmogorov spectrum may occur. We will have some brief remarks about this at the conclusion of this paper.

For small $k$ we investigate the case $\Gamma(0) \neq 0$ in detail. From a consideration of mixing forces which act primarily in a local spatial region, for example, $F_j(\vec{x},t) = c_j(t) \times \theta(\vec{R}^2-\vec{x}^2)$ with $c_j(t)$ some vector function of time, or $F_j$ which falls exponentially at large distances,
we find by direct calculation that \( \Gamma(k = 0) \neq 0 \). Also an analysis\(^5\) of the behavior of non-stationary turbulence long into the decay period suggests, on the basis of the equipartition theorem, that \( \Gamma(k = 0) \neq 0 \).

There are arguments that \( \Gamma \) should behave as \( k^2 \) for small \( k \). These are based on the use of the fluctuation-dissipation theorem and carry with them an implicit view that the turbulence arises from the thermal motions responsible for viscosity.\(^5\) This appears to be a physically uninteresting possibility so we focus on \( \Gamma(0) \neq 0 \).

With such behavior for \( \Gamma(0) \), there are long wavelength divergences at every order of perturbation theory. These divergences can be summed up using the renormalization group. This we do explicitly. We find, in analogy to the theory of second order phase transitions, a critical exponent in the theory of turbulence. It arises from the renormalization of the viscosity. In terms of the energy spectral function, \( E(k) \), the critical exponent \( \rho \), enters as

\[
E(k) \sim k^{D-3-\rho}
\]

for small \( k \) and \( D < 4 \). In lowest order \( \rho = -(4-D)/3 \).

Finally in the case of strong mixing, we show how to use the renormalization group to construct the velocity-velocity correlation function for all \( k \).\(^6\) It is crucial in this construction that perturbation theory be accurate for large \( k \).
II. FORMULATION OF THE PERTURBATION THEORY

We begin with the Navier-Stokes equation for an incompressible Newtonian fluid

\[ \frac{\partial}{\partial t} \mathbf{v}_j(x^+, t) + \frac{1}{2} (\Delta_{jn}(\nabla) \nabla_j + \Delta_{jn}(\mathbf{v}) \nabla_j) \mathbf{v}_n \nabla_j = \nu_0 \nabla^2 \mathbf{v}_j + \mathbf{F}_j, \]  

where the pressure gradient has been eliminated using \( \nabla_j \mathbf{v}_j = 0; \Delta_{jn}(\nabla) \) is the operator insuring incompressibility.

\[ \Delta_{jn}(\nabla) = \delta_{jn} - \frac{1}{\nu_0^2} \nabla_j \nabla_n; \]  

\( \nu_0 \) is the kinematic viscosity, and \( \mathbf{F}_j \) is the random mixing force needed to balance the viscous dissipation in maintaining stationary turbulence.

The generating functional for velocity correlation functions

\[ \langle \mathbf{v}_{j_1}(x^+, t_1) \cdots \mathbf{v}_{j_N}(x^+, t_N) \rangle, \]  

where \( \langle \rangle \) is an average over the forcing, is given by the functional integral

\[ Z[\eta_j(x^+, t)] = \int \pi_{x^+, t, j} dF_j(x^+, t) P[F_j] e^{-\int d^D x dt \eta_j(x^+, t) \mathbf{v}_j(x^+, t)} \]  

in which \( \mathbf{v}_j \) is to be expressed as a functional of \( \mathbf{F}_j \) via (5). \( P[F_j] \) is the probability distribution functional for the random mixing.

Into this integral we introduce \( 1^7 \) in the form

\[ 1 = \Delta \int d\mathbf{v}_j(x^+, t) \delta(0 \mathbf{v}_j - \mathbf{F}_j) \]  

\( \Delta \)
where

\[ 0v_j = \partial_t v_j + \frac{1}{2} (\Delta_{jn}(\nabla)\nabla \n_j + \Delta_{j\ell}(\nabla)\n_n) \n_j \n_n - \nu_0 \n^2 v_j \]  (10)

and \( \Delta \) is the functional determinant

\[ \Delta = \text{det}(\partial_t - \nu_0 \n^2 \delta^D(x-y) \delta(t-\tau) \delta_{ij} + \n_1 \n_j \delta^D(x-y) \delta(t-\tau)). \]  (11)

Because of retardation and \( \n \cdot \n = 0 \), this determinant is a constant, independent of \( v_j \) (or \( F_j \)), so in the various functional integrals we encounter, we may drop it. Exponentiating the delta functional in (9) with an auxiliary field \( \n_j(x,t) \) we write

\[ Z[\eta_j] = \int d\n_j d\n_j^* dF_j P[F_j] \times \exp \left[ -\int d^Dxdt \left[ \frac{1}{2} \nabla_j \nabla_j^* v_j + \nu_0 \n_0 \n_j \n_j v_j + F_j \n_j + \eta_j v_j \right] \right] \]  (12)

where \( A^\tau B = A(\partial_t B) - (\partial_t A)B \).

The integral over \( F_j \) is just the characteristic functional for \( P[F_j] \) and can be carried out explicitly in the case of a gaussian. Henceforth we take \( F_j \) to be gaussian. With the correlation function (1) we have for the \( v_j, \n_j \) generating functional

\[ Z[\eta_j, \n_j] = \int d\n_j d\n_j^* \exp \left[ -\int d^Dxdt \left[ I + \n_j \n_j^* + \eta_j v_j, \right] \right], \]  (13)
with the effective lagrangian

\[ \mathcal{L} = \frac{1}{2} \nabla_j \dot{x}_j + \nu_0 \nabla_n \nabla_n v_j \]

\[ - \frac{\gamma_0}{8} \nabla_j \nabla_j - \frac{1}{2} \left( (\Delta_j n(\nabla) \nabla_j + \Delta_j n(\nabla) \nabla_n) \nabla_j \nabla_n v_j \right). \] (14)

It is convenient now to rescale the velocity \( v_j \) and anti-velocity \( \bar{v}_j \) to place \( \gamma_0 \) in front of the non-linear term in (14). So set

\[ x_j(\bar{x},t) = \frac{2}{\gamma_0} v_j(\bar{x},t), \]

and

\[ \bar{x}_j(\bar{x},t) = \frac{\gamma_0}{2} \bar{v}_j(\bar{x},t), \] (15)

resulting in

\[ \mathcal{L}(x_j,\bar{x}_j) = \frac{1}{2} \nabla_j \dot{x}_j + \nu_0 \nabla_n \nabla_n x_j - \frac{1}{2} \nabla_j \nabla_j \]

\[ - \frac{\gamma_0}{4} \left( (\Delta_j n(\nabla) \nabla_j + \Delta_j n(\nabla) \nabla_n) \nabla_j \nabla_n x_j \right). \] (16)

This makes it quite clear that a perturbation expansion of correlation functions of \( x_j \) and \( \bar{x}_j \) is an expansion in \( \gamma_0 \). From

\[ Z[\eta_j,\bar{\eta}_j] = \int d\chi d\bar{\chi} \exp - \int dt \mathcal{L}(\chi,\bar{\chi}) + \chi_j \bar{\eta}_j + \chi_j \eta_j \]

the correlation functions

\[ g(n,m) = \langle x_{j_1}^{+}(x_{i_1},t_1) \cdots x_{j_n}^{+}(x_{i_n},t_n) \bar{x}_{\eta_1}(y_{\eta_1},\tau_1) \cdots \bar{x}_{\eta_m}(y_{\eta_m},\tau_m) \rangle, \] (17)
are derived by functional differentiation with respect to \( \eta \) and \( \overline{\eta} \). The velocity correlation functions (7), are found from \( G^{(N,0)} \) by multiplication by \( (\gamma_0/2)^N \).

Before becoming involved in the perturbation theory in \( \gamma_0 \) it is useful to identify the dimensionless parameter on which the perturbation series will depend. We proceed by noting that the "action"

\[
\int d^Dx dt \mathcal{L}
\]

is dimensionless. Giving the dimensions of quantities in powers of wave number \( k \), and frequency \( \omega \), we find

\[
\text{dimensions of } \gamma_0 = \omega^{3/2} k^{-(D+2)/2}
\]

The dimensionless combination of \( \gamma_0, \nu_0 \), and \( k_0 \)

\[
R_0 = \frac{\gamma_0}{\nu_0^{3/2} k_0^{(D-4)/2}}
\]

is the Reynolds number based on the external length scale \( k_0^{-1} \).

In perturbation theory we encounter integrals which are functions of wave number, so it is convenient to introduce a wave number scale \( k_N \) which tells us where we are probing in \( k \) space. We then form the dimensionless quantity

\[
\hat{g}_0 = \frac{\gamma_0}{\nu_0^{3/2} k_N^{(D-4)/2}}
\]
which we will call the bare coupling constant. It tells how the modes $\nu_j(k,\omega)$ are coupled through the non-linearity. We also introduce

$$\sigma = \frac{k_N^2}{k_0^2}$$

which indicates the scale of wave number relative to the mixing length $k_0^{-1}$.

Perturbation theory now proceeds by expanding the expressions (17) in powers of $\gamma_0$. A diagrammatic representation of the correlation functions results, with the following ingredients:

1. Associate a solid line with each $\chi_j$, a dotted line with each $\overline{\chi}_j$.
2. An arrow on a line pointing left (right) indicates a positive (negative) frequency.
3. Two propagators or bare correlation functions are required

$$G_{0j\ell}(k,\omega) = \int d^D x dt e^{i(k \cdot \vec{x} - \omega t)}\langle \chi_j(\vec{x},t)\overline{\chi}_\ell(0,0)\rangle,$$

$$= \Delta_{j\ell}(k)G_{0}^{(1,1)}(k^2,\omega),$$

$$= \Delta_{j\ell}(k)/(-i\omega + \nu_0 k^2)$$

and

$$G_{0j\ell}^{(2)}(k,\omega) = \int d^D x dt e^{i(k \cdot \vec{x} - \omega t)}\langle \chi_j(\vec{x},t)\chi_\ell(0,0)\rangle,$$

$$= \Delta_{j\ell}(k)G_{0}^{(2,0)}(k^2,\omega),$$

$$= \Delta_{j\ell}(k)\Gamma(k^2/k_0^2)/(-i\omega + \nu_0 k^2)(i\omega + \nu_0 k^2).$$

These are illustrated in Figure 1.
There is a fusion vertex where $X_j^+(k_1,\omega_1)$ and $X_\ell^+(k_2,\omega_2)$ join to form $X_n^+(k,\omega)$. It gives a factor

$$G^{(1,2)}(k,\omega,k_1,\omega_1,k_2,\omega_2) = G^{(1,1)}(k_1,\omega_1)G^{(1,1)}(k_2,\omega_2)G^{(1,1)}(k,\omega)$$

$$\times \frac{-i\gamma_0}{D+1} \left( \delta_{\alpha\nu} k_\beta + \delta_{\beta\nu} k_\alpha \right), \quad \omega = \omega_1 + \omega_2$$

This suggests defining a one line irreducible vertex $\Gamma_{0,\alpha\beta}(k,\omega,k_1,\omega_1,k_2,\omega_2)$ by removing the $G^{(1,1)}$ factors in (29). This is shown in Figure 2.

(5) Draw all graphs at any order of $\gamma_0$ using $G^{(1,1)}$, $G^{(2,0)}$ and $\Gamma_0$ as needed.

(6) Integrate $d^Dk\omega$ around loops; conserve $k,\omega$ at each vertex.

(7) With each graph associate a weight of unity except when there is a loop with precisely two factors of $G^{(1,1)}$ or $G^{(2,0)}$—these have weight 1/2.

These rules agree with Wyld\(^2\) and Martin, Rose, and Siggia.\(^9\) In the later paper, the Dyson equations representing the formal sum of all perturbation theory graphs is presented.

In order $\gamma_0^2$, the graphs in Figure 3 enter the evaluation of $G^{(1,1)}(k,\omega)$. Writing

$$G^{(1,1)}(k,\omega) = \Delta^+(k)G^{(1,1)}(k^2,\omega),$$

which follows from isotropy and incompressibility, we have
\[ G(1,1)(k^2,\omega)^{-1} = -i\omega + \nu_0 k^2 - \sum(k^2,\omega), \]  

(31)

and at order \( \gamma_0^2 \)

\[ \sum(k^2,\omega) = \frac{-\gamma_0^2}{4(2\pi)^D} \int \frac{d^Dq \Gamma(q^2/k_0^2)q_L n(k) q_n}{2\nu_0 q^2 [-i\omega + \nu_0 (q^2 + (q-k)^2)]} \]

\[ \times \left[ \frac{k^2}{q^2} - \frac{2k^2}{(\Delta - 1)(k^2-q^2)^2} \right]. \]

(32)

If we take the limit \( k \to \infty, \gamma_0, \nu_0, k_0 \) fixed, then

\[ \sum(k^2,\omega) + \frac{\gamma_0^2 (D-1)}{8\nu_0^2 (2\pi)^D} \int \frac{d^Dq}{q^2} \Gamma(q^2/k_0^2) \]

(33)

which is finite for \( D > 2 \) and \( \Gamma \) falling rapidly in \( q \) for large \( q \).

This means \( \sum(k^2,\omega) \) becomes negligible with respect to \( \nu_0 k^2 \) for large \( k \), so perturbation theory is excellent for \( k \to \infty \). Further contributions to \( \sum(k^2,\omega) \) behave similarly or fall as powers of \( k \). Perturbation theory for \( G(1,1) \) is good for large \( k \) because each term in the series in \( \gamma_0^2 \) becomes negligible with respect to the dominant dissipative term.

In the limit \( k^2, \omega \to 0 \), the story is different. \( \sum(k^2,\omega) \) vanishes as \( k^2 \) (to all orders actually) and the expression (32) behaves as

\[ \frac{\sum(k^2,\omega)}{\nu_0^2 k^2} \to 0 \quad \text{as} \quad \nu_0^2 k^2, \omega \to 0 \int \frac{d^Dp}{4} \Gamma(p^2). \]

(34)

For any \( \gamma_0, \nu_0 \) and \( k_0 \), when \( \Gamma(0) \neq 0 \) this integral is divergent for \( D \leq 4 \). The "correction term" to \(-i\omega + \nu_0 k^2\) swamps the unperturbed expression, and perturbation theory is inadequate. The situation
in higher orders is even worse with powers of \((34)\) appearing. This problem will be cured by the use of the renormalization group in the next section.

Next we turn to the expression for \(G^{(2,0)}(k^2,\omega)\). In lowest order of perturbation theory the graphs in Figure 4 appear. As noted by Martin, Rose, and Siggia, the expression for \(G^{(2,0)}(k^2,\omega)\) may be organized as

\[
G^{(2,0)}(k^2,\omega) = |G^{(1,1)}(k^2,\omega)|^2 (\Gamma(k^2/k_0^2) + \Sigma_2(k^2,\omega)) \tag{35}
\]

so the new feature in the velocity-velocity correlation function is \(\Sigma_2\).

At order \(\gamma_0^2\) it reads

\[
\Sigma_2(k^2,\omega) = \frac{\gamma_0^2}{8(2\pi)^D} \int \frac{d^Dq}{2\nu_0 q^2} q_i \Delta_{jn}(k) q_n \Gamma(q^2/k_0^2) \Gamma((\hat{q}-\hat{\omega})^2/k_0^2) \times \left( \frac{(D-2)k^2}{(D-1)} + \frac{k \cdot \hat{q}}{(D-1)(\hat{k} \cdot \hat{q})^2} \right) \left\{ \frac{1}{q^2} \left[ \frac{1}{i\omega + \nu_0 (q^2 + (\hat{q} \cdot \hat{k})^2)} - \frac{1}{-i\omega + \nu_0 ((\hat{k} - \hat{q})^2 + q^2)} \right] + \hat{q} \leftrightarrow \hat{k} - \hat{q} \right\} \tag{36}
\]

For \(\omega, k^2 \rightarrow 0\) the divergence of \(\Sigma_2\) is just the same as for \(\Sigma(k^2,\omega)\) above. For large \(k\), the matter is more interesting.

As \(k\) becomes large, \(\Gamma(k^2/k_0^2)\) decreases rapidly; so the question we must face is how small \(\Sigma_2\) becomes relative to the unperturbed quantity \(\Gamma(k^2/k_0^2)\). This is in striking contrast to the study of \(G^{(1,1)}\) where the unperturbed quantity \(-i\omega + \nu_0 k^2\) grows relative to the finite \(\Sigma(k^2,\omega)\).
For large \( k \) the behavior of \( \Sigma_2 \) is determined by the behavior of \( \Gamma(k^2/k_0^2) \). If \( \Gamma(k^2,k_0^2) \) falls as a power of \( k \), say

\[
\Gamma(k^2/k_0^2) = \left( \frac{k^2}{k_0^2} + 1 \right)^{-\lambda}, \quad \lambda > 0,
\]

then the dominant domains of the \( q \) integration in (36) are \( q \approx 0 \) or \( q \approx k \) where one or the other of the \( \Gamma \) factors is constant. Then the order \( \gamma_0^2 \) form of \( \Sigma_2 \) behaves as \( (k^2)^{-\lambda-1} \) and is smaller than \( \Gamma(k^2/k_0^2) \). Higher orders in \( \gamma_0 \) yield higher powers of \( (k^2)^{-1} \) and are less important.

For the situation where \( \Gamma \sim k^\alpha e^{-k}, \alpha > 0 \), the integral is dominated by \( q \approx k/2 \). Counting powers of \( k \) in the integral we find that when \( \alpha = 6-D \) the fall off of \( \Gamma \) is just matched by that of \( \Sigma_2 \). At higher orders in \( \gamma_0 \) this is also the break even point. If \( \alpha < 6-D \), perturbation theory at \( k \to \infty \) is accurate.

When \( \Gamma \) falls faster than exponential times a power, the result is different. A gaussian, for example, where \( \Gamma \sim \exp - (k^2/k_0^2) \) makes the integral peak again at \( q = k/2 \) but now \( \Sigma_2 \) behaves as

\[
\exp - (k^2/4k_0^2) \exp - (k^2/4k_0^2) \approx \exp - \frac{k^2}{2k_0^2}
\]

and this is much larger than \( \Gamma \) itself. So perturbation theory fails. At higher order in \( \gamma_0 \) the terms fall off slower and slower in \( k \) making perturbation theory worse and worse. For a sharp cutoff \( \Gamma \alpha_0(k^2-k_0^2) \), say, the behavior is similar. This behavior is reminiscent of large momentum transfer scattering in non-relativistic quantum theory where
qualitatively the same phenomenon is observed in the study of the Born series or the eikonal expansion. 10

The situation where perturbation theory is good for large $k$ we call strong mixing. It represents physically the case where the inertial transfer is not strong enough to overcome the tail of the random mixing at large wave numbers. The existence of strong mixing is clearly a quantitative property of the coupling inherent in the Navier-Stokes equation. When the mixing is extremely weak at large $k$, say $\Gamma \sim \exp \left(-\frac{k^2}{k_0^2}\right)$, then perturbation theory fails for the velocity-velocity correlation function because the non-linear inertial transfer becomes greater than the linear response to the mixing. The proper treatment of the weakly mixed turbulence will be discussed in the last section.

In either strongly mixed or weakly mixed turbulence the $k \to 0$ limit needs rectification; that is the subject of our next section.
III. USING THE RENORMALIZATION GROUP TO INVESTIGATE THE SMALL k REGIME

We saw in the previous section that for both strong and weak mixing in the regime \( \omega, k^2 \to 0 \), the perturbation theory for the Navier-Stokes equation fails for \( D < 4 \). We might have anticipated this from looking at the bare coupling constant

\[
g_0 = \frac{\gamma_0}{\nu_0^{3/2}} \frac{k_N^{(D-4)/2}}{}
\]

which becomes infinite as \( k_N \to 0 \) for \( D < 4 \). Varying \( k_N \) probes various parts of \( k \) space. The growth of \( g_0 \) as \( k_N \to 0 \) matches the singular behavior of perturbation theory as \( k^2, \omega \to 0 \).

What we require is a resummation procedure or a modified coupling constant which remains finite as \( k_N \to 0 \). The renormalization group provides what we desire\(^3,5\). It gives us non-perturbative differential constraints on the \( \chi, \bar{\chi} \) correlation functions \( G^{(n,m)} \) which enable us to survey various regimes of \( k \) space in a uniform fashion. In particular it identifies an effective coupling constant, dependent on wave number, which remains finite as \( k \to 0 \) and provides the infrared expansion parameter we desire.

The details of the renormalization group have been described at length elsewhere.\(^3\) We content ourselves here with the necessary outline. The procedure is to rescale the quantities appearing in the effective Lagrangian: \( \chi_j, \bar{\chi}_j, \gamma_0, \nu_0 \), and possibly \( k_0 \) at some arbitrary point \( \omega_N, k_N \) in frequency, wave number space. Then we require that the correlation functions \( G^{(n,m)} \) considered as functions of \( \gamma_0, \nu_0, k_0 \), are independent of \( \omega_N \) and \( k_N \). From the chain rule this leads us to a
differential equation (the Lie equation for the group of rescaling or renormalizations induced by changing \( \omega_N \) and \( k_N \)).

The rescaling procedure we follow in this section is as follows:

Set

\[
X_j(x,t) + \chi^+_R(x,t) = z^{-1/2}X_j(x,t) \tag{39}
\]

\[
\bar{X}_j(x,t) + \bar{\chi}^+_R(x,t) = \bar{z}^{-1/2}\bar{X}_j(x,t) \tag{40}
\]

\[
\nu_0 + \nu = z\nu_0 \tag{41}
\]

\[
\gamma_0 + \gamma = z\gamma_0. \tag{42}
\]

We do not rescale \( k_0 \). The rescaled or renormalized coupling constant

\[
g = \frac{\gamma}{\nu^{3/2}} k^{(D-4)/2} = \frac{z\gamma}{z\nu^{3/2}} g_0 = zg_0 \tag{43}
\]

is a function of \( g_0 \) and \( \sigma = k^2/k^2_N \).

To determine the scaling factors, the \( Z \)'s, we require the following normalization conditions on \( G^{(n,m)}_{R} = \langle \chi^+_R \ldots \chi^+_R \bar{\chi}^+_\bar{R} \ldots \bar{\chi}^+_\bar{R} \rangle_{n,m} \):

(1) \[
\left. \frac{\partial}{\partial \omega} G^{(1,1)}_{R}(k^2,\omega)^{-1} \right|_{k^2=0, \omega=\pm i\omega_N} = -1, \tag{44}
\]

(2) \[
\left. \frac{\partial}{\partial k^2} G^{(1,1)}_{R}(k^2,\omega)^{-1} \right|_{k^2=0, \omega=\pm i\omega_N} = \nu = z\nu_0, \tag{45}
\]

and

(3) \[
\left. \frac{3}{\partial \omega} G^{(2,0)}_{R}(k^2,\omega)^{-1} \right|_{k^2=0, \omega=\pm i\omega_N} = 2i\omega_N. \tag{46}
\]
These conditions enable us to determine $Z$, $\bar{Z}$, and $Z^\gamma$. Because we renormalize at $k^2 = 0$, $Z$ and $\bar{Z}$ are each unity. The origin of this is the incompressibility condition. $Z^\gamma$ is determined by a condition on the renormalized fusion vertex $\Gamma_{R\nu\alpha\beta}$:

$$
\frac{k_2 \Gamma_{R\nu\alpha\beta}(k, \omega, k_1, \omega_1, k_2, \omega_2)}{k_1} \bigg|_{k_2 = k_1 = 0} = -\frac{i \gamma}{(2\pi)^{D+1/2}} \quad (47)
$$

Next we require that $\phi(n, m) = \left(\frac{\gamma_0}{2}\right)^{n-m} G(n, m)$, the velocity and anti-velocity correlation functions, be independent of $\omega_N$. Using the chain rule and introducing $k^2_N = \omega_N / \nu$, we find the renormalization group equation

$$
\left[ \frac{2}{2} A(g, \sigma) \frac{\partial}{\partial g} + \frac{B(g, \sigma)}{1-B(g, \sigma)} \frac{\partial}{\partial \nu} + \sigma \frac{\partial}{\partial \sigma} + C_n, m(g, \sigma) \right] \phi(n, m)(k^i, \omega, g, \nu, \sigma, k^2_N) = 0,
$$

with the renormalization group functions

$$
A(g, \sigma) = \omega_N \frac{\partial}{\partial \omega_N} g \bigg|_{\gamma_0, \nu_0, k_0} \quad \text{fixed}, \quad (49)
$$

$$
B(g, \sigma) = \omega_N \frac{\partial \nu}{\partial \omega_N} \bigg|_{\gamma_0, \nu_0, k_0} \quad \text{fixed}, \quad (50)
$$

and

$$
C_n, m(g, \sigma) = (m-n) \left[ 1 - \frac{D}{4} + \frac{D+2}{4} B(g, \sigma) + \frac{A(g, \sigma)}{g} \right]. \quad (51)
$$
The solution to the renormalization group equation is given in terms of effective $g, \nu,$ and $\sigma$: 

\[ \frac{dg(\tau)}{d\tau} = -\frac{A(g(\tau), \nu(\tau))}{B(g(\tau), \sigma(\tau))}, \]  

\[ \frac{d\nu(\tau)}{d\tau} = -\frac{1}{A-B(g(\tau), \sigma(\tau))}, \]  

and 

\[ \frac{d\sigma(\tau)}{d\tau} = -\sigma(\tau), \]  

with the boundary conditions $\tilde{g}(0) = g, \tilde{\nu}(0) = \nu, \tilde{\sigma}(0) = \sigma$; so $\tilde{\sigma}(\tau) = \sigma e^{-\tau}$.

After some standard dimensional analysis we find

\[ \phi_{n,m}^{(n,m)}(\sqrt{k^2} k_i, \omega_i, \tau, \nu, \sigma, k_N^2) = \phi_{n,m}^{(n,m)}(k_i, \omega_i, \tilde{g}(-\log \xi), \tilde{\nu}(-\log \xi), \xi, \sigma, k_N^2) \times \exp \int_{-\log \xi}^{0} d\tau \gamma_{n,m}(\tilde{g}(\tau), \sigma e^{-\tau}), \]  

with

\[ \gamma_{n,m}(g, \sigma) = \frac{(m-n)(2-D) + 2D(1-n)}{4} \]  

\[ + \frac{(m-n)}{1-B(g, \sigma)} \left[ 1 + \frac{B(g, \sigma)}{2} + \frac{A(g, \sigma)}{g} \right]. \]
This is the stage where we see that the behavior of the correlation functions of $v_j$ and $\bar{v}_j$ are governed by an effective coupling $\tilde{g}(-\log \xi)$, an effective viscosity $\tilde{v}(-\log \xi)$, and an effective wave number ratio $\tilde{\alpha}(-\log \xi) = \sigma\xi$, which vary with the scale, $\xi$, in wave number space. These variations in $\xi$ of the parameters in the theory are a compact non-perturbative expression of the $k$ (and $\omega$) dependence of the integrals appearing in perturbation theory.

To determine the renormalization group functions we turn to the evaluation of $\Sigma(k^2, \omega)$ and $\Gamma_{\nu, \alpha \beta}$ in perturbation theory. (Through the renormalization group equation (48), knowledge of $A(g, \sigma)$ and $B(g, \sigma)$ in perturbation theory allows us to extract non-perturbative behavior for the correlation functions.) With the normalization condition (47) the graphs contributing to $\Gamma$ to order $\gamma_0^3$ as shown in Figure 5 have a neat cancellation and yield $Z_\gamma = 1$ to this order. From the expression (32) for $\Sigma$ we have

$$B(g, \sigma) = -\frac{g^2 \sigma^{3-D/2}}{32D(D+2)} \int \frac{d^D q}{(2\pi)^D} \frac{\Gamma(q^2)}{q^2} \left[ \frac{q^2 (2^D - D+2) + g^2 (2^D - D-2)}{(q^2 + \sigma/2)^3} \right]$$

$$= -\frac{g^2}{D+2} F(\sigma).$$

For $A(g, \sigma)$ we then find

$$A(g, \sigma) = -\frac{(4-D)}{4} g + \frac{F(\sigma)}{4} g^3.$$  

Before turning to $\tilde{g}(\tau)$ we examine the relationship between $g_0$ and $g$. Since $Z_\gamma = 1$, $g = Z^{-3/2}_\nu g_0$. From (49) and (50)
\[
\frac{3}{\delta g} \log \zeta \nu (g, \sigma) = B(g, \sigma)/A(g, \sigma),
\]

which yields

\[
\zeta \nu (g, \sigma) = (1-g^2/g_1^2)^{-2/D+2}
\]

with the boundary condition \(\zeta \nu (0, \sigma) = 1\). Here

\[
g_1^2 = (4-D)/F(\sigma)
\]

which is the location of the non-trivial zero of \(A(g, \sigma)\) as a function of \(g\). Then we learn

\[
g^2 = g^2(1-g^2/g_1^2)^{-2/D+2}
\]

As \(g_0^2 \rightarrow 0\), \(g_2 \rightarrow g_0^2\), which is good. As \(g_0^2 \rightarrow \infty\), namely in the small \(k\) regime, \(g^2 \rightarrow g_1^2\) which is finite. So as an expansion parameter \(g\) is much preferable to \(g_0\).

A study of the equation for \(\tilde{g}(\tau)\), (52), with \(A(g, \sigma)\) and \(B(g, \sigma)\) as given, shows that as \(\tau = -\log \xi \rightarrow \infty\), namely for small wave number, \(\tilde{g}(\tau) \rightarrow g_1\). As \(\tau \rightarrow -\infty\), namely large wave number, \(\tilde{g}\) can approach zero or it can grow without bound. We have seen in the previous section that when the fluid is strongly mixed, perturbation theory works for large \(k\), so one must choose the branch of \(\tilde{g}\) which approaches zero as \(\tau \rightarrow -\infty\). In the case of weakly mixed fluids, the opposite branch is indicated. A formal procedure to yield this latter result would introduce a renormalization of the external scale \(k_0\): \(k_0 + Z_k k_0 = k_R\).
then we would see that in the weakly mixed fluid \( k_\tau \to \infty \) as
\[
\tau = -\log \xi \to -\infty
\]
and this would drive us away from the fixed point at \( k_\tau = k_0, \ g = 0 \). In the strongly mixed fluid we are driven to that short wavelength fixed point. For the remainder of this section, we focus on the low \( k \) behavior of the theory.

At low \( k \) the effective coupling lies in the neighborhood of \( g_1 \). We take, henceforth, \( \tilde{g} = g_1 \) precisely. Corrections are straightforward to evaluate. When \( \tilde{g}(\tau) = g_1 \), \( A(g, \sigma) = 0 \) and the coupling remains fixed at \( g_1 \). The solution to the renormalization group equation near \( \xi = 0 \) now reads

\[
\phi(n, m)(\sqrt{\xi k^2}, \omega, g_1, \nu, \sigma, k^2) \\
= \xi^{(m-n)(2-D)+2D(1-n)/4+(m-n)(1-B_1)(1+B_1/2)} \\
\times \phi(n, m)(k^2, \omega, g_1, \nu, \sigma, 0, k^2)
\]

where \( B_1 = B(g_1, \sigma = 0) \). Using the dimensional analysis sketched before, (64) leads to the scaling form for \( \phi(n, m) \)

\[
\phi(n, m)(\omega, k, k_\tau, g_1, \nu, \sigma, k_N^2) \\
= \nu^{1-2m(k_N^2)}[2-n-3m+D(1-n)]/2 \\
\times (k^2/k_N^2)^{1/4}[[m-n)(2-D)+2D(1-n)+1-2m-1/1-B_1] \\
\times F_{n, m}(\frac{\omega}{k}, \frac{\omega k}{v k_N^2}, g_1)
\]

(65)
where \( F_{n,m} \) is a dimensionless function of its dimensionless arguments.

In this any one of the frequencies has been singled out and called \( \omega \) and any one of the wave numbers, \( k \).

For the velocity-velocity correlation function \( n = 2, m = 0 \) (65) yields for the scalar quantity \( \phi(2,0) \)

\[
\phi_{1,0}^{(2,0)}(\omega, k, \xi, \nu, \sigma, k^2_N) = \Delta_{1,0}(k) \phi^{(2,0)}(\omega, k^2, \xi, \nu, \sigma, k^2_N),
\]

(66)

\[
\phi^{(2,0)}(\omega, k^2, \xi, \nu, \sigma, k^2_N) = \nu k^{-D}(k^2_N)^{-2/(1-B_1)}
\]

\[
\times F_{2,0}(\omega, 1/1-B_1, \xi, \nu, \sigma, k^2_N)
\]

(67)

The energy spectral function \( E(k) \) is related to \( \phi^{(2,0)} \) by

\[
E(k) = \frac{(D-1)k^{D-1}}{\Gamma(D/2)(4\pi)^{D/2}} \int_{-\infty}^{\infty} d\omega \phi^{(2,0)}(\omega, k^2, \xi, \nu, \sigma, k^2_N),
\]

(68)

and (67) implies it behaves as

\[
E(k) \sim k^{D-3-\rho}
\]

(69)

with

\[
\rho = 2B_1/(1-B_1)
\]

(70)

From (58) and (59) we have

\[
B_1 = -(4-D)/(D+2),
\]

(71)
\[ \rho = -(4-D)/3. \]  

All these steps, of course, are accurate only for \( k \) small.

The scaling law and the result (72) for the "critical exponent" \( \rho \) are the key conclusions of this section.
IV. THE BEHAVIOR OF STRONGLY MIXED TURBULENCE AT ALL WAVE NUMBERS

In the case of strongly mixed turbulence we know that perturbation theory is accurate at large \( k \) and in the previous section we learned the behavior of correlation functions at small \( k \). It is then natural to inquire whether given both ends of the \( k \) spectrum we can devise a means to interpolate between them. Again the renormalization group comes to our aid. We will use it to derive formulae for the velocity correlation function \( G^{(2,0)} \) accurate at large \( k \) and at small \( k \).

The idea is to establish differential equations for the dimensionless parameter \( Z_v \), which rescales the unrenormalized viscosity \( \nu_0 \)

\[
\nu = Z_v \nu_0, \quad \text{(73)}
\]

for the parameter \( Z \) which relates the bare coupling \( g_0 \)

\[
g = Z g_0, \quad \text{(74)}
\]

and for \( \zeta \), which expresses the correction to the unperturbed velocity correlation function

\[
G^{(2,0)}(k^2, \omega, \gamma_0, \nu_0, k_0) = \frac{\Gamma(k^2/k_0^2)}{\omega^2 + \nu_0^2 k^4} - \zeta(k^2, \omega, \gamma_0, \nu_0, k_0). \quad \text{(75)}
\]

Each of these depends on the three dimensionless variables

\[
g_0 = \frac{\gamma_0}{\nu_0^{3/2} q_N^{(D-4)/4}}, \quad \text{(76)}
\]

\[
y_0 = i \omega_N / \nu_0 q_N^2, \quad \text{(77)}
\]
and

$$\sigma = q_N^2 / k_0^2$$  \hspace{1cm} (78)

which enter the evaluation of \( G(n,m)(k_1, \omega_1, \gamma_0, \nu_0, k_0) \) at an arbitrary point \( q_N, \omega_N \).

The differential equations we seek express \( Z_V, Z, \) and \( \zeta \) in terms of \( g, y = i \omega_N / \nu q_N^2 \), and \( \sigma \). These equations are solved with the boundary condition \( Z_V, Z, \zeta = 1 \) when \( g = 0 \), and the solutions are used to solve for \( g \) and \( y \) in terms of \( g_0, \gamma_0, \nu_0 \) and \( \sigma \) and then express the correlation functions in terms of the physical parameters \( \gamma_0, \nu_0 \) and \( k_0 \).

We want to determine the \( \omega \) and \( k \) dependence of the scalar functions \( G(1,1) \) and \( G(2,0) \), so we choose their normalizations at a general point \( k^2 = q_N^2, \omega = i \omega_N \). First we examine \( Z_1 \) which relates

$$G(1,1)(k^2, \omega, \gamma_0, \nu_0, k_0) = Z_1^{-1} G(1,1)(k^2, \omega, g, \nu, \sigma, q_N^2, \omega_N). \hspace{1cm} (79)$$

We choose

$$\frac{\partial}{\partial \omega} G_R(1,1)^{-1} \bigg|_{k^2 = q_N^2, \omega = i \omega_N} = -1. \hspace{1cm} (80)$$

This means

$$Z_1(g, \sigma, y) = \frac{\partial}{\partial \omega} G(1,1)(k^2, \omega, \gamma_0, \nu_0, k_0)^{-1} \bigg|_{k^2 = q_N^2, \omega = i \omega_N} \hspace{1cm} (81)$$

and \( G(1,1) \) is typically evaluated as a perturbation series in \( g_0 \).
With $Z_1$ known, we can determine $Z_v$ from the requirement

$$G^{(1,1)}_R(k^2, \omega, g, \nu, \sigma, q^2_N, \omega'_v) = G^{(1,1)}_R(k^2, \omega, g, \nu, \sigma, q^2_N, \omega'_v) = Z_v^{-1}(\omega, \nu, q^2_N Z_v)$$

which means

$$Z_v(g, \sigma, y) = 1 - \frac{1}{\nu_0 q^2_N} \sum_{\nu_0} \left( \frac{k^2}{\omega_0 q^2_N} \right)^2 \zeta(g, \sigma, y).$$

And finally we have

$$G^{(2,0)}(k^2, \omega, \nu_0, \sigma, k_0) = \frac{\Gamma(\sigma)}{(-\omega^2 + \nu_0^2 q^2_N)} \zeta(g, \sigma, y).$$

To acquire the differential equations we will need the renormalization group functions

$$A_\omega = \omega N \frac{\partial}{\partial \omega} g \mid_{\gamma_0, \nu_0, k_0 \text{ fixed}},$$

$$A_\nu = \nu_0 \frac{\partial}{\partial \nu} g \mid_{\gamma_0, \nu_0, k_0 \text{ fixed}},$$

and

$$A_0 = k_0 \frac{\partial}{\partial k_0} g \mid_{\gamma_0, \nu_0 \text{ fixed}}.$$

For $Z_v$ we'll also need

$$(B_\omega, B_\nu, B_0) = \left( \omega N \frac{\partial}{\partial \omega N}, q^2_N \frac{\partial}{\partial q^2_N}, k_0^2 \frac{\partial}{\partial k_0^2} \right) \log Z_v.$$
These functions $A_\alpha$ and $B_\alpha$ ($\alpha = \omega, q, 0$) are to be determined as follows: in perturbation theory (or whatever method you prefer) find $\gamma(y_0, v_0, k_0, w_N, q_N)$ and $\nu(y_0, v_0, k_0, w_N, q_N)$ using the normalization conditions just above plus (47). After performing the indicated derivatives, replace $y_0$ and $v_0$ by $\gamma$ and $\nu$ consistent with the order of approximation employed. This yields $A_{\alpha}(g, \sigma, y)$ and $B_{\alpha}(g, \sigma, y)$.

Now we use the chain rule to express general conditions on going over from $y_0, v_0, k_0$ to $g, \sigma, y$. We have

$$\frac{A_\omega}{g} = \omega_N \frac{\partial}{\partial \omega_N} \log Z \bigg|_{y_0, v_0, k_0 \text{ fixed}} = A_{\omega} \frac{\partial}{\partial g} \log Z(g, \sigma, y) + (1 - B_\omega) \gamma \frac{\partial}{\partial y} \log Z(g, \sigma, y), \quad (89)$$

$$\frac{A_q}{g} + \frac{4-D}{4} g = A_q \frac{\partial}{\partial g} \log Z - (1 + B_q) \gamma \frac{\partial}{\partial y} \log Z + \sigma \frac{\partial}{\partial \sigma} \log Z, \quad (90)$$

and

$$\frac{A_0}{g} = A_0 \frac{\partial}{\partial g} \log Z - B_0 \gamma \frac{\partial}{\partial y} \log Z - \sigma \frac{\partial}{\partial \sigma} \log Z. \quad (91)$$

Solving for $\frac{\partial}{\partial g} \log Z$ we find

$$\frac{\partial}{\partial g} \log Z(g, \sigma, y) = \left[ \frac{\tilde{A}(g, \sigma, y)}{g} + \frac{4-D}{4} (1 - B_\omega) (g, \sigma, y) \right] / \tilde{A}(g, \sigma, y), \quad (92)$$

with $\tilde{A} = (1 - B_\omega) (A_0 + A_q) + (1 + B_0 + B_q) A_\omega$. \quad (93)
Since \( Z(0, \sigma, y) = 1 \),

\[
Z(g, \sigma, y) = \exp \int_{0}^{g} \frac{du}{A(u, \sigma, y)} \left[ \frac{\dot{A}(u, \sigma, y)}{u} + \frac{4-D}{4}(1-B_0(u, \sigma, y)) \right].
\]  

(94)

Similarly

\[
Z_v(g, \sigma, y) = \exp \int_{0}^{g} \frac{du}{\tilde{B}(u, \sigma, y)/\check{A}(u, \sigma, y)},
\]  

(95)

where

\[
\tilde{B} = (1-B_\omega)(B_0 + B_q) + (1 + B_0 + B_q)B_\omega = B_0 + B_q + B_\omega,
\]  

(96)

and

\[
\zeta(g, \sigma, y) = \exp \int_{0}^{g} \frac{du}{\check{C}(u, \sigma, y)/\check{A}(u, \sigma, y)},
\]  

(97)

with

\[
\check{C} = (1-B_\omega)(C_0 + C_q) + (1 + B_0 + B_q)C_\omega,
\]  

(98)

and the \( C_\alpha \) are gotten as in (88) with \( \log \zeta \) substituted for \( \log Z_v \).

Our strategy will be to solve for \( g = Z(g, \sigma, y)g_0 \) and

\[
y = Z_v^{-1}(g, \sigma, y)y_0
\]  

as functions of \( g_0, y_0 \) and \( \sigma \) using what knowledge we have of the \( A_\alpha \) and \( B_\alpha \). Then we express \( \zeta(g, \sigma, y) \) as \( \zeta(g(g_0, \sigma, y_0), \sigma, y(g_0, \sigma, y_0)) \). Since \( q_N \) and \( \omega_N \) were arbitrary we'll have determined the \( \omega \) and \( k^2 \) dependence of \( G(2,0) \) by noting

\[
G(2,0)(k^2, \omega, y_0, v_0, k_0) = \frac{\Gamma(k^2/k_0^2)}{2^{2k^2/4} \omega + \nu_0 k_0^2} \zeta(g(\bar{g}_0, \bar{\sigma}, \bar{y}_0), \bar{\sigma}, y(\bar{g}_0, \bar{\sigma}, \bar{y}_0))
\]  

(99)

with
At this stage all reference to $\omega_N$ and $q_N$ will have disappeared, thus emphasizing their rule as arbitrary auxiliary quantities. What will have been achieved by their coming and going is a resummation of series in $g_0$ by the solution of differential equations expressing the arbitrariness of the rescalings involved.

Now we can be a bit more concrete. From the expressions given earlier for $\Sigma$ and $\Sigma_2$, we can calculate $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ to first non-trivial order in $g$:

$$\tilde{A}(g,y,\sigma) = -\frac{(4-D)}{4} g + a(y,\sigma)g^3,$$

$$\tilde{B}(g,y,\sigma) = -b(y,\sigma)g^2,$$

$$\tilde{B}_\omega(g,y,\sigma) = b_\omega(y,\sigma)g^2,$$

and

$$\tilde{C}(g,y,\sigma) = c(y,\sigma)g^2,$$

with $a$, $b$, $b_\omega$, and $c$ some functions of $y$ and $\sigma$. The solution for $Z$ reads

$$Z(g,y,\sigma) = [1 - \frac{g^2}{g_1(y,\sigma)}]^{\frac{1}{2}} - \frac{4-D}{8} \frac{b_\omega}{a}$$

with
\begin{equation}
g_1^2(y, \sigma) = (4-D)/4a(y, \sigma).
\end{equation}

Similarly

\begin{equation}
Z_v(g, y, \sigma) = (1 - g^2/g_1^2)^{-b/2a}
\end{equation}

and

\begin{equation}
\zeta(g, y, \sigma) = (1 - g^2/g_1^2)c/2a.
\end{equation}

With our normalization condition on the fusion vertex we found

\[ Z_y = 1 \] to the present order of accuracy, so \[ Z = Z_v^{-3/2} \] and

\[ a = \frac{3b}{2} + \frac{4-D}{4} b_\omega. \]  \hfill (111)

Each of the ratios \( b/a, b_\omega/a, \) and \( c/a \) is independent of \( y \) and \( \sigma \).

To see this compare, for example, the derivative of (107) with respect to \( y \)

\begin{equation}
\frac{3}{\gamma_3 y} \log Z = -\frac{(4-D)}{8} \log (1-g^2/g_1^2) \frac{3}{\gamma_3 y} \frac{b_\omega}{a} + \\
+ \left( \frac{1}{2} - \frac{(4-D)b_\omega}{8 a} \right) (1-g^2/g_1^2)^{-1} \frac{2g_2^2}{3} \frac{3}{\gamma_3 y} g_1(y, \sigma)
\end{equation}

with the general formula derived from (89)-(91). In the latter the logarithm is absent, so \( b_\omega/a \) is independent of \( y \). The independence of \( y \) and \( \sigma \) of all ratios follows in the same fashion. Using this observation we may choose \( y \) and \( \sigma \) at our convenience in these ratios only. Setting \( y \to 0, \sigma \to \infty \) yields \( b_\omega/a = 0 \), so
\[ a = \frac{3b}{2} \]  

and then

\[ g^2 = g_0^2 \left( 1 - \frac{g^2}{g_1^2} \right)^{-1}. \]  

or

\[ 1 - \frac{g^2}{g_1^2} = \left( 1 + \frac{g^2}{g_0^2} \right)^{-1}. \]

In the same limit we find by numerical integration

\[ \frac{c}{b} = 1.1496 \approx 8/7. \]

We have the functional relation \( g(g_0, y, \sigma) \) from (114) and for \( y(g_0, y_0, \sigma) \) we have the implicit formula

\[ y = y_0 \left[ 1 + \frac{4R_0^2}{4-D} a(y, \sigma) \sigma (D-4)/2 \right]^{-1/3}, \]

where we note the presence of the Reynolds number based on the external scale \( k_0 \)

\[ R_0 = \frac{\gamma_0}{v_0^{3/2}} (k_0^2 (D-4)/2) \]

From (114) and (117) we may evaluate

\[ \zeta(k^2, \omega, \gamma_0, v_0, k_0) = \left( 1 + \frac{4a(y, \sigma)}{4-D} R_0^2 \sigma (D-4)/2 \right)^{-8/21} \]
which expresses $G^{2(2,0)}$ directly in terms of the input parameters $\gamma_0$, $v_0$, and $k_0$.

This is the result we sought. One may directly check that as $k \to 0$ this expression for $\zeta$ (or $G^{(2,0)}$ via (99)) yields precisely the scaling limit derived earlier. Also $\zeta \to 1$ as $k \to \infty$, so it expresses the emergence of a good perturbation theory at $k \to \infty$ for strongly mixed turbulence. Since this formula has the correct $k \to 0$ and $k \to \infty$ limits one can hope to use it (or improved versions reflecting more knowledge of the A, B, and C functions) as an interpolating formula.

I have done extensive numerical calculations for $k_0 = 1$, $\Gamma(k^2) = (k^2+1)^{-3}$, and $R_0$ ranging from 10 to $10^6$ to observe the qualitative behavior of $E(k)$. Nothing surprising happens. For $k \leq k_0$, $E(k)$ rises gently away from zero. Then even for the largest values of $R_0$, $E(k)$ rapidly approaches $\Gamma(k)$ for $k > k_0$. As expected no Kolmogorov $k^{-5/3}$ spectrum emerges for strongly mixed turbulence as no inertial range is able to develop.
V. CONCLUSIONS

By examining the perturbation series in $(\nabla \cdot \mathbf{v}) \mathbf{v}$ for stationary, homogeneous, isotropic turbulence we have found a regime of wave number behavior for the correlation function of the random mixing force which permits perturbation theory to be used at large wave number $k$. If this correlation function $\Gamma(k)$ falls as a power

$$\Gamma(k) \sim (k^2)^{-\lambda} \quad \text{as } k \to \infty$$

(120)

where $\lambda > D/2$ in $D$ spatial dimensions so the net viscous dissipation proportional to $\int d^Dk \Gamma(k)$, is finite or behaves as

$$\Gamma(k) \sim k^\alpha e^{-k} \quad \text{as } k \to \infty, \alpha < 6-D$$

(121)

then we call the fluid strongly mixed because the turbulent flow never forgets how it is being mixed even for $k >> k_0$, the external mixing scale. In (117) the restriction on $\alpha$ comes because the exp $-k$ behavior alone makes all orders of perturbation theory fall at the same rate in $k$, namely exp $-k$, up to powers of $k$. If $\alpha \geq 6-D$, then the integrals of perturbation theory yield up additional powers of $k$ for large $k$ and perturbation theory must fail.

In the case of strong mixing the fluid tracks the forcing for large $k$ and perturbation theory becomes accurate, with calculable power corrections in $k^{-1}$. When $\Gamma(k)$ falls very rapidly in $k$, say

$\Gamma(k) \sim \exp -k^2$ or $\Gamma(k) \sim \theta (1-k^2)$, then perturbation theory fails for the velocity-velocity correlation function: each order of perturbation falls more slowly than the previous orders and so overwhelms the lower
orders at large $k$. (Interestingly, the velocity-anti-velocity correlation function is still calculable in these cases.) We call this situation \textit{weak mixing} as the mixing "goes away" at large $k$ leaving a self-consistent problem for the determination of various velocity correction functions.

As $k \to 0$ the theory has infrared divergences in perturbation theory. We showed how to cure these divergences using the renormalization group to sum up the leading singularities. As in the theory of second order phase transitions there emerges a critical exponent governing the $k \to 0$ limit. Here it arises from the renormalization of the viscosity.

For strongly mixed turbulence we gave a renormalization group derivation of an interpolating formula for the velocity-velocity correlation function which has the correct limits as $k \to 0$ or $k \to \infty$.

Weakly mixed turbulence is more difficult to deal with. One must perform a resummation of the theory to exhibit exact relations (the \textit{Dyson equations}) among the various correlation functions and then treat these equations in some approximate fashion. The Direct Interaction Approximation (DIA)\textsuperscript{9,11} is an example of an approximated form of these Dyson equations. It, in essence, keeps the exact form of the two point functions $G_{1/1}$ and $G_{2/0}$ in the notation of this paper, and sets all three line vertices to their unperturbed values. The DIA suffers from incorrect behavior in the infrared limit\textsuperscript{12} where large scale eddies ($k \to 0$) sweep along in a random fashion the small scale eddies. The work in this paper suggests a possible remedy for this feature of the DIA. Since we are able, using the renormalization group, to calculate the small $k$ behavior of correlation functions, we may use this information in the Dyson equations to approximate the three line vertices so the infrared limit of this modified DIA is accurate. Since only the
weakly mixed turbulence can have a natural inertial range (i.e. the fluid forgets how it was stirred), one might well expect from the extensive discussions of the DIA that a Kolmogorov spectrum would emerge.

NOTE

After this paper was written, H. Rose brought to my attention a short note by C. DeDominics and P. C. Martin who treat a fluid mixed as in the present paper using very similar renormalization group techniques. In the notation of the present paper they take a forcing function correlation $\Gamma(k)$ which behaves as $\Gamma(k) \sim k^{4-D-y}$ for large $k$ and then study $0 \leq y \leq 4$. As shown by E. A. Novikov (Soviet Physics-JETP 20, 1290(1965)) the net energy dissipation with a gaussian forcing is proportional to $\int d^Dk \Gamma(k)$ which requires $y > 4$ to converge. The theory of DeDominicis and Martin corresponds to a very different kind of mixing than that treated here.

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REFERENCES


7. This method is similar to the one used by R. Phythian, *J. Phys. A* 10, 777 (1977). His argument for constancy of the determinant fails when the propagation is not retarded. I enjoyed several discussions with H. A. Rose on this reference.

8. If $P[F_j]$ is not gaussian, a cumulant expansion of the integral is called for.


FIGURE CAPTIONS

Figure 1  The graphical representation of the bare velocity -
anti-velocity correlation function, $G^{(1,1)}$, and
the bare velocity-velocity correlation function,
$G^{(2,0)}$.

Figure 2  The bare fusion vertex

Figure 3  The terms in the series for $G^{(1,1)}$ to order $\gamma_0^2$

Figure 4  The terms in the series for $G^{(2,0)}$ to order $\gamma_0^2$

Figure 5  The terms in the series for the fusion vertex
to order $\gamma_0^3$
Figure 1

\[ G^{(1,1)}_{0j\ell}(k,\omega) \quad \xrightarrow{\Delta_{j\ell}(k)} \quad k,\omega \quad \frac{\Delta_{j\ell}(k)}{-i\omega + \nu_0 k^2} \]

\[ G^{(2,0)}_{0j\ell}(k,\omega) \quad \xrightarrow{\Delta_{j\ell}(k) \Gamma(k^2/k_0^4)} \quad k,\omega \quad -k,-\omega \quad \frac{\Delta_{j\ell}(k) \Gamma(k^2/k_0^4)}{\omega^2 + \nu_0^2 k^4} \]
\[ \Gamma_{0\nu,\alpha\beta}(\hat{k}, \omega, \hat{k}_1, \omega_1, \hat{k}_2, \omega_2) \]

\[ \begin{array}{c}
\alpha_{k_1}, \omega_1 \\
\Rightarrow \nu k, \omega \\
\Rightarrow \beta_{k_2}, \omega_2 \\
\hline
\end{array} \]

\[ \omega = \omega_1 + \omega_2 \]

\[ \hat{k} = \hat{k}_1 + \hat{k}_2 \]

\[ \frac{-i\gamma_0}{(2\pi)^{D+1}/2} (\delta_{\alpha\nu} k_\beta + \delta_{\beta\nu} k_\alpha) \]

**Figure 2**
$g^{(1,1)} = \quad + \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Figure 4
Figure 5
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