FISHER CENTER FOR REAL ESTATE AND URBAN ECONOMICS

WORKING PAPER SERIES

WORKING PAPER NO. 00-271

OPTIMAL LOAN INTEREST RATE CONTRACT DESIGN

By

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Optimal Loan Interest Rate Contract Design

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Working Paper 00-271

December 2000
Optimal Loan Interest Rate Contract Design*

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December 20, 2000

Abstract

Our paper develops a theoretical framework for analyzing optimal loan interest rate contracts under conditions of risky, symmetric information. We obtain a series of closed form solutions for one-period (static) and multi-period (dynamic) optimal contracts. The optimal design for loan interest rate contracts depends upon the volatility of, and co-variation among the market interest rate, borrower collateral, and borrower income, as well as the loan contract time horizon and the risk preferences of lenders and borrowers. Our analysis demonstrates that for a risk averse borrower with stochastic collateral, variable interest rate contracts, if structured properly, are, in general, Pareto optimal. If the collateral value and/or borrower’s future income are positively correlated with the market interest rate, optimal loan interest rate contracts will allocate more interest rate risk sharing to the borrower vis-a-vis the lender than would be the case in the absence of such correlation. This result occurs because the positive correlations make it “easier” for the borrower to repay loans when market interest rates rise, and vice versa. This, in turn, would enable the lender to reduce his risk exposure by receiving higher loan interest rate payments as market interest rates rise, and vice versa. The opposite would be true when the correlation between total borrower’s wealth and the market interest rates is negative.

While the specific optimal loan interest rate contract may be sensitive to the set of assumptions made, for plausible sets of assumptions, the optimal loan interest rate contract for the multi-period (dynamic) model often exhibits “muted” responses to changes in the market interest rate, making fixed rate loan contracts a reasonable approximation for the optimal design. That may explain why, in the absence of optimal contracts, long term borrowers tend to prefer fixed rate contracts, while short term borrowers tend to prefer variable rates contracts. These conclusions are reinforced when exogenous prepayments and defaults are incorporated into our analysis.

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*This is a preliminary draft. Please do not quote without authors consent.
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‡We are indebted to Ashok Bardhan, Peter DeMarzo, Dwight Jaffee, Lionel Martellini, John Quigley, Nancy Wallace, Nick Wonder, participants of the Seventh Asia-Pacific Finance Conference in Shanghai, and participants of the Ph.D. Seminar in Real Estate at the Haas School of Business, for useful comments. All remaining errors are ours.
1 Introduction

Our paper develops a theoretical framework for analyzing optimal loan interest rate contracts under conditions of risky, symmetric information. The analysis demonstrates that optimal loan interest rate contracts are dependent upon the volatility of, and co-variation among the market interest rate, borrower collateral, and borrower income, as well as the loan contract time horizon and the risk preferences of lenders and borrowers. The analysis commences with a one period loan contract, and proceeds by increasing the number of stochastic variables in a multi-period setting.

Recent studies of lending contracts largely focus on the role of the asymmetric information between the borrower and the lender. Dunn and Spatt (1988) demonstrate that asymmetric information generates a basic tension between the borrower and the lender, each trying to fully protect themselves against idiosyncratic events such as default and/or prepayment. The impact of asymmetric information is further elaborated in Chari and Jagannathan (1989), Kazarian (1993), Brueckner (1994), Stanton and Wallace (1995) and LeRoy (1996), among others. The issues addressed span the spectrum from the reasons for the existence of loan points and "due on sale" clauses\(^1\) to the optimal selection of interest rate contracts (fixed versus variable) for different borrower types. Fried and Howitt (1980), Stiglitz and Weiss (1981) and Williamson (1986) show the linkages among asymmetric information, the possibility of the borrower's default, and credit rationing. Jaffee and Stiglitz (1990) present a comprehensive review of the literature\(^2\).

While some degree of asymmetric information always exists when two different agents interact, the preponderance of asymmetric information models may engender a mis-perception that all of the interesting features of lending contracts can be, in one way or the other, attributed to that assumption. That is not so. For example, Jones and Nickerson (2000) demonstrate that symmetric information can lead to the possibility of default, endogenous prepayment risk, and credit rationing. It is important to explore, therefore, how far one can go in developing a realistic symmetric information model.

The closest analysis to our approach is that of Arvan and Brueckner (A-B)(1986). They discuss efficient loan interest rate contracts subject to market interest rate risk in a static, symmetric information framework, where collateral value is predetermined\(^3\). An implicit assumption that

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1 Due-on-sale clause is discussed, also, in Dunn and Spatt (1985).
2 Recently, Yavas (2000) demonstrates that borrowers with different levels of default risk can self-select different loan-to-value ratios, based on the magnitude of their default costs.
3 The A-B model is intimately related, in form, to the paper on the design of an optimal insurance policy Raviv
A-B use (and we maintain) is that there are situations in which it is desirable, or necessary, for both parties to hedge market interest rate risk exposure by entering into an explicit interest rate contract\(^4\). A-B show that, if the borrower's income is known with certainty by both the borrower and the lender at origination, the optimal loan contract interest rate is positively related to changes in the market interest rate, and is proportional to the ratio of lender-borrower risk aversion coefficients.

In this article we extend the A-B model along two dimensions:

a) We allow loan collateral value\(^5\) (as well as borrower's income) to be correlated with the market interest rate. These inter-correlations significantly modify the optimal loan interest rate contract curve since the parties have to take into account the direct impacts of the market interest rate risk and the indirect impacts on the collateral value and borrower's income.

b) We introduce multi-period dynamics into the optimal loan determination, with the possibility of early prepayments and default. Optimal contract curves for the long and short term borrower's can differ, both quantitatively and qualitatively. In addition, long-term (dynamic) contracts in our model are, generally, less sensitive to the market interest rate volatility than the short term (static) contracts. Therefore, while neither static nor dynamic optimal contracts are, generally speaking, fixed rate, dynamic loan contract curves are better approximated with a fixed rate contract than those in the static setting\(^6\).

Our approach is to increase complexity step by step. We commence in Section 2 by examining a static model where the collateral value is a deterministic function of the market interest rate, (1979). Dokko and Edelstein (1991) develop an alternative static model, and show that the full protection of the borrower against the interest rate risk would likely be suboptimal. In addition, they argue that the optimal design of adjustable rate mortgages should include an interest rate CAP provision.

\(^4\)Our model is complementary to Jones and Nickerson (2000), who assume the opposite, namely, that the collateral and the interest risk can be hedged away using standard hedging techniques. Clearly, there are situations when it is not feasible to hedge interest rate risk because of either prohibitive transaction costs or the unavailability of appropriate hedge vehicles. In addition, hedging through diversification may not be feasible because of institutional investment-underwriting limitations.

\(^5\)Stochastic collateral is discussed by Chan and Kanatas (1985) (asymmetric valuation setting), and Jones and Nickerson (2000) (symmetric information valuation setting), among others.

\(^6\)In the context of real estate loans, this is consistent with stylized empirical facts. Empirical examinations of the choice between fixed and adjustable rate mortgages (Brueckener and Follain (1988), Shilling, Dhillon and Sirrns (1987), Kazarian (1993)) show that more mobile borrower's are more likely to choose adjustable-rate mortgages, while the less mobile ones tend to choose fixed rate mortgages. LeRoy (1996) and Stanton and Wallace (1998) find that asymmetric information is a likely cause of this behavior. Our model provides an explanation in a simpler, symmetric information setting.
and the borrower's future income is known to all parties at origination. In Section 3, we consider a general static model where collateral value, borrower's income, and the market interest rate are jointly distributed random variables. In Section 4, we construct a dynamic extension of the basic static model, followed by the general dynamic model when no prepayments are allowed (Section 5). Section 6 considers various modelling extensions, such as the introduction of exogenous prepayments and default. Section 7 is a brief summary. Some proofs are relegated to the Appendix.

2 Basic Static Model

In this section, we present a simple static model of optimal lending interest rate contract when the loan collateral value is stochastic. We assume that the value of the loan collateral is a deterministic function of the real market interest rate $s$, $V(s)$, known, along with the future borrower's income $y_1$, to all parties at origination. Our aim is to design an interest rate contract which allows all parties to optimally share interest rate risk\(^7\). Our approach takes the earlier results of A-B as a benchmark. For A-B, the slope of the optimal contract rate function (Equation 2.6) depends only on the ratio of the lender's and borrower's absolute risk aversion coefficients because the value of the collateral does not depend on the market interest rate.

We assume:

A1) All information is revealed cost-free to all agents in the economy, and is common knowledge at originations.

A2) The loan contract has a due-on-sale clause. The market for loans is competitive so that there is a minimum non-zero spread between contract and market interest rate below which no lending occurs.

A3) The agents in the economy are expected utility maximizers with strictly increasing preferences (i.e., they prefer more to less). In addition, the lender's utility function $v$ is weakly concave (i.e., the lender may be risk-neutral or risk-averse). In contrast, the borrower's utility function $U$ is strictly concave (i.e., the borrower is strictly risk-averse)\(^8\).

\(^7\)In Jones and Nickerson (2000), the ability to hedge collateral value risk is necessary, along with perfect markets, for the construction of the model. Neither of these assumptions is required in our model. In fact, we implicitly assume that there are situations in which it may not be desirable, or even possible, for the parties to hedge the collateral value (and, later in the paper, the borrower's income) risk. In that case, an optimal risk sharing contract has to take such risks into account.

\(^8\)If agent's preferences are described by a utility function $u(.)$, the absolute risk aversion coefficient is given by
A5) The lender’s utility depends on his net lending profits. The borrower’s utility depends upon his collateral and other consumption\(^9\).

A6) Initial value of the collateral \(V_0\) and the loan amount \(L_0\) are known\(^{10}\).

A7) All parties, at origination, agree on the contract interest rate corresponding to the market interest rate at that time, \(r(s_0) = r_0\)\(^{11}\). For convenience, we subsume that, at origination, the real interest rate is zero \((s_0 = 0)\).

A8) Market interest rate probability density function, \(f(s)\), is known.

In addition, in this section we assume that:

B1) The borrower buys the collateral at time 0 and sells it at time 1. Default is not allowed\(^\,\,\,12\).

B2) There exists a deterministic functional relationship between the collateral value and the real interest rate, \(V(s)\), known to all parties.

B3) Borrower’s future (first period) income \(y_t\) is common knowledge.

This is our Basic Static Model (BSM). At origination, the lender both borrows in the marketplace and lends to the borrower an amount \(L_0\), which the lender pays back at the end of period 1, at the then prevailing market interest rate \(s\). The borrower repays the loan at the contract interest \(r(s)\) at time 1, in full. The lender’s net profit at origination is, therefore, 0. At time 1, the lender earns net profits of \((r(s) - s)L_0\) since he receives \((1 + r(s))L_0\) from the borrower, but pays for his cost of funds \((1 + s)L_0\). The lender’s expected utility function is\(^{13}\):

\[
Ev = \int v((r(s) - s)L_0) f(s) \, ds
\]

(2.1)

On the other side of the transaction, the borrower receives the amount \(L_0\) at origination. The remainder of the collateral purchase, \(V_0 - L_0\), is funded from the borrower’s initial endowment. At time 1, the borrower receives income \(y_t\), sells the collateral for the price \(V(s)\), and consumes

the expression \(\gamma = \frac{\alpha u''}{u'}\). In the case of Constant Absolute Risk Aversion (CARA) preferences, \(\gamma\) is a constant. CARA preferences can be described by a negative exponential with the exponent equal to \(\gamma\). The agent is risk-neutral if \(\gamma = 0\). (See any standard text, for example, Ingersoll (1987)).

\(^9\)For example, if the collateral is a house, then the borrower receives utility from housing services by purchasing the house and from other, non-housing, consumption.

\(^{10}\)\(L_0\) is an exogenous quantity since strategic (endogenous) default is not allowed and information is symmetric.

\(^{11}\)Without this assumption, the model predicts only the slope of the optimal contract rate as a function of the market interest rate, but not the level of the contract interest rate. More formally, assumption A7 allows us to obtain the unique solution for the optimal contract rate differential equation (see below).

\(^{12}\)The default assumption can be relaxed to include the possibility of exogenous default (see Section 6). The shape of the optimal static contract would not change, however.

\(^{13}\)If default is allowed, the lender’s expected utility would, also depend, on the value of the collateral as well as borrower’s income. See Section 6.
the residual. The borrower’s real collateral consumption during period 1 is assumed to be fixed by his collateral purchase $V_0$. The borrower’s other consumption at time 0 is:

$$ C_1 = V(s) + y_1 - (1 + r(s))L_0 $$

Since the collateral consumption is fixed, the borrower’s utility depends upon the amount of money he has for other consumption, after he finances his collateral purchase. The borrower’s expected utility function is:

$$ EU(C_1) = \int U(y_1 + V(s) - (1 + r(s))L_0) f(s) ds $$

(2.2)

In an optimal loan interest rate contract the lender offers, and the borrower accepts, a contract interest rate as a function of the market interest rate $r = r(s)$, which maximizes the lender’s expected utility function $E_u$ while providing the borrower his reservation expected utility $EU(C_1) = \text{const}$. This is a standard constraint maximization problem which can be solved using the Lagrange-Hamilton variation method\(^{14}\).

The following proposition holds:

**Proposition 1** The optimal contract $r(s)$ for the problem stated above satisfies the following differential equation:

$$ \frac{dr}{ds} = \frac{1}{L_0} \frac{dV}{ds} \gamma_U + \gamma_v \quad \gamma_v + \gamma_U $$

(2.3)

where the coefficients of the absolute risk aversion are defined as:

$$ \gamma_U = -\frac{U''}{U} \text{ and } \gamma_v = -\frac{v''}{v'} $$

For a risk-neutral lender, the optimal contract rate satisfies:

$$ \frac{dr}{ds} = \frac{1}{L_0} \frac{dV}{ds} \quad \gamma_v + \gamma_U $$

(2.4)

The solution to the equation 2.3 in the case of Constant Absolute Risk Aversion (CARA) preferences is given by:

$$ r_{st}(s) = \left( \frac{V(s) - V_0}{L_0} \right) \frac{\gamma_U}{\gamma_v + \gamma_U} + \frac{s\gamma_v}{\gamma_v + \gamma_U} + r_0 $$

(2.5)

\(^{14}\)It can be shown that the first order conditions are both necessary and sufficient for the existence of a global maximum.
Results of Arvan-Brueckner (1986) are recovered if \( V(s) \) is a constant, in which case:

\[
\frac{dr}{ds} = \frac{\gamma_v}{\gamma_v + \gamma_u} \\
\tau_{bench}(s) = \tau_0 + \frac{\gamma_v s}{\gamma_v + \gamma_u}
\]  

(2.6)

Proof.

The constrained maximization problem satisfies the following first order condition:

\[
0 = v'((r(s) - s)L_0) - \lambda U'(y_1 + V(s) - (1 + r(s))L_0) \\
\lambda = \text{const} = \frac{v'((r(s) - s)L_0)}{U'(y_1 + V(s) - (1 + r(s))L_0)}
\]

Taking the derivative of the first expression with respect to \( s \) we obtain:

\[
\left( \frac{dr}{ds} - 1 \right) L_0 v''((r(s) - s)L_0) - \lambda \left( \frac{dV}{ds} - L_0 \frac{dr}{ds} \right) U''(y_1 + V(s) - (1 + r(s))L_0) = 0
\]

Substituting for \( \lambda \) and solving for \( \frac{dr}{ds} \):

\[
\frac{dr}{ds} = \frac{\frac{1}{L_0} \frac{dV}{ds} \gamma_U + \gamma_v}{\gamma_v + \gamma_u}
\]

When \( \gamma_v = 0 \) Equation 2.4 immediately follows. With CARA preferences, \( \gamma_U \) and \( \gamma_v \) are constants. In that case, Equation 2.3 can be shown to yield the solution, Equation 2.5. Equation 2.6 is recovered if we set \( V(s) = V_0 \) in Equation 2.3 and Equation 2.5.

Let us compare our model, Equation 2.3, with the benchmark, Equation 2.6. In both models the relationship between changes in the loan contract interest rate and the changes in the market interest rate (i.e., the slope of the optimal contract rate) depends on the ratio of the lender's and borrower's risk aversion coefficients. In addition, the optimal contract rate in our model depends on the change in the value-to-loan ratio (VTL), \( \frac{V(s)-V_0}{L_0} \).

In particular, when the lender is risk neutral, the Equation 2.4 implies that the slope of the contract curve equals the marginal change in VTL. Therefore, if the value of the collateral rises when the interest rate rises, the optimal contract would be 'upward sloping' (i.e., the interest rate paid by the borrower should be adjusted upwards when the market interest rate rises, and downwards, when the market interest rate falls). Intuitively, if an increase in the market rate is associated with an increase in the collateral value, the borrower would find it easier to repay the loan\(^{15}\). The opposite would be true if the collateral value were to fall as the interest rate rises. In

\(^{15}\) As an extreme example, if \( V(s) = V_0 \left( 1 + \frac{\delta^0 s}{V_0} \right) \) we would have: \( r(s) = r_0 + s \). The collateral serves as a perfect interest rate hedge, so the borrower would completely absorb the interest rate risk.
that case, the optimal loan interest rate contract should be negatively sloped. In other words, the optimal contract interest rate would have to be adjusted downwards for market rate increases, and upwards for market rate declines in order to take into account the "total risk" which now includes the collateral value risk. The A-B benchmark, in contrast, always predicts a fixed rate contract curve when the lender is risk-neutral.

In the case of a risk-averse lender, from Equation 2.3, the optimal contract curve is upward-sloping if the value of the collateral is not declining too fast as a function of the market interest rate, or, more precisely, whenever \( \frac{1}{L_0} \frac{dV}{ds} > -\frac{\gamma_s}{\gamma_U} \). When \( \frac{1}{L_0} \frac{dV}{ds} = -\frac{\gamma_s}{\gamma_U} \), the slope is zero, which corresponds to a fixed-rate loan contract. If \( \frac{1}{L_0} \frac{dV}{ds} < -\frac{\gamma_s}{\gamma_U} \), the optimal loan interest rate contract will be a decreasing function of the market interest rate.

In Figure 1, we plot the optimal loan interest rate curves for four situations: a) \( V(s) \) is an increasing function; b) \( V(s) \) is a decreasing function; c) \( V(s) \) is a constant (the benchmark); and d) Fixed interest rate loan contract\(^{16}\). As expected, for each value of the market interest rate \( s \), the optimal loan interest rate is highest when the collateral value increases with market rate increases, and lowest when the situation is reversed. In the latter case, the loan interest rate declines with increases in the market interest rate.

In summary, conclusions of the A-B benchmark model that the optimal contract loan rate never declines when the market rate increases, and that a risk-neutral lender should absorb the entire interest rate risk (i.e., that an adjustable rate loan is only optimal if the lender is risk averse) do not hold if the value of the collateral depends upon the market interest rate. In fact, the optimal loan interest rate could decline when the interest rate increases. What is more, if the value of the property is an increasing function of the market interest rate, so that the ownership of the property provides a hedge against interest rates, it may be optimal for the borrower to assume the entire interest rate risk even if the lender is risk-neutral.

In the following sections we extend our model in two directions, first, by considering more general dependencies between the collateral value, borrower's income and the interest rate, and, second, by developing a multi-period (dynamic) model.

\(^{16}\)We assume the following parametrization: \( V(s) = 0.5(\exp(\pm 0.5 s) + 1) \), \( L_0 = 0.5 \), \( \gamma_0 = 2\% \), \( \gamma_s = 0.5 \), and \( \gamma_U = 3 \).
General Static Model

In the previous section we have shown that the shape of the optimal contract interest rate curve, in a one-period model in which the future value of the collateral is a deterministic function of the market interest rate, and the future borrower’s income is known at origination, is determined by the marginal value-to-loan ratio, and the lender-to-borrower risk aversion coefficients ratio. How reliant are the conclusions on the exact model specification? In order to analyze this question, consider a more general model which contains the results of the previous section as a special case. Keeping the principal assumptions (A1 to A8) and B1, we replace assumptions B2 and B3 with:

C1) Collateral value \( V \), borrower’s income \( y \)\(^{17}\), and the market interest rate \( s \) are, possibly correlated, random variables whose joint density distribution function is \( k(V, y, s) \)\(^{18}\), where the conditional density functions are defined as follows:

\[
k(V, y, s) \equiv c(V, y|s)f(s) \equiv l(V|y)g(y|s)f(s)
\] (3.1)

We refer to such a model as General Static Model (GSM). Utilizing Equation 3.1, the expected utility functions for the lender and the borrower can be written as:

\[
Ev = \int v((r(s) - s)L_0)f(s)ds
\] (3.2)

\[
EU(C_1) = \int \left( \int U(y + V - (1 + r(s))L_0)c(V, y|s)dVdy \right)f(s)ds
\]

The expression for \( Ev \) corresponds to Equation 2.1 and is independent of the borrower’s income and the collateral value\(^{19}\). On the other hand, the expression for \( EU(C_1) \) is a generalization of Equation 2.2. In particular, if we were to choose \( l(V|y, s) = l(V|s) = \delta(V - V(s)) \), and \( g(y|s) = \delta(y - y_1) \), where \( \delta \) is the so-called Dirac’s Delta function\(^{20}\), the second of the Equations 3.2 becomes:

\[
EU(C_1) = \int \left( \int U(y + V - (1 + r(s))L_0)c(V, y|s)dVdy \right)f(s)ds =
\]

\[
= \int \left( \int U(y + V - (1 + r(s))L_0)\delta(V - V(s))\delta(y - y_1)dVdy \right)f(s)ds =
\]

\[
= \int U(y_1 + V(s) - (1 + r(s))L_0)f(s)ds
\]

\(^{17}\)Borrower’s income \( y \) is the component of the total borrower’s income unrelated to the (explicit or implicit) rent from the collateral and, thus, uncorrelated with the collateral value \( V \).

\(^{18}\)Consistency requires that \( f(s) = \int k(V, y, s)dVdy \), where \( f(s) \) is the market interest rate density function.

\(^{19}\)See Footnote 11.

\(^{20}\)Dirac’s Delta function belongs to the class of generalized functions, and can be, informally, thought of as an infinitely ”skinny” Gaussian. It has the property that, for any sufficiently smooth \( p(x) \), \( \int p(x)\delta(x - x_0)dx = p(x_0) \). This and other useful properties of the Dirac’s delta function can be found, for example, in Kolmogorov and Fomin (1999).
which is the equivalent of Equation 2.2. Therefore, BSM in Section 2 is a special case of GSM.

The following proposition holds:

**Proposition 2** In a one-period economy where the borrower’s income, collateral value and the market interest rate are random variables with joint probability density function \( k(V, y, s) = c(V, y|s) f(s) \), and the assumptions A1-A8 and C1 hold, the optimal static loan interest rate contract satisfies the following equation:

\[
\frac{dr}{ds'} = \frac{L_0 \gamma_v \int U'(C_1) c(V, y|s) dV - \int U'(C_1) \frac{dc(V, y|s)}{ds} dV dy}{L_0 \int U'(C_1) (\gamma_v + \gamma_U(C_1)) c(V, y|s) dV dy}
\]

\( C_1 = y + V - (1 + r(s))L_0 \)

When the lender is risk-neutral (\( \gamma_v = 0 \)) the slope of the optimal contract rate would be:

\[
\frac{dr}{ds} = \frac{- \int U'(C_1) \frac{dc(V, y|s)}{ds} dV dy}{L_0 \int U'(C_1) \gamma_U(C_1) c(V, y|s) dV dy}
\]

In the case of CARA preferences, Equation 3.3 simplifies to be:

\[
\frac{dr}{ds} = \frac{\gamma_v}{(\gamma_v + \gamma_U)} + \frac{\gamma_U \Omega_1(r(s), s)}{L_0 (\gamma_v + \gamma_U)}
\]

\[
\Omega_1(r(s), s) = \frac{- \int U'(C_1) c_y(V, y|s) dV dy}{\gamma_U \int U'(C_1) c(V, y|s) dV dy}
\]

Quantity \( \Omega_1 \) in Equation 3.5 is a measure of a marginal change in collateral and income, in response to the change in the market interest rate. Since \( \frac{dc(V, y|s)}{ds} = \frac{dV}{ds} g(y|s) + l(V|y) \frac{dg(y|s)}{ds} \), we can divide the marginal change into its component parts:

\[
\Omega_1 = \Omega_1^Y + \Omega_1^\gamma
\]

where \( \Omega_1^Y \) is the marginal change in the value of the collateral, and \( \Omega_1^\gamma \) is the marginal change in the borrower’s income. In particular, for BSM model from Section 2, \( \Omega_1^Y = \frac{dV}{ds} \) and \( \Omega_1^\gamma = 0 \), so that \( \Omega_1 = \frac{dV}{ds} \). Therefore \( \frac{\Omega_1}{L_0} \) can be thought of as a generalization of the marginal value-to-loan ratio. The slope of the loan interest rate contract curve is determined by \( \frac{\Omega_1}{L_0} \) and the ratio of risk aversion coefficients. Namely:

- When \( \frac{\Omega_1}{L_0} > \frac{\gamma_v}{\gamma_U} \), the contract curve is upward sloping
- When \( \frac{\Omega_1}{L_0} = \frac{\gamma_v}{\gamma_U} \), the contract curve is flat (fixed rate loan)
- When \( \frac{\Omega_1}{L_0} < \frac{\gamma_v}{\gamma_U} \), the contract curve is downward sloping
In the case of a risk-neutral lender, the sign of the optimal interest rate slope is the same as the sign of $\Omega_1$, and is opposite of the sign of $\frac{d\langle V_y(s) \rangle}{ds}$. When $\Omega_1 = 0$, collateral and borrower’s income are stochastically independent of the market interest rate, and the GSM, Equation 3.5, transforms into the benchmark model, Equation 2.6. When $\Omega_1 \neq 0$, Equation 3.5 may not be easy to solve in closed form since $\Omega_1 (r(s), s)$ can be a very complex function of $r(s)$.

Our analysis dramatically simplifies when all parties have CARA preferences, and the income and the collateral value are either deterministic functions of the market interest rate, or jointly normally distributed. In that case, $\Omega_1 (r(s), s)$ does not depend on the contract interest rate $r(s)$. As we have shown previously, when the collateral value is a deterministic function of the market interest rate $\Omega^V_1 = \frac{d\langle V \rangle}{ds}$. Similarly, when the borrower’s income is a deterministic function of the market interest rate $\Omega^I_1 = \frac{d\langle I \rangle}{ds}$. The direction of the collateral (income) effect (i.e., whether it would raise or lower the slope of the optimal contract curve) is determined by the sign of $\Omega^I_1$. The magnitude of the effect is determined by the absolute value $|\Omega^I_1|$. Similarly, when the collateral (income) is stochastically correlated with the market interest, we obtain $\Omega^I_1 = K_{i,s} \frac{\sigma_I}{\sigma_s}$. The sign of the correlation coefficient $K_{i,s}$ determines the direction of the effect of the collateral (income) change on the contract rate slope, and the ratio of the standard deviations $\frac{\sigma_I}{\sigma_s}$ determines its magnitude\textsuperscript{21}.

In Table 1, we summarize the computation of $\Omega_1$ when the collateral (income) is uncorrelated with $s$ (denoted by U, for “uncorrelated”), deterministic function of $s$ (denoted by D), and stochastically correlated with $s$ (denoted by S)

<table>
<thead>
<tr>
<th>Income</th>
<th>Collateral</th>
<th>U</th>
<th>D</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td></td>
<td>0</td>
<td>$\frac{d\langle V \rangle}{ds}$</td>
<td>$K_{\langle V \rangle, s} \frac{\sigma_V}{\sigma_s}$</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td>$\frac{d\langle I \rangle}{ds}$</td>
<td>$\frac{d\langle I \rangle}{ds} + \frac{d\langle V \rangle}{ds}$</td>
<td>$\frac{d\langle I \rangle}{ds} + K_{\langle V \rangle, s} \frac{\sigma_V}{\sigma_s}$</td>
</tr>
<tr>
<td>S</td>
<td>$K_{\langle I \rangle, s} \frac{\sigma_I}{\sigma_s}$</td>
<td>$K_{\langle I \rangle, s} \frac{\sigma_I}{\sigma_s} + \frac{d\langle V \rangle}{ds}$</td>
<td>$K_{\langle I \rangle, s} \frac{\sigma_I}{\sigma_s} + K_{\langle V \rangle, s} \frac{\sigma_V}{\sigma_s}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: $\Omega_1$ in the General Static Model

The row index represents the borrower’s income, and the column index represents the collateral. Each element of the matrix, then, gives us a specific “model”. For example, UU cell corresponds to the benchmark model (Equation 2.6) and UD cell to the BSM model (Equation

\textsuperscript{21}If the collateral value (income) is uncorrelated with the market interest rate (or is known at origination), it has no bearing on the optimal contract curve.
2.3. Knowing that $\Omega_1$ is a function only of the market interest rate, an explicit solution for Equation 3.5 is:

$$r_{st}(s) = r_0 + \frac{\gamma_v s}{\gamma_v + \gamma_U} + \frac{\gamma_U}{L_0(\gamma_v + \gamma_U)} \int_0^s \Omega_1(s') ds'$$  \hspace{1cm} (3.8)

where $\int_0^s \Omega_1(s') ds' = i(s) - i_0$ in the deterministic case, and $\int_0^s \Omega_1(s') ds' = K_{i,s} \xi_s$ in the stochastic case ($i = V, y$). Therefore, together with Table 1, the Equation 3.8 completely describes solutions to the general static model (modulo restrictions discussed above).

As an example, the “model” DD in Table 1, where both income and the collateral value are deterministic functions of the interest rate, would yield $\Omega_1 = \frac{dV}{ds} + \frac{dy}{ds}$, and the Equation 3.8 would be:

$$r_{st}(s) = r_0 + \frac{\gamma_v s}{\gamma_v + \gamma_U} + \left( \frac{\gamma_U}{\gamma_v + \gamma_U} \right) \frac{V(s) - V_0 + y(s) - y_0}{L_0}$$  \hspace{1cm} (3.9)

A change in the market interest rate creates the change in the optimal loan rate in two ways: directly, through the market interest rate risk sharing (the first term); and indirectly by affecting the value of the collateral and the borrower’s income (the second term). The collateral and income effects can amplify each other, if $\frac{dV(s)}{ds}$ and $\frac{dy}{ds}$ have the same sign, or dampen each other if the signs are opposite. Figure 2 illustrates that, for a given realization of the market interest rate, the optimal loan interest rate is largest when both the collateral and the borrower’s income increase with the market interest rate, and smallest, if both of these functions are decreasing with the market interest rate$^{22}$. The A-B benchmark is midway between these loan contract curves, and corresponds to the situation when income and collateral effects exactly cancel out$^{23}$.

Equation 3.7 can be used to determine the sign of the optimal loan contract interest rate slope. For a risk-neutral lender, when $\Omega_1 = \frac{dV}{ds} + \frac{dy}{ds} > 0$, the total borrower’s wealth increases with an increase in the market interest rate, enhancing his ability to repay the loan, and, thus, making it optimal for the lender to require a higher loan interest rate when the market rate increases. In such circumstances, the contract curve would be upward sloping. The opposite would be true if $\frac{dV}{ds} + \frac{dy}{ds} < 0$. Finally, the contract curve is flat when $\frac{dV}{ds} + \frac{dy}{ds} = 0$, because the marginal changes in collateral value and the borrower’s income cancel.

*Comparative Statics for GSM:* With simple risk sharing, such as our A-B benchmark, Equation 2.6, increasing the lender’s risk-aversion coefficient $\gamma_v$ would *increase* the slope of the optimal contract rate; while increasing the borrower’s risk aversion $\gamma_U$ would *decrease* it, ceteris paribus.

$^{22}$We use $y(s) = 0.3 (\exp(\pm 0.3s) + 1)$.

$^{23}$This corresponds to $\Omega_1 = \frac{dV}{ds} + \frac{dy}{ds} = 0$, yielding Equation 2.6.
When the collateral and/or income are stochastic (other cells in Table 1), the following holds:

\[ \text{sign} \left( \frac{\partial}{\partial \gamma_t} \frac{dr_{st}}{ds} \right) = -\text{sign} \left( \frac{\partial}{\partial \gamma_t} \frac{dr_{st}}{ds} \right) = \text{sign} \left( \frac{\Omega_1}{L_0} - 1 \right) \]  

(3.10)

While an increase in the lender’s risk aversion has the opposite effect of an increase in the borrower’s risk aversion, ceteris paribus, there is a significant contrast with the A-B benchmark. The \text{sign} of the overall effect of such a change can be positive, negative, or zero, depending on the size of the change in the generalized value-to-loan ratio \( \frac{\Omega_1}{L_0} \). In particular, an increase in lender’s risk aversion coefficient increases the optimal loan interest rate contract only if the borrower’s total wealth does not increase too rapidly with the increase in the market interest rate (i.e., when \( \frac{\Omega_1}{L_0} < 1 \)). If \( \frac{\Omega_1}{L_0} = 1 \), a change in the lender (borrower) risk aversion coefficient would not change the optimal loan interest rate contract curve. Finally, if \( \frac{\Omega_1}{L_0} > 1 \), an increase in the lender’s risk aversion would decrease the optimal loan contract interest rate. Intuitively, increases in both the lender’s risk aversion and the generalized value-to-loan ratio tend to work in opposition to each other. When the increase in the lender’s risk aversion prevails, the overall impact is positive (i.e., the optimal loan contract interest rate increases); if the increase in the generalized value-to-loan ratio prevails, the overall impact is negative (i.e., the optimal loan contract interest rate decreases); finally, when \( \frac{\Omega_1}{L_0} = 1 \) the two effects balance, with a zero net impact on the optimal loan interest rate contract.

4 Basic Dynamic Model

How does an optimal multi-period loan contract differ from that of one period? To address this issue, we extend BSM model from Section 2 to two periods. While we explicitly consider only two periods, it is straightforward to extend the model to any number of periods. Adopting assumptions A1 through A8, and B2 from Section 2, we substitute for assumptions B1 and B3 with the following:

B1') The borrower buys the collateral at time 0 and sells it at time 2. The borrower is not allowed to repay the loan prior to the expiration of the contract (time 2). No default is allowed\(^{24}\).

B3') Borrower’s future incomes, \( y_1 \) and \( y_2 \), are common knowledge at origination. We shall refer to this model as a Basic Dynamic Model (BDM).

At origination, the borrower purchases the collateral using his endowment and borrowed funds.

\(^{24}\)No-prepayment and no-default assumption can be relaxed to include the possibility of exogenous prepayment and/or default (see Section 6).
Thus, his consumption at time 0 is zero. At time 1, the borrower receives income $y_1$, and pays the interest due on the loan, $r(s_1)L_0$, so his consumption is $y_1 - r(s_1)L_0$. At time 2, he sells the collateral at $V(s_2)$, receives income $y_2$, and repays the second period interest and the principal on the loan, $(1 + r(s_2))L_0$. His consumption at the end of the period 2 is: $y_2 + V(s_2) - (1 + r(s_2))L_0$. If $\rho$ is the borrower’s subjective discount factor, the borrower’s expected utility function will be:

$$EU = \int U(y_1 - r(s)L_0) + \rho U(y_2 + V(s) - (1 + r(s))L_0) f(s) ds$$

If the lender’s subjective discount factor is $\delta$, his expected utility function is:

$$Ev = (1 + \delta) \int v((r(s) - s)L_0) f(s) ds$$

The shape of the optimal contract curve is described by the Proposition 3:

**Proposition 3** Optimal loan interest rate contract function for a two-period BDM satisfies the following equation:

$$\frac{dr}{ds} = \frac{\gamma_v}{\gamma_v + \gamma_u} + \frac{\gamma_u}{\gamma_v + \gamma_u} \frac{\Omega_2}{L_0}$$

$$\Omega_2(s) = \Omega_2^Y = \frac{\Omega_1^Y}{1 + \rho \exp \gamma_u I(s)}$$

where:

$$\Omega_1^Y(s) = \frac{dV}{ds}$$

$$I(s; x) = y_2 - y_1 + V(s) - L_0$$

The solution for Equation 4.3 is:

$$r_{d;np}(s) = r_{st}(s) + \frac{1}{(\gamma_v + \gamma_u) L_0} \ln \left( \frac{1 + \frac{1}{\rho} \exp \gamma_u I(0; x)}{1 + \frac{1}{\rho} \exp \gamma I(s; x)} \right)$$

where the subscript $d; np$ signifies “dynamic curve with no prepayments”, and $r_{st}(s)$ is the contract interest rate for the static loan analysis (Equation 2.5).

**Proof.** See the Appendix ■

Dynamic BDM is similar to the BSM from Section 2 because the sign of the slope of the optimal contract interest rate function is determined by the relative size of $\Omega_2$, a quantity analogous to $\Omega_1$, and the risk aversion ratio $\frac{\gamma_u}{\gamma_v}$. However, in a dynamic model, income and collateral affect the loan collateral value asymmetrically, because $\Omega_2^Y = \Omega_1^Y = 0$, and, yet, $\Omega_2^V \neq \Omega_1^V$ when $\Omega_1^V \neq 0$. In the next section we shall give a precise characterization of the relationships among the dynamic and static contract curves; as we shall demonstrate, it is possible for the slope of the static loan interest rate contract function to be downward, and for the dynamic model to be sloping upward, *ceteris paribus.*
We now construct a general dynamic model without prepayments by adopting assumptions A1 to A8 from Section 2, assumption B1' from Section 4 (disallowing prepayments and default), as well as the following:

Assumption C1') Collateral value $V$, borrower’s income $y$, and the market interest rate $s$ are correlated, random variables with a joint density distribution function $k(V, y, s)$. The distribution function $k(V, y, s)$ is assumed to be inter-temporally invariant, a.s. (i.e., except, possibly, on the set of measure zero). We shall refer to this analysis as the Generalized Dynamic Model Without Prepayments or GDM(np).

Expected utility function for the borrower and the lender read:

$$EU = \int \left( \int (U(C_1) + \rho U(C_2)) c(V, y|s) dV dy \right) f(s) ds \equiv$$

$$C_1 = y - r(s)L_0$$

$$C_2 = y + V - (1 + r(s))L_0$$

and:

$$Ev = (1 + \delta) \int v((r(s) - s)L_0) f(s) ds \quad (5.1)$$

For the GDM(np), Proposition 4 delineates the shape of the optimal loan contract.

**Proposition 4** The optimal loan interest rate contract, when agents have CARA preferences, satisfies the following equations:

$$\frac{dr}{ds} = \frac{\gamma_v}{\gamma_v + \gamma_U} + \frac{\gamma_U \Omega_2(s)}{L_0 (\gamma_v + \gamma_U)} \quad (5.2)$$

$$\Omega_2(s) = -\int \frac{U'(C_1) + \rho U'(C_2)}{\gamma_U \int (U'(C_1) + \rho U'(C_2)) c(V|s) dV dy} \frac{dc(V|s)}{ds} dV dy \quad (5.3)$$

where:

$$\Omega_2(s) = \Omega_2^V + \Omega_2^y \quad (5.3)$$

The analysis mimics the static case, with the marginal utility of consumption for the two periods, $U'(C_1) + \rho U'(C_2)$, replacing the marginal utility for one-period consumption, $U'(C_1)$, and the marginal change of income and collateral for 2 periods ($\Omega_2$) replacing the corresponding

25Like in the GSM, we assume that borrower’s income and the collateral value are uncorrelated.
marginal change for one period \((\Omega_1)^{26}\). Similarly, the relative size of \(\frac{\Omega_2(\theta)}{L_0}\) and \(-\frac{\gamma_u}{\gamma_U}\) determines the slope of the optimal contract curve\(^{27}\):

When \(\frac{\Omega_2}{L_0} > -\frac{\gamma_u}{\gamma_U}\), the contract curve is upward sloping \(\text{(5.4)}\)

When \(\frac{\Omega_2}{L_0} = -\frac{\gamma_u}{\gamma_U}\), the contract curve is flat (fixed rate loan)

When \(\frac{\Omega_2}{L_0} < -\frac{\gamma_u}{\gamma_U}\), the contract curve is downward sloping

The following expressions relate dynamic and static models:

\[
\begin{align*}
\Omega_2^D & = \Omega_1^D \quad (5.5) \\
\Omega_2^V & = \frac{\Omega_1^V}{1 + \frac{1}{\rho} \exp \gamma_u I(s; x)}
\end{align*}
\]

where \(\Omega_1^i\) are mathematical expressions taken from the static analysis in Table 1. The expressions for \(I(s; x)\) are listed in Table 2\(^{28}\):

<table>
<thead>
<tr>
<th>Collateral</th>
<th>(U)</th>
<th>(D)</th>
<th>(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U)</td>
<td>(\Delta y + V(s) - L_0)</td>
<td>(\Delta y + V_m + \frac{\sigma_{V,}}{\sigma_s} s - L_0 - \frac{\gamma_u}{2} (1 - K_{V,s}^2))</td>
<td></td>
</tr>
<tr>
<td>(D)</td>
<td>(V(s) - L_0)</td>
<td>(V_m + \frac{\sigma_{V,}}{\sigma_s} s - L_0 - \frac{\gamma_u}{2} (1 - K_{V,s}^2))</td>
<td></td>
</tr>
<tr>
<td>(S)</td>
<td>(V(s) - L_0)</td>
<td>(V_m + \frac{\sigma_{V,}}{\sigma_s} s - L_0 - \frac{\gamma_u}{2} (1 - K_{V,s}^2))</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: \(I(s; x)\) in the General Static Model Without Prepayments

\(V_m\) is the expected collateral value, and, for simplicity, the expected real market interest rate is assumed to be zero. If we denote by \(r_{d, np}(s)\) optimal dynamic contract rate without the possibility of prepayment, and by \(r_{st}(s)\) the corresponding optimal static contract rate, Equation 3.8, the following relation holds:

\[
r_{d, np}(s) = r_{st}(s) + \frac{1}{(\gamma_v + \gamma_u) L_0} \ln \left( \frac{1 + \frac{1}{\rho} \exp \gamma_u (I(0; x))}{1 + \frac{1}{\rho} \exp \gamma_u (I(s; x))} \right) \quad (5.6)
\]

\(^{26}\)Equation 5.2 coincides with the benchmark, Equation 2.6, when \(V\) and \(y\) are stochastically independent of the market interest rate.

\(^{27}\)For a risk-neutral lender, the sign of the slope of the loan interest rate contract curve is opposite of the sign of \(\frac{dc(V,y,s)}{ds}\).

\(^{28}\)Vector \(x\) stands for the collection of all variables, such as the change in the future borrower's income, for example, except for the interest rate \(s\).
Relationship between the dynamic and static contract curves: The following propositions discuss the ordering of the static and the corresponding dynamic optimal interest rate contract curve slopes.

**Proposition 5** Whether \( r_{d, np}(s) \) has a slope greater, equal, or smaller than the corresponding static contract \( r_{st}(s) \) is determined by the sign of \( \Omega_1^V \):

\[
\frac{dr_{d, np}(s)}{ds} < \frac{dr_{st}(s)}{ds} \iff \Omega_1^V > 0
\]
\[
\frac{dr_{d, np}(s)}{ds} = \frac{dr_{st}(s)}{ds} \iff \Omega_1^V = 0
\]
\[
\frac{dr_{d, np}(s)}{ds} > \frac{dr_{st}(s)}{ds} \iff \Omega_1^V < 0
\]

**Proof.** Since \( \Omega_2^V = \frac{\Omega_1^V}{1+\frac{\rho \exp (\gamma_I(s;x))}{\beta}} \), and \( \Omega_2^V = \Omega_1^V \), then \( \Omega_2 - \Omega_1 = \Omega_2^V - \Omega_1^V = -\frac{\frac{1}{\beta} \exp (\gamma_I(s;x))}{1+\frac{\rho \exp (\gamma_I(s;x))}{\beta}} \Omega_1^V \), so the assertion, Equation 5.7, follows immediately. ■

Thus, whether a long-term borrower would be optimally pay higher, lower, or the same contract interest rate as a short term borrower depends upon the sign of the correlation between the collateral and the market interest rate.29 Figures 3 and 4 illustrate Proposition 5 for the model DD (i.e., when both income and the collateral value are deterministic functions of the market interest rate). Figure 3 corresponds to \( \frac{dV}{ds} > 0 \), so the optimal static interest rate contract curves are above the dynamic ones. The opposite is true on Figure 4, where \( \frac{dV}{ds} < 0 \).30

**Proposition 6** Let \( \Omega_1^V = 0 \) and the lender is risk-averse. If, in addition, there is a component \( x_i \) such that \( I(s;x_i) \) is an increasing function in \( x_i \), \( I(s;x) \) is bounded from below as a function of \( s \), and \( |\Omega_1^V| \neq 0 \) is bounded as a function of \( s \), then the following holds:

i) If \( \frac{dr_{st}(s)}{ds} > 0 \) then \( \frac{dr_{d, np}(s)}{ds} > 0 \).

ii) If \( \frac{dr_{st}(s)}{ds} = 0 \) then \( \frac{dr_{d, np}(s)}{ds} > 0 \).

iii) If the lender is risk-averse and \( \frac{dr_{st}(s)}{ds} < 0 \) then \( x_i^* \) exists such that \( (x_i^* \text{ can be infinite})\):

\[
\frac{dr_{d, np}(s)}{ds} < 0 \text{ when } x < x^*
\]
\[
\frac{dr_{d, np}(s)}{ds} = 0, \text{ when } x = x^*
\]
\[
\frac{dr_{d, np}(s)}{ds} > 0, \text{ when } x > x^*
\]

29Strictly speaking, the proposition compares slopes rather than the levels of the contract curves.

30Note that the relationship holds independently of the sign of \( \Omega_1^V \)

31Condition \( \Omega_1^V \neq 0 \) serves to exclude the previously studied case \( \Omega_1^V = \Omega_1^V = 0 \), when \( r_{d, np} = r_{st} = r_{bench} \).

Similar proposition holds for monotonically decreasing functions of \( x \).
iv) In the limit \( x_i \to +\infty \), \( r_{d, np} \) converges to \( r_{bench} \);

v) In the limit \( x_i \to -\infty \), \( r_{d, np} \) converges to \( r_{st} (s) \).

According to Proposition 6, when the income is uncorrelated with the market interest rate, positive slope of the static curve assures that the dynamic curve will be upward sloping. If the static curve is flat, on the other hand, the corresponding dynamic curve is positively sloped. Interestingly, if the optimal static loan interest rate curve slopes downward, the dynamic curve can slope upward. Thus, static and dynamic optimal contract interest rate curves can have opposite sign of the slopes\(^{32}\). If \( x_i \) is very large, effectively, \( \Omega_2^\nu = \Omega_2^V = 0 \), which corresponds to the A-B benchmark model. When \( x_i \) is very small, \( \Omega_2^V \to \Omega_1^V \), so the optimal static and dynamic contract interest rate curves coincide. Proposition 6 does not generalize to the situation with stochastic income, where \( \Omega_1^\nu \neq 0 \)\(^{33}\). In that case, if the slope of the optimal static curve is non-negative, the slope of the dynamic curve can have any sign.

Proposition 6 is illustrated for the model corresponding to the UD cell in Table 2. Figure 5 demonstrates the static and dynamic optimal loan interest rate contracts when the income is the same in each period. When \( \frac{dV}{ds} > 0 \), the dynamic optimal interest rate contract is below the static one, although both are increasing (part i) of Proposition 6). When \( \frac{dV}{ds} < 0 \), the dynamic optimal loan contract is not only above the corresponding static optimal loan function, but is positively sloped, while the static loan interest rate function is negatively sloped (part iii of the Proposition 6). There would be, therefore, very little difference between the dynamic models with increasing and decreasing collateral values, and both are very closely approximated by the A-B benchmark. When \( y_2 - y_1 = 1 \) (Figure 6), these conclusions are strengthened: the dynamic models cannot be, for all practical purposes, distinguished from the A-B benchmark (part iv) of Proposition 6). Intuitively, if the borrower’s income is known to significantly increase in the second period, collateral value would play much smaller role in establishing the optimal loan interest rate contract. In contrast, when \( y_2 - y_1 = -1.8 \), i.e., the second period income falls dramatically (Figure 7), collateral becomes very important so that the dynamic and static optimal contract curves practically coincide (part v) of Proposition 6).

\(^{32}\)For a risk-neutral, a flat optimal static contract interest rate curve corresponds to the flat dynamic curve, while a negatively sloped static curve corresponds to the negatively sloped dynamic curve, so parts ii) and iii) of Proposition 6 are not preserved.

\(^{33}\)One exception is Proposition 6 part v) since the static case obtains when \( x_i \) is very small. When \( x_i \) is very large (part iv), instead of the A-B benchmark solution, in the limit we obtain the static SU or DU model (Table 1).
Intuitively, Propositions 5 and 6 state that dynamic optimal loan contracts often provide more muted response to the changes in the market interest rate in comparison with the corresponding static optimal loan functions. In other words, fixed rate loans often better approximate long term rather than short term contracts. This may provide at least a partial explanation of why more "mobile" borrowers tend to choose adjustable rate loans (corresponding to the static case in our model), while the less "mobile" long term borrowers tend to select fixed-rate loans (corresponding to an approximation of a dynamic contract in our model). They do so, according to our analysis, because they are trying to select, from the available contracts menu, the contract which best approximates optimality.

Comparative Statics: In the static model (Equation 3.10), increased lender’s and borrower risk aversion coefficients have opposite effects on the slope of the contract curve, and the sign of the effect depends on the sign of \( \frac{\Omega_1}{L_0} - 1 \). In the dynamic case, the situation is more complex.

\[
\frac{\partial}{\partial \gamma_U} \frac{dr_{d, np}}{ds} = \frac{\gamma_v}{\gamma_U + \gamma_v} \left( 1 - \frac{\Omega_2}{L_0} \right)
\]

\[
\frac{\partial}{\partial \gamma_U} \frac{dr_{d, np}}{ds} = -\frac{\gamma_v}{\gamma_U} \frac{\partial}{\partial \gamma_U} \frac{dr}{ds} - \frac{\gamma_U \Omega_2 I}{(\gamma_U + \gamma_v) \rho L_0}
\]

The sign \( \frac{\partial}{\partial \gamma_U} \frac{dr_{d, np}}{ds} \) is determined in, essentially, the same way as in the static case, expect \( \Omega_1 \) is replaced by \( \Omega_2 \). On the other hand, \( \frac{\partial}{\partial \gamma_U} \frac{dr_{d, np}}{ds} \) contains a novel term, \( -\frac{\gamma_U \Omega_2 I}{(\gamma_U + \gamma_v) \rho L_0} \), which complicates the analysis. In order to simplify, consider the fixed income case, \( \Omega_2^U = 0 \) and assume that \( \frac{\Omega_2}{L_0} = 1 \). In such circumstances, \( \frac{\partial}{\partial \gamma_U} \frac{dr_{d, np}}{ds} = 0 \), and the sign of \( \frac{\partial}{\partial \gamma_U} \frac{dr_{d, np}}{ds} \) is opposite of the sign of \( I \). Thus, due to the extra term, increases in the borrower’s and the lender’s risk aversion coefficients do not always have opposite effects, as they do in the static model.

In this section we have generalized the model BDM to include much more general distribution functions for collateral, borrower’s income, and the market interest rate, and discussed inter-relationships among the dynamic and static solutions. While in some situations static and dynamic optimal contracts overlap, or are very close to each other, in other situations they can be dramatically different, leading even to opposite directional predictions about optimal interest rate contract curves. Essentially, long-term contract functions typically have more muted responses to the changes in the market interest rate than do short term loan contract functions.
**Exogenous prepayments:** A significant constraint for our analysis relates to our assumptions that prohibit prepayments and default. We now allow exogenous prepayments\(^{34}\). Let us, for simplicity, consider BDM (Section 4) in which assumption B1' is replaced by the following:

B1') The borrower is allowed to prepay his loan at time 1, if and only if he is forced to sell the collateral due to some unforeseen circumstances at time 1. Such an event occurs with probability \(p\), independent of the other parameters of the system. The borrower pays the exogenously-fixed penalty \(\Gamma\) at time of prepayment. Otherwise, the collateral is sold at time 2. No defaults or voluntary (endogenous) prepayments are allowed.

The only difference with the original BDM model is that, now, at period 1, with probability \(p\) the borrower sells the collateral, prepayments the loan, and pays the prepayment penalty \(\Gamma\). The lender’s and borrower’s expected utilities are:

\[
Ev = p \int u((r(s) - s)L_0 + \Gamma)f(s)ds + (1 - p)Ev_{np}
\]

\[
EU = p \int U(y_1 + V(s) - (1 + r(s))L_0 - \Gamma)f(s)ds + (1 - p)EU_{np}
\]

where index \(np\) stands for "no-prepayment" expressions (Equations 4.1 and 4.2). In the case of CARA utility, the optimal loan contract interest rate satisfies the following equation:

\[
\frac{dr}{ds} = \frac{\gamma_v}{\gamma_v + \gamma_u} + \frac{\gamma_u}{\gamma_v + \gamma_u} \frac{\Omega_2}{L_0}
\]

\[
\Omega_2(s) = \frac{\Omega_2^V}{1 + \frac{1}{\rho} \left( \exp \gamma_u I(s) + \frac{p}{1 - p} \exp \gamma_u I_p \right)}
\]

where:

\[
\Omega_1^V(s) = \frac{dV}{ds}
\]

\[
I(s; x) = y_2 - y_1 + V(s) - L_0
\]

\[
I_p = y_2 - y_1 + \Gamma
\]

The solution is:

\[
r_{d, np}(s) = r_{\text{bench}}(s) + \frac{1}{(\gamma_v + \gamma_u) b L_0} \ln \left( \frac{1 + \frac{1}{\rho} \exp \gamma_u (I(0; x))}{1 + \frac{1}{\rho b} \exp \gamma_u (I(s; x))} \right)
\]

\[
b = \left( \frac{p}{1 - p} \right) \exp \gamma_u I_p + 1
\]

\(^{34}\)Prepayments caused by unforeseen events such as selling of the collateral because of illness, for example.
The general form of the solution 6.4 is similar to the case when prepayment is forbidden; and when \( p = 0 \), Equation 4.6 and 6.4 are identical.

**Comparison of models:** since \( \text{sign} \left( \frac{\partial}{\partial T} \frac{d_{\text{dwp}}}{d_s} \right) = \text{sign} \left( \frac{\partial}{\partial p} \frac{d_{\text{dwp}}}{d_s} \right) = -\text{sign} \left( \Omega_1^V \right) \), where \( d_{\text{dwp}} \) denotes "dynamic, with prepayment", if the collateral is positively correlated with the market interest rate, an increase in either probability of prepayment or the prepayment penalty, ceteris paribus, reduces the slope of the optimal contract interest rate curve, and vice versa. Since \( p = 0 \) corresponds to the situation when prepayment is not allowed, Equations 4.3 through 4.5, the optimal contract interest rate has an even more muted response to the change in the market interest rate than the dynamic contract curve when prepayments are not allowed (Sections 4 and 5). Using Proposition 5, if the collateral is positively correlated with the market interest rate, \( \frac{d_{\text{dwp}}}{d_s} < \frac{d_{\text{dwp}}}{d_s} < \frac{d_{\text{opt}}}{d_s} \) (Figure 8). The signs of the inequalities reverse when the correlation is negative (see Proposition 5) (Figure 9). Finally, when the collateral and the market interest rate are uncorrelated, \( \frac{d_{\text{dwp}}}{d_s} = \frac{d_{\text{dnp}}}{d_s} = \frac{d_{\text{opt}}}{d_s} \) and coincide with the A-B benchmark.

**Exogenous default:** We can easily incorporate exogenous default into our analysis. For each level of the borrower's future income \( y \), there is a probability of default \( q (y) \). If the loan is non-recourse, in the case of default the lender receives \( V (s) (1 - \alpha) \), where \( \alpha \) is the proportional cost of bankruptcy proceedings, while the borrower keeps his other income \( y \). In the Appendix we demonstrate that the optimal contract function for a static model with the possibility of exogenous default is identical to the no-default case (Equation 3.8). That is not true for the dynamic model. The possibility of bankruptcy changes the shape of the optimal contract interest rate functions. For example, consider the case of the UD model with no prepayment (Section 5), generating the optimal contract curve:

\[
\frac{dr}{ds} = \frac{g_v}{g_v + g_u} + \frac{1}{g_v + g_u} \frac{\Omega_1^V}{L_0} \frac{\exp \gamma_u I (s)}{1 + \frac{1}{\rho} \exp \gamma_u I (s)}
\]

\[
I (s) = V (s) - L_0
\]

\[
\rho' = \rho \int dy g (y) \exp (-y \gamma_u) \frac{(1 - q (y))^2}{\int dy g (y) \exp (-y \gamma_u) (1 - q (y))}
\]

If borrower's income is uncorrelated with the market interest rate, exogenous default renormalizes

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35 We plot the Figures 8 and 9 for \( p = 0.66 \), \( \Delta y = -1 \), and \( \Gamma = 0.5 \).

36 This precludes the possibility of strategic borrower's default, as in Jones and Nickerson (2000) or, in other words, credit rationing. Thus, we assume that the borrower can borrow enough to purchase any given collateral.

37 The results do not depend on these assumptions. All that really matters is that, in the case of default, the borrower's and lender's expected utility do not depend on the contractual interest rate.

38 Income is uncorrelated with the market interest rate so that \( g (y | s) = g (y) \), and collateral is a deterministic function of the market interest rate.
the borrower's discount rate $\rho \rightarrow \rho' < \rho$: possibility of default makes the optimal contract curve less steep, and, thus, has the same directional effect on the shape of the optimal contract curve as the possibility of prepayment\textsuperscript{39}. It is straightforward to extend the model to simultaneously include exogenous prepayments and default, and we leave it as an exercise to the reader.

7 Conclusions

We have presented a series of models that extend existing research relating to optimal loan contract interest rate design in the presence of risky, symmetric information. We obtain a series of closed form solutions for one-period (static) and multi-period (dynamic) optimal loan interest rate contracts. The design of the optimal loan contract depends on the assumed inter-temporal dynamics, the risk preference characteristics of the borrower and the lender, as well as the joint probability distribution of the collateral value, the borrower's income, and the market interest rate. While it has been argued by some (see Dymski and Isenberg (1998)) that fixed rate mortgages are more efficient and equitable for low income households, our analysis demonstrates that for risk averse households with stochastic collateral, variable interest rate contracts, if structured properly, may be Pareto optimal. In particular, if the collateral value and/or the borrower's future income are positively correlated with the market interest rate, optimal loan interest rate contracts will allocate more interest rate risk sharing to the borrower vis-a-vis the lender than in the absence of such correlation. This result occurs because positive correlation makes it easier for the borrower to repay her loan payments if the market interest rate were to rise. This, in turn, will enable the lender to receive higher loan contract interest rates in order to reduce her interest rate risk exposure without impairing the well-being of the borrower. The opposite is true when the correlation between the total borrower's wealth and the market interest rate is negative.

While in certain situations static and dynamic optimal loan contract interest rate functions coincide, or are very close to each other, in other situations they can be dramatically different, leading even to diametrically opposed terms for optimal loan interest rate contracts. For plausible assumptions, the optimal loan interest rate contract for the (inter-temporal) dynamic model often exhibits "muted" responses to the changes in the market interest rate, making fixed rate loan contracts close to the optimal design. That may explain why, in the absence of optimal contracts,\textsuperscript{39}If income and the market interest rate are correlated, however, the shape of the dynamic contract curve can change profoundly due to the interaction between the probability of default and the market interest rate. This case may warrant further study.

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long term borrowers tend to prefer fixed rate contracts, while the short term borrowers tend to prefer variable rate contracts. These conclusions are reinforced when exogenous prepayments and defaults are allowed.

Our analysis is suggestive of future avenues for extending optimal loan interest rate contracts research. Our model can easily be extended from two to many periods. Preliminary analyses shows that, qualitatively, all of the major results of the paper remain unchanged. It is equally easy to incorporate transaction costs and taxes into the analysis. Relaxing the assumption that the joint probability distribution functions are the same in each period is, in principle, straightforward, and may lead to interesting added insights. A more challenging extension would be to endogenize the prepayment penalty $\Gamma$ and probability of prepayment $p$. Finally, the probability of default may depend on the contract interest rate. Determining the optimal contract rate when endogenous default is possible requires solving a non-trivial feedback system\textsuperscript{40}. Introducing endogenous default risk into the model is potentially an important link for our model to those of credit rationing (see Jones and Nickerson (2000), and the literature therein).

8 References


\textsuperscript{40}This amounts to simultaneously endogenizing the choice of loan size $L_0$. 

23


The proof of the Proposition 2

**Proof.** Following Section 2, the optimal contract is found by solving the constrained maximization problem. The first order conditions are:

\[
0 = (1 + \delta) V_0' v'((r(s) - s)L_0) - \lambda V_0(U'(C_1) + \rho U'(C_2)) \\
\lambda = \text{const} = \frac{(1 + \delta) v'((r(s) - s)L_0)}{(U'(1) + \rho U'(2))}
\]

Taking a derivative with respect to \(s\), substituting for \(\lambda\), and simplifying yields the following expression:

\[
\frac{dr}{ds} = \frac{L_0 \gamma_v + dV_0 \gamma_v^{(2)} U''(2)}{d_U(U'(1) + \rho U'(2))}
\]

For CARA preferences, this can be re-written as:

\[
dr = \frac{\gamma_v ds}{\gamma_v + \gamma_u} + \frac{\gamma_u 1}{\gamma_v + \gamma_u} \frac{dV_0}{L_0} \frac{1}{1 + \frac{1}{\rho} \exp \gamma_u (y_2 - y_1 + V(s) - L_0)}
\]

Introducing

\[
a = \frac{1}{\rho} \exp \gamma_u (y_2 - y_1 + V_0 - L_0)
\]

\[
x = \exp \gamma_u (V(s) - V_0)
\]

the equation becomes

\[
dr = \frac{\gamma_v ds}{\gamma_v + \gamma_u} + \frac{\gamma_u}{\gamma_v + \gamma_u} \frac{dx}{L_0 x (1 + ax)} = \frac{dx}{(\gamma_v + \gamma_u) aV_0} \left( \frac{1}{x} - \frac{a}{1 + ax} \right)
\]

Performing integrations, fixing the boundary conditions \(V(0) = V_0\) and \(r(0) = r_0\) and simplifying will yield:

\[
r(s) = r_0 + \frac{\gamma_u V(s) - V_0}{\gamma_v + \gamma_u} L_0 + \gamma_v s + \frac{1}{(\gamma_v + \gamma_u) L_0} \ln \left( \frac{1 + \frac{1}{\rho} \exp \gamma_u (y_2 - y_1 + V_0 - L_0)}{1 + \frac{1}{\rho} \exp \gamma_u (y_2 - y_1 + V(s) - L_0)} \right)
\]

Proof that introducing exogenous default does not change the shape of the optimal static contract curve in the case of UD model (other models in Table 1 (Section 3) can be treated in exactly the same fashion.

Expected lender’s utility reads:

\[
Ev_v = \left(1 - \int q(y) g(y) \, dy\right) \int v(r(s) - s) L_0 \, f(s) \, ds + Ev_b
\]

\[
Ev_b = \int q(y) g(y) \, dy \int v(V(s) (1 - \alpha) - (1 + s) L_0)
\]
The first term is the expected utility of the lender conditional that default does not occur, weighted by the probability of no-default. The second term is the expected lender’s utility conditional on default occurring, weighted by the probability of default. Since default is triggered exogenously, \( q(y) \) and \( Ev_b \) do not depend on the contract rate \( r(s) \). Thus, the exact form of \( Ev_b \) has no relevance to the shape of the optimal contract curve (it drops out from the calculation). In the same fashion, the borrower’s expected utility is \( EU_b \) again drops out from the calculation:

\[
EU = \int \left( U(y + V(s) - (1 + r(s) L_0)(1 - q(y)) g(y) dy \right) f(s) ds + EU_b
\]

Performing the optimization with respect to \( r(s) \) the first order condition reads:

\[
\left(1 - \int q(y) g(y) dy\right) v'((r(s) - s) L_0) - \lambda \int U'(y + V(s) - (1 + r(s) L_0)(1 - q(y)) g(y) dy = 0
\]

Taking the derivative of both sides with respect to \( s \), and substituting the expression for \( \lambda, q(y) \) completely drops out and we obtain:

\[
\frac{dr}{ds} = \frac{1}{L_0} \frac{dV}{ds} \gamma_U + \frac{\gamma_v}{\gamma_v + \gamma_U}
\]

which coincides with the no-default case Equation 2.3. ■
Figure 1: Static Contract Curves (Basic Static Model)
Figure 3: Static Versus Dynamic Curve with Prepayment (Collateral and Income Deterministic; Increasing Collateral)
Figure 5: Static Versus Dynamic Contract Without Prepayment
(Known Income; Change in Income = 0.0)
Figure 6: Static Versus Dynamic Contract Without Prepayment

(Known Income; Change in Income = 1.0)
Figure 7: Static Versus Dynamic Contract Without Prepayment
Figure 8: Contract With and Without Prepayment (Increasing Collateral)
Known Income (d,w,p)
Increasing Collateral
Known Income (d,p)
Increasing Collateral
Known Income (st)
Increasing Collateral
Benchmark
Figure 9: Contract With and Without Prepayment (Decreasing Collateral; Change In Income = -1.0, p=0.66, Gamma=0.5)