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Author
Ma, Yiping

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Localized Structures in Forced Oscillatory Systems

by

Yiping Ma

A dissertation submitted in partial satisfaction of the requirements for the degree of
Doctor of Philosophy
in
Physics
in the
Graduate Division
of the
University of California, Berkeley

Committee in charge:
Professor Edgar Knobloch, Chair
Professor Jonathan Wurtele
Professor Andrew Szeri

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Localized Structures in Forced Oscillatory Systems

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Yiping Ma
Abstract
Localized Structures in Forced Oscillatory Systems
by
Yiping Ma
Doctor of Philosophy in Physics
University of California, Berkeley
Professor Edgar Knobloch, Chair

This dissertation studies nonlinear partial differential equations (PDEs) describing pattern formation, using a combination of analytical and numerical techniques. A major part focuses on spatially localized states in the 1:1 forced complex Ginzburg-Landau equation (FCGLE)

\[ A_t = (\mu + i\nu)A - (1 + i\beta)|A|^2A + (1 + i\alpha)\nabla^2A + \gamma, \quad A = U + iV \in \mathbb{C}, \quad U, V \in \mathbb{R}, \]

which is the normal form for a 1:1 resonantly forced Hopf bifurcation in spatially extended systems.

One-dimensional (1D) steady localized states can be fruitfully studied using tools from dynamical systems theory. In particular, the localized states consisting of a Turing pattern embedded in a background equilibrium are shown to grow along its bifurcation curve via a new mechanism called defect-mediated snaking (DMS). In this growth mechanism new rolls are nucleated from the center of the wavetrain, in contrast to standard homoclinic snaking observed in the generalized Swift-Hohenberg equation. The temporal dynamics of localized states outside the snaking region are mediated by successive phase slips resulting from the Eckhaus instability. The spatial dynamics of DMS are explained by an asymptotic theory near a saddle-center bifurcation in a planar reversible map. Aside from DMS, a new class of steady localized states consisting of phase-winding states, namely spatially periodic states with \( U \) and \( V \) out of phase, are shown to grow via collapsed snaking mediated by a central pacemaker defect rather than standard homoclinic snaking. Parameter regimes exhibiting localized spatiotemporal chaos (STC) are also identified, and weak STC is interpreted in terms of its underlying coherent structures.

Two-dimensional (2D) spatially localized states that exist in the same parameter regimes as their 1D counterparts include localized ring patterns, planar and circular localized hexagons, as well as 2D localized STC. These solutions are studied in detail using numerical continuation and direct numerical simulations.

The final chapter derives a nonlocal pattern equation for weakly nonlinear Rayleigh-Bénard convection with large aspect ratio using the Bogoliubov method coupled with a diagrammatic technique. A Lyapunov functional for the nonlocal pattern equation is found and a necessary condition for finding stable localized states in variational PDEs is formulated.
To my parents.
# Contents

List of Figures vi

List of Tables xvii

1 Introduction 1
  1.1 Oscillons: from experiment to theory ..................................... 1
  1.2 An overview of methods ......................................................... 5
    1.2.1 Asymptotic methods ...................................................... 5
    1.2.2 Numerical continuation .................................................. 6
    1.2.3 Numerical solution of PDE .............................................. 8
  1.3 A brief summary of results .................................................. 9

2 Background 12
  2.1 Local bifurcations .................................................................. 12
    2.1.1 Center manifold reduction ............................................... 14
    2.1.2 Normal form theory ....................................................... 15
    2.1.3 Local bifurcations in flows .......................................... 18
    2.1.4 Local bifurcations in maps ........................................... 21
  2.2 Global bifurcations .............................................................. 22
    2.2.1 The saddle case ............................................................ 23
    2.2.2 The saddle-focus case ................................................... 25
  2.3 Homoclinic orbits in reversible systems ................................... 28
    2.3.1 Linear theory ............................................................... 28
    2.3.2 The saddle case ............................................................ 29
    2.3.3 The saddle-focus case ................................................... 30
    2.3.4 The saddle-center case .................................................. 31
    2.3.5 Other homoclinic orbits ................................................. 32
  2.4 Homoclinic snaking and spatially localized states ...................... 32
    2.4.1 Standard homoclinic snaking ........................................... 33
    2.4.2 Depinning in variational systems ..................................... 36
    2.4.3 Recent work on localized states ...................................... 40
6.2.2 Interaction between front and saddle-node ......................................... 118
6.2.3 Phase winding states (PWS) ................................................................. 119
6.2.4 Simple localized PWS ............................................................................ 123
6.3 Temporal dynamics ............................................................................. 127
  6.3.1 Traveling waves .............................................................................. 127
  6.3.2 Dynamics of defects ....................................................................... 128
6.4 Discussion ............................................................................................ 130

7 Classification of 1D localized states ......................................................... 132
  7.1 Introduction ......................................................................................... 132
  7.2 The forced complex Ginzburg-Landau equation ..................................... 133
  7.3 Properties of the uniform states .......................................................... 134
    7.3.1 Bifurcation analysis ....................................................................... 135
    7.3.2 Spatial eigenvalues ...................................................................... 138
    7.3.3 Temporal stability ........................................................................ 145
  7.4 Heteroclinic cycles ............................................................................ 148
    7.4.1 Localized phase-winding states (LPWS) \((\alpha = 4, \beta = 1, \mu = 1)\) ...... 149
    7.4.2 Defect-mediated snaking (DMS) \((\alpha = -1.5, \beta = 6, \mu = -1)\) ....... 152
  7.5 Discussion ........................................................................................... 156
  7.6 Appendix: Weakly nonlinear analysis near \(\gamma = \gamma_{SN}\) .................... 157
  7.7 Appendix: Weakly nonlinear analysis near \(\gamma = \gamma_T\) .................... 159

8 2D localized states ...................................................................................... 163
  8.1 Introduction ......................................................................................... 163
  8.2 Planar localized hexagons ................................................................... 164
  8.3 Circular traveling fronts ...................................................................... 166
  8.4 Localized ring patterns ...................................................................... 168
  8.5 Circular localized hexagons ............................................................... 170
  8.6 Discussion ........................................................................................... 173

9 Localized spatiotemporal chaos ................................................................. 175
  9.1 Introduction ......................................................................................... 175
  9.2 Instabilities of uniform oscillations .................................................... 176
  9.3 LSTC from a Hopf instability ............................................................. 177
  9.4 LSTC from DMS branches .................................................................. 180
  9.5 Discussion ........................................................................................... 182

10 Derivation of pattern equations ............................................................... 184
  10.1 Introduction ......................................................................................... 184
  10.2 Rayleigh-Bénard Convection ............................................................. 187
    10.2.1 Boussinesq Equations .................................................................. 187
    10.2.2 Linear Theory ............................................................................ 188
    10.2.3 Nonlinear Terms ....................................................................... 190
  10.3 The Expansion Procedure .................................................................. 191
    10.3.1 The Bogoliubov Approach .......................................................... 191
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.3.2</td>
<td>Diagrammatic Notation</td>
<td>192</td>
</tr>
<tr>
<td>10.3.3</td>
<td>An Illustrative Calculation</td>
<td>195</td>
</tr>
<tr>
<td>10.4</td>
<td>A Convective Pattern Equation</td>
<td>197</td>
</tr>
<tr>
<td>10.4.1</td>
<td>Setup</td>
<td>197</td>
</tr>
<tr>
<td>10.4.2</td>
<td>Second Order</td>
<td>198</td>
</tr>
<tr>
<td>10.4.3</td>
<td>Third Order</td>
<td>199</td>
</tr>
<tr>
<td>10.4.4</td>
<td>The Evolution Equation</td>
<td>200</td>
</tr>
<tr>
<td>10.5</td>
<td>The Variational Structure and Its Applications</td>
<td>201</td>
</tr>
<tr>
<td>10.5.1</td>
<td>Variational Structure of the Nonlocal Pattern Equation</td>
<td>201</td>
</tr>
<tr>
<td>10.5.2</td>
<td>Generalized Clausius Theorem</td>
<td>203</td>
</tr>
<tr>
<td>10.5.3</td>
<td>Least Action Principle</td>
<td>206</td>
</tr>
<tr>
<td>10.5.4</td>
<td>A Conservation Law</td>
<td>207</td>
</tr>
<tr>
<td>10.6</td>
<td>A generalized nonlocal pattern equation</td>
<td>207</td>
</tr>
<tr>
<td>10.6.1</td>
<td>Second Order</td>
<td>208</td>
</tr>
<tr>
<td>10.6.2</td>
<td>Third Order</td>
<td>209</td>
</tr>
<tr>
<td>10.6.3</td>
<td>Result</td>
<td>209</td>
</tr>
<tr>
<td>10.7</td>
<td>Hopf bifurcations in spatially extended systems</td>
<td>210</td>
</tr>
<tr>
<td>10.8</td>
<td>Pattern equation of the Eckhaus instability</td>
<td>212</td>
</tr>
<tr>
<td>10.8.1</td>
<td>The Unreduced Equations</td>
<td>212</td>
</tr>
<tr>
<td>10.8.2</td>
<td>Linear Theory and the Bogoliubov Method</td>
<td>213</td>
</tr>
<tr>
<td>10.8.3</td>
<td>Second Order</td>
<td>214</td>
</tr>
<tr>
<td>10.8.4</td>
<td>Third Order</td>
<td>214</td>
</tr>
<tr>
<td>10.8.5</td>
<td>Result</td>
<td>214</td>
</tr>
<tr>
<td>10.9</td>
<td>Discussion</td>
<td>215</td>
</tr>
</tbody>
</table>

11 Conclusion | 217

Bibliography | 219
List of Figures

1.1 Experimental observation of oscillons in vertically vibrated granular media. (a) Top and side views of an individual oscillon; the two profiles are separated by one forcing period. (b) Examples of patterns exhibited by collections of oscillons – (chains of) dipoles, triangles and squares. Reproduced from Figures 1 and 3 of [174]. .................................................. 2

1.2 (a) Light propagation in an infinite Kerr medium. (b) Configuration that generates cavity solitons. Reproduced from Figure 1 of [3] .................. 4

1.3 (a-b) Examples of steady localized states in 1D studied in (a) Chapter 3 and (b) Chapter 6, shown in terms of $U(x)$ and $V(x)$ where $U$ and $V$ are the real and imaginary parts of $A$ in Eq. (1.11); (c) A sample time evolution of a 1D localized state studied in Chapter 4, shown in terms of a space-time plot of $V(x,t)$; (d) The phase portrait of the 2D map studied in Chapter 5; (e-f) Sample snapshots during time evolutions of 2D localized states studied in (e) Chapter 8 and (f) Chapter 9, shown in terms of $V(x,y)$ on square domains of size $[-L,L] \times [-L,L]$ where $L = 100$ in (e) and $L = 200$ in (f) .......... 11

2.1 Bifurcation diagrams for the saddle-node bifurcation (2.33). (a) $\epsilon_1 = \epsilon_2 = 1$; (b) $\epsilon_1 = \epsilon_2 = -1$; (c) $-\epsilon_1 = \epsilon_2 = 1$; (d) $\epsilon_1 = -\epsilon_2 = 1$. Solid (dashed) branches are stable (unstable). ........................................... 18

2.2 (a) Bifurcation diagrams for the transcritical bifurcation (2.35). (a) $\epsilon_1 = \epsilon_2 = 1$; (b) $\epsilon_1 = \epsilon_2 = -1$; (c) $-\epsilon_1 = \epsilon_2 = 1$; (d) $\epsilon_1 = -\epsilon_2 = 1$. Solid (dashed) branches are stable (unstable). (b) Perturbed transcritical bifurcation diagrams. ........................................... 19

2.3 (a) Bifurcation diagrams for the pitchfork bifurcation (2.37). (a) $\epsilon_1 = -\epsilon_2 = 1$; (b) $-\epsilon_1 = \epsilon_2 = 1$; (c) $\epsilon_1 = -\epsilon_2 = 1$; (d) $\epsilon_1 = \epsilon_2 = 1$. Solid (dashed) branches are stable (unstable). (b) Perturbed pitchfork bifurcation diagrams. 20

2.4 Bifurcation diagrams for the Hopf bifurcation (2.38). (a) The supercritical case ($a_1 < 0$); (b) the subcritical case ($a_1 > 0$). ............................... 20

2.5 Bifurcations in a Poincaré return map. (a) Period-doubling; (b) Neimark-Sacker. ................................................................. 22

2.6 Breakup of the homoclinic orbit $\Gamma$ to a saddle as $\mu$ varies around 0. ....... 24

2.7 Deformation of a rectangle $D$ by the Poincaré map $P$. (a) The orientation reversing case. (b) The orientation preserving case. ...................... 24
2.8 (a) Poincaré sections for two homoclinic orbits \( \Gamma_r \) and \( \Gamma_l \) to a saddle. (b) Horizontal and vertical strips that generate horseshoe dynamics. ..................................................... 25
2.9 Poincaré maps in the saddle-focus case. (a) The local map \( P_0 \). (b) The composite map \( P = P_1 \circ P_0 \). ................................................................. 27
2.10 Breakup of the homoclinic orbit \( \Gamma \) to a saddle-focus as \( \mu \) varies around 0. ......................................................... 27
2.11 Bifurcation diagrams of periodic orbits near the homoclinic orbit \( \Gamma \) to a saddle-focus. (a) \( \delta > 1 \). (b) \( \delta < 1 \). ......................................................... 27
2.12 Linearization around the equilibrium 0 in the 4D reversible ODE (2.60). Reproduced from Figure 1 of [46]. ................................................................. 29
2.13 Floquet multipliers for reversible periodic orbits in 4D reversible ODEs. (a,b,c): configuration (I,II,III). Reproduced from Figure 2.8 of [35]........... 30
2.14 The orbit flip bifurcation in a 4D reversible ODE. Reproduced from Figure 2 of Ref. [46] ................................................................. 31
2.16 Heteroclinic tangle in a 2D reversible map. Reproduced from Figure 1.3 of Ref. [18]. ................................................................. 34
2.17 (a) The horizontal (vertical) lines labeled by \( n^u \) (\( n^s \)) represent segments of \( W^u(0) \) (\( W^s(0) \)). The red (blue) dots correspond to symmetric localized states with maxima (minima) at \( x = 0 \), while the black crosses correspond to asymmetric localized states. (b) The snakes-and-ladders bifurcation diagram where \( r \) is the control parameter and \( N = (n^u + n^s)/2 \) is the norm. The solid red, blue and black segments represent respectively the red dots, blue dots and black crosses in (a). ................................................................. 35
2.18 The snakes-and-ladders bifurcation diagram and selected profiles of localized states in the nonlinear optical model \((1 + \partial_x)^2 f + \epsilon^4 f + 3\epsilon E_{hg} f^2 + \epsilon^2 f^3 = 0\) which is a rescaled version of the steady state of SH23 (2.64). The bifurcation diagram has been computed at \( \epsilon = 0.55 \) for the optical model with control parameter \( E_{hg} \), but qualitatively the same bifurcation diagram with control parameter \( r \) is present in the steady state of (2.64) at suitable \( b_2 \) (e.g. \( b_2 = 1.8 \)). The two snakes are shown in black while the ladders are shown in green. Reproduced from Figure 2 of Ref. [52]........................................... 37
2.19 Space time plots showing the depinning of a localized state in SH23 (2.64) (a) \( |\delta r| = 0.0005 \) to the left of \( r_- \) and (b) \( |\delta r| = 0.0005 \) to the right of \( r_+ \). Parameters: \( b_2 = 1.8 \). Also shown is the depinning period \( T \) as a function of \( r \) with numerical values in circles and predicted values in a solid curve. Reproduced from Figures 3.10 and 3.11 of Ref. [35]. ................................................................. 39
3.1 (a) The critical curves and limits of snaking regions in the \((\nu, \gamma)\) plane. The lines \( \gamma^{SN} \) delimit the region with three homogeneous equilibria while the line \( \gamma^T \) corresponds to Turing bifurcations on \( A^+ \). Belyakov-Devaney points on \( A^- \) occur along the line \( \gamma^{BD} \). The transition from collapsed snaking at \( \gamma^{CS} \) to defect-mediated snaking between \( \gamma_1^{DMS} \) and \( \gamma_2^{DMS} \) occurs at the codimension-two point \((\nu^*, \gamma^*) \approx (5.983, 2.339)\), marked with an open circle. (b) At larger \( \nu \), defect-mediated snaking has an additional limit \( \gamma_3^{DMS} \). ................................................................. 48
3.2 (a) Bifurcation diagram corresponding to the \( \nu = 5 \) slice of Fig. 3.1, where the branch of localized states undergoes collapsed snaking towards \( \gamma = \gamma^{CS} \approx 1.842 \). The branch of localized states is computed on a domain of half-length \( \ell/2 = 50 \). Stable (unstable) segments are shown in solid (dashed) lines. For clarity, only the first three unstable segments are shown. On the homogeneous branches, solid (dotted) lines correspond to stable (unstable) solutions, and the labels indicate the spatial eigenvalue configurations. (b) A sample solution high up the \( L_0 \) branch.

3.3 (a) Bifurcation diagram corresponding to the \( \nu = 7 \) slice of Fig. 3.1, where the branch of localized states undergoes defect-mediated snaking between \( \gamma_1^{DMS} \approx 2.8949 \) and \( \gamma_2^{DMS} \approx 2.8970 \). Stability on the homogeneous branches is as in Fig. 3.2, but stability on \( L_0 \) is not indicated. (b) A sample solution high up the \( L_0 \) branch.

3.4 (a) Detail of the \( L_0 \) snaking branch in Fig. 3.3. Temporally stable (unstable) segments are shown as solid (dashed) lines and coincide with uniform (defect) segments. (b) Five sample profiles at \( \gamma = 2.896 \). The spatial phase \( \Phi \) is indicated for each profile on a uniform amplitude segment.

3.5 Section of the surface of spatially periodic states for the parameters used in Fig. 3.3. The surface is bounded by the neutral stability curve of \( A^+ \). The configuration of the Floquet multipliers in each shaded region is indicated. The curve \( C \) shows the wavenumber \( k(\gamma) \) of the patterns included in defect-mediated snaking at this value of the parameters and spans the width of the Eckhaus band (inset).

3.6 The phase \( \Phi(\gamma) \) of the solution across two complete turns of the snaking branch in Fig. 3.4a. The labels mark the phases of the profiles shown in Fig. 3.4b.

3.7 Profiles from the defect segments across the snaking region of Fig. 3.4. The plot range is truncated to focus on the changes in \( V(x) \) near the center of the domain at \( x = 0 \). The direction of decreasing \( \gamma \) is indicated by an arrow, with the upper (lower) frames taken at the right (left) boundary of the snaking region.

3.8 (a) Bifurcation diagram corresponding to the \( \nu = 36 \) slice of Fig. 3.1, where the branch of localized states undergoes three-limit defect-mediated snaking. Stability on the homogeneous branches is as in Fig. 3.2, but stability on \( L_0 \) is not indicated. (b) A sample solution high up the \( L_0 \) branch.

3.9 (a) One complete turn of the snaking branch \( L_0 \) in Fig. 3.8. Solid (dashed) lines show uniform (defect) segments of the branch. Temporal stability is not indicated. The limiting values for the saddle-node bifurcations are \( \gamma_1^{DMS} \approx 30.872 \), \( \gamma_2^{DMS} \approx 30.695 \) and \( \gamma_3^{DMS} \approx 30.285 \). (b) Five sample solutions at \( \gamma = 30.5 \). The spatial phase \( \Phi \) is indicated for profiles on the uniform amplitude segments. As \( \Phi \) is defined in terms of \( V(x) \), the change in the shape of \( U(x) \) between (i) and (ii) does not change \( \Phi \).
3.10 Profiles $U(x)$ at six locations straddling a fold at $\gamma_{3}^{DMS}$ on a uniform segment (solid line in Fig. 3.9) showing the transition from a state with a maximum at $x = 0$ (panel (a)) to one with a minimum at $x = 0$ (panels (d)–(f)) in the vicinity of the fold. During this transition $V(x)$ remains qualitatively unchanged. The locations (a)–(f) are indicated in Fig. 3.15 below.

3.11 A section of the surface of spatially periodic states for the parameters used in Fig. 3.8. Refer to the text for notation.

3.12 The phase $\Phi(\gamma)$ of the solution across two complete turns of the snaking branch in Fig. 3.9a. The labels mark the phases of the profiles shown in Fig. 3.9b.

3.13 Space-time plot of $V(x, t)$ for $\nu = 7$ showing the depinning of the fronts bounding a localized state at (a) $10^{-3}$ to the left of $\gamma_{1}^{DMS}$ and (b) $10^{-4}$ to the right of $\gamma_{2}^{DMS}$. The final state in (a) is a steady localized state consisting of two wavelengths; in (b) the expanding state eventually fills the whole domain.

3.14 Plots of $1/T^2$, where $T$ is the time between the first two phase slips, as a function of the distance $|d\gamma|$ from the saddle-node for (a) $\gamma < \gamma_{1}^{DMS}$ and (b) $\gamma > \gamma_{2}^{DMS}$. The initial conditions are taken at the 69-th and 70-th saddle-nodes, respectively, and the $|d\gamma|$’s are measured relative to the location of the 67-th and 72-nd saddle-nodes. The solid lines show the best fit lines (a) $T^2 = 4.533 \times 10^{-4}|d\gamma| - 5.612 \times 10^{-10}$, and (b) $T^2 = 6.766 \times 10^{-4}|d\gamma| - 3.606 \times 10^{-10}$. Parameters: $\alpha = -1.5$, $\beta = 6$, $\mu = -1$, $\nu = 7$.

3.15 (a) A branch of uniform amplitude localized states when $\nu = 36$ with different symbols indicating different final states reached from small amplitude perturbations (see text). The solution profiles at locations (a)–(f) straddling the fold at $\gamma_{3}^{DMS}$ are shown in Fig. 3.10. (b) Space-time plot of $V(x, t)$ showing the fast transition from the localized state (i) on the upper branch to the corresponding state on the lower branch.

3.16 Time evolution of the localized state (ii) in Fig. 3.15a. (a) Snapshots of the solution profile at two instants half an oscillation period apart ($t_1 - t_0 \approx 0.3$), corresponding to maxima of the oscillation amplitude. (b) The time series $V(x_0, t)$ at fixed $x_0 \approx 6.05$.

3.17 Space-time plot showing the evolution of $V(x, t)$ corresponding to the localized state (iii) in Fig. 3.15a into a Hanoi tower. The final state is a stable spot at $x \neq 0$.

3.18 The condition (3.32) in the $(E/E_c, L/L_c)$ plane. The red dots indicate the Eckhaus points. Quasiperiodic solutions exist in the shaded region, consistent with [100].

4.1 Space-time plots of $V(x, t)$ showing Type-I LS depinning at (a) $d\gamma = 0.04$; (b) $d\gamma = -0.04$; (c) $d\gamma = -0.24$. 
4.2 Plot of the front speed \( c \) as a function of \( \gamma \) at \( \nu = 5 \). Open circles: DNS of (1.11). Solid line: continuation of heteroclinic orbits in (4.2). Vertical dashed lines: SN bifurcations (left/right: \( \gamma_{SN}^+/- \)). Horizontal dot-dashed line: RB bifurcations on \( A^+ \). The insets show the eigenvalue configuration of \( A^+ \) in (4.2) for different signs of \( c \). ........................................ 73

4.3 The first two phase slips during the depinning of Type-II LS, plotted on the left half-domain for \( N = 67 \). (a) Slow depinning (\( d\gamma = -2 \times 10^{-5} \)): phase slips take place at the center \( x = 0 \). (b) Fast depinning (\( d\gamma = -4 \times 10^{-3} \)): phase slips take place at a constant distance from the moving front. (c) Intermediate case (\( d\gamma = -1 \times 10^{-3} \)): phase slips gradually move towards the front. Figure (b) introduces the quantities \( T, \Lambda \) and \( X \) defined in the text. 76

4.4 Plots of \( T, \Lambda \) and \( X \) as functions of \( |d\gamma| \) for (a,c,e) \( \gamma_{DMS}^1 \) (cross: \( N = 67 \), circle: \( N = 175 \)); (b,d,f) \( \gamma_{DMS}^2 \) (cross: \( N = 72 \), circle: \( N = 162 \)). The horizontal scale for \( |d\gamma| \) is logarithmic. The vertical scale is logarithmic for \( T \) but linear for \( \Lambda \) and \( X \)............................. 77

4.5 Fast depinning of a Type-II LS as a pulsating front. In each case the space-time plot is drawn in a frame comoving with the front. (a) Sink in the 1:1 FCGLE (1.11) at \( d\gamma = -4 \times 10^{-3} \); (b) source in (1.11) at \( d\gamma = 1 \times 10^{-2} \); (c) sink in SH23; (d) source in SH23, where SH23 refers to the PDE \( ut = [r - (1 + \partial_{xx})^2]u + b_2u^2 - u^3 \). Depinning is shown for (c) \( r = -0.3410 \) and (d) \( r = -0.2580 \) at fixed \( b_2 = 1.8 \). For comparison, the snaking region of steady Type-II LS is \( r \in [-0.3390, -0.2593] \). ..................... 79

4.6 (a) Plot of the front speed \( c \) as a function of \( \gamma \) at \( \nu = 7 \). Crosses: DNS of (1.11) for depinning of Type-II LS. Solid line: continuation of heteroclinic orbits in (4.2). Dashed: SN bifurcations (left/right: \( \gamma_{SN}^+/- \)). Dot-dashed: RB bifurcations on \( A^+ \). (b) Front profiles at (i) \( c = -0.01 \); (ii) \( c = 0.01 \). The insets show that the tail wavenumber in (ii) is larger than (i), consistent with the stable eigenvalues of \( A^+ \) depicted in (a). ..................... 80

4.7 Space-time plot of \( V(x,t) \) showing the time evolution of (a) a Type-II LS at \( (\gamma,k) = (2.8947, 1.85) \); (b) a Type-I LS at \( \gamma = 2.87 \). ..................... 82

4.8 The Eckhaus, defect and mixed modes \( (A_E(x), A_d(x) \) and \( A_m(x)) \) at (a,c,e) \( N = 67 \); (b,d,f) \( N = 72 \). The Eckhaus and defect modes are computed at \( |d\gamma| = 2 \times 10^{-5} \)............................. 84

4.9 Growth rates of the Eckhaus and defect modes \( (\sigma_E \) and \( \sigma_d \)) as functions of \( |d\gamma| \) at (a,c) \( N = 67 \); (b,d) \( N = 72 \). ..................... 85

4.10 The slope \( m \) and the intercept \( \tilde{\gamma} \) in (4.18) as functions of \( N \) for (a) & (c) \( \gamma_1^{DMS} \) and (b) & (d) \( \gamma_2^{DMS} \). ‘\( \circ \)’: DNS; ‘\( + \)’: weakly nonlinear theory. .... 88

5.1 The dynamics of the normal form (5.1)-(5.2). ........................................ 93

5.2 (a) The assumed intersection between \( W^u(B) \) and \( W^s(A_0) \). (b) The corresponding case for \( \lambda > 0 \) showing the infinite number of intersections between \( W^u(B) \) and \( S \). .... 94
5.3 (a) A typical set $\Gamma$ (blue curve) and its computed forward iterates $\Gamma^n$, $n = 1, 2, \ldots$ (red curves) for $\lambda = 0.01$ with the appropriate level set of $H$ shown as a thin solid line. (b) A similar computation for $\lambda = -0.002$.

5.4 (a) The set $\Gamma$ (blue curve) and its first 10 iterates $\Gamma^n$, $n = 1, 2, \ldots$ (red curves) for $\lambda = 0.01$ with the appropriate level set of $H$ shown as a thin solid line. (b)–(d) Similarly depicted $\Gamma$ and its tenth iterate $\Gamma^{10}$ for: (b) $\lambda = -0.002$, $\beta = 0$; (c) $\lambda = 0.02$, $\beta = 0.02$; (d) $\lambda = 0.02$, $\beta = -0.0022$. Other parameters are $\kappa = 50$, $\delta = 0.2$ and $\varepsilon = 0.02$.

5.5 Two-parameter continuation of curves of inner tangency between $\Gamma^n$ and the symmetric section $S$ for each of $n = 7, 8, 9, 10$. The slanted parabolas are the tip tangency curves, while the horizontal parabolas are the side tangency curves. The parameters are the same as Fig. 5.4(a) except that $\kappa = -50$.

5.6 (a) Bifurcation diagrams showing the two primary snaking branches ($L_0$ and $L_\pi$) at $\gamma = -0.04$. The “ladders” are not shown here. (b) Sample profiles on the 6-th C-shaped segment of the $L_0/L_\pi$ branch are shown in (i-ii)/(iii-iv).

5.7 (a) A bifurcation diagram showing the hybrid snaking branch $L^h_0$ at $\gamma = -0.04$. (b) Sample profiles on the 6-th C-shaped segment of the $L^h_0$ branch are shown in (i-ii), with their cores and fronts featured in (iii-iv).

5.8 (a) A bifurcation diagram showing the hybrid snaking branch $L^h_\pi$ at $\gamma = -0.04$. (b) Sample profiles on the 6-th C-shaped segment of the $L^h_\pi$ branch are shown in (i-ii), with their cores and fronts featured in (iii-iv).

5.9 (a) A bifurcation diagram showing the stack of isolas created from mating between the $L_0$ and $L^h_0$ snakes at $\gamma = -0.64$. Only the first 9 figure-8 isolas are shown. (b) Sample profiles on the 9-th figure-8 isola are shown in (i-iv).

5.10 (a) A bifurcation diagram showing the stack of isolas created from mating between the $L_\pi$ and $L^h_\pi$ snakes at $\gamma = -0.64$. Only the first 9 figure-8 isolas are shown. (b) Sample profiles on the 9-th figure-8 isola are shown in (i-iv).

5.11 Bifurcation diagrams showing (a) the two periodic branches $P_1$ and $P_2$ at $\gamma = -0.4$ reconnecting into (b) the two periodic branches $P_3$ and $P_4$ at $\gamma = -0.44$.

5.12 The family of invariant submanifolds $\Omega(\lambda)$ near an Eckhaus bifurcation. On each $\Omega(\lambda)$ the dynamics are equivalent to a saddle-center normal form. Also shown is the unstable manifold segment $\Gamma$ which generically does not lie within a single $\Omega(\lambda)$.

5.13 (a) Accumulation of the $n$-th fold $\gamma_n$ on $L_0$ for odd $n$ to the limiting value $\gamma_\infty \equiv \gamma_1^{DMS}$ near the left edge of the DMS region. (b) Plot of $n$ as a function of $\sigma \equiv |\gamma_n - \gamma_\infty|^{-1/4}$ for $n = 23$ to 71 in steps of 2. The black circles are the numerical result, and the blue line is the best fit based on Eq. (5.31). (c) Plot of $n'(\sigma)$ as a function of $\Gamma$ over the same range of $\Gamma$ as (b).
5.14 (a) The segments comprising $W^u(A^-) \cap \Sigma$ (where $\Sigma \equiv \{(U,V,U_x,V_x): U_x = 0\}$) computed at $\gamma = 2.8948$ (vertical slice in Fig. 5.13(a)) near the $n$-th fold for $n = 5, \ldots, 27$ in steps of 2. As $n$ increases, the color of the corresponding segment changes linearly from red to blue. For visualization purposes, these segments have been projected onto the $(V,V_x)$ plane. (b/c) Details of all segments in the left/right cluster near $V_x = 0$. (d/e) Further details of larger $n$ segments in the left/right cluster.

6.1 Typical profiles of steady-state solutions studied in §6.2 and their phase portraits in the $(U,V)$ plane. (a) A phase kink (PK). (b) A phase-winding state (PWS). (c) A primary localized phase-winding state (LPWS). (d) A secondary LPWS.

6.2 The branch following from the saddle-node bifurcation ($\alpha = 4, \beta = 1, \mu = 1$, and $\nu = 1.2$). (a) Bifurcation diagram in terms of the $L^2$-norm $N$. The black and blue curves are respectively the branches of equilibria and localized states. (b) Sample solution profiles plotted for $U$ and $V$ as functions of $x$ (left) and on the $(U,V)$ phase plane (right). On the left panels $U$ and $V$ are respectively shown in red and green. On the right panels the red/green/blue dot marks the location of $A^+/A^0/A^-$. .

6.3 (a) The loci of saddle-node bifurcations $\gamma^\text{SN}_\nu$, Turing/BD bifurcations $\gamma^\text{T}_\nu$ and Hopf bifurcations $\gamma^\text{H}_\nu$ of the equilibria on the $(\nu,\gamma)$ plane. Also shown are the existence curves $\gamma^*$ for the fronts between $A^\pm$, and $\gamma^\text{LP}_1$ for the primary LPWS. (b) Sample profiles during the transition from $\gamma^*$ to $\gamma^\text{LP}_1$.

6.4 (a) Details of the transition from $\gamma^*$ to $\gamma^\text{LP}_1$ for the two-parameter continuation in $(\nu,\gamma)$ as shown in Fig. 6.3(a). Since $\gamma$ appears to be a single-valued function of $\nu$, in this figure we have plotted $\nu$ as a function of the continuation step. (b) Sample profiles during the transition from localized phase kinks to localized phase-winding states (the sequel to Fig. 6.3(b)).

6.5 The existence curve $\gamma^\text{PK}_1^P(\nu)$ for the family of PK embedded in $A^+$, and sample profiles on this curve.

6.6 PWS branches that follow from (a) the primary and (b) the secondary PK at $\nu = 1.2$. .

6.7 PWS branch at $\nu = 1.5$. .

6.8 Bifurcation diagram for the $\phi = \pi$ branch of localized states when $\nu = 1.2$. The branch originates from the (subcritical) Turing bifurcation at $\gamma^\text{T}_-$. .

6.9 Bifurcation diagram for the $\phi = \pi$ branch of localized states created in the Turing bifurcation when $\nu = 1.5$. .

6.10 The locus $\gamma^\text{LP}_2$ of secondary LPWS on the $(\nu,\gamma)$ plane, together with $\gamma^*$ and $\gamma^\text{LP}_1$. (i) $\nu = 1.5$. (ii) $\nu = 1.2$. (iii) Vicinity of $\gamma^*$. .

6.11 Bifurcation diagram for the $\phi = \pi$ branch of localized states created in the Turing bifurcation when $\nu = 3$. (i) $\gamma = 1 \times 10^{-3}$, (ii) $\gamma = 1 \times 10^{-4}$, (iii) $\gamma = 1 \times 10^{-5}$. .

6.12 The TW branch at $\gamma^\text{LP}_2$ for $\nu = 1.5$. .

6.13 Sample profiles of the initial condition $24^+12^-24^+$ assembled at (a) $\nu = 1.5$, $\gamma = 0.15$, $\Phi^\pm = -\pi$; (b) $\nu = 1.5$, $\gamma = 0.15$, $\Phi^\pm = 0$. .
6.14 Space-time plots of $V(x, t)$ for $\nu = 1.5$ showing the symmetric defects at (a) $\gamma = 0$, $\Phi^\pm = -\pi$; (b) $\gamma = 0$, $\Phi^\pm = 0$. ........................................ 129

6.15 Space-time plots of $V(x, t)$ for $\nu = 1.5$ showing the symmetric defects at (a) $\gamma = 0.15$, $\Phi^\pm = -\pi$; (b) $\gamma = 0.15$, $\Phi^\pm = 0$. ........................................ 130

6.16 Space-time plots of $V(x, t)$ for $\nu = 1.5$ showing the symmetric defects at (a) $\gamma = 0.2$, $\Phi^\pm = 0$; (b) $\gamma = 0.35$, $\Phi^\pm = 0$. ........................................ 131

7.1 The uniform states. The first and second columns show, respectively, the critical curves in the $(\nu, \gamma)$ plane and $(\nu, R_2)$ plane. Blue curves denote saddle-node bifurcations while green curves denote Hopf bifurcations. The gray curves in the second column denote projections of the saddle-node bifurcations onto the competing branch. The third column shows the bifurcation diagrams corresponding to constant $\nu$ slices (dotted vertical lines in the first two columns) in matching colors. Hopf bifurcations are denoted, respectively, by a dashed magenta line for Hopf bifurcations on the left (i.e. smaller $\nu$) and dashed black line for Hopf bifurcations on the right (i.e. larger $\nu$). First row: $\mu = 1$, $\beta = 0$ (region: $\mu > 0$, $0 \leq \beta < 1/\sqrt{3}$); slices: $\nu = 0.3$ (red), $\nu = 0.54$ (green), $\nu = 0.6$ (blue). The symmetry-related Hopf bifurcation on the left is not shown. Second row: $\mu = 1$, $\beta = 1$ (region: $\mu > 0$, $1/\sqrt{3} < \beta < \sqrt{3}$); slices: $\nu = 0.28$ (cyan), $\nu = 1.5$ (red), $\nu = 3$ (green), $\nu = 4$ (blue). Third row: $\mu = 1$, $\beta = 2$ (region: $\mu > 0$, $\beta > \sqrt{3}$); slices: $\nu = 0.75$ (cyan), $\nu = 3$ (red), $\nu = 4$ (green). Fourth row: $\mu = -1$, $\beta = 3$ (region: $\mu < 0$, $\beta > \sqrt{3}$); slices: $\nu = 15$ (red). The black dashed lines in the $(\nu, R_2)$ plane in the last two rows are the asymptotes for the blue hyperbola. ....................... 139

7.2 Equilibria in the $(\nu, R_2)$ plane. The bold face digits label the spatial type in the region delimited by $R_{SN}^\nu$ and $R_{2+}^\nu$ using the scheme in Fig. 2.12. The red (orange) portion of the saddle-node conic section represents $R_{2+}^SN$ ($R_{2-}^SN$). In the first two rows, the black dashed lines are the asymptotes for the hyperbola represented by $R_{2+}^SN$. The green (cyan) Turing line represents $R_{2+}^SP$ ($R_{2-}^SP$). The red (green) dashed line represents $\nu_T^+$ ($\nu_T^-$). Parameters: (a) $\alpha = -0.3$, $\beta = 3$, $\mu = 1$. (b) $\alpha = 3$, $\beta = 3$, $\mu = 1$. (c) $\alpha = -2$, $\beta = 3$, $\mu = 1$. (d) $\alpha = -0.3$, $\beta = 3$, $\mu = -1$. (e) $\alpha = 3$, $\beta = 3$, $\mu = -1$. (f) $\alpha = -3$, $\beta = 3$, $\mu = -1$. (g) $\alpha = 2$, $\beta = 1$, $\mu = 1$. (h) $\alpha = 2$, $\beta = 1$, $\mu = -1$. ........................................ 142

7.3 The critical curves in the $(\beta, \alpha)$ plane. Different groups of critical curves are represented by different colors: $\alpha_{SN}$ (blue), $\alpha_{HY}$ (purple), $\alpha = \beta$ (gray), $\alpha_{ST}$ (cyan), $\alpha_{ST}$ (red), $\alpha_{ST}$ (green). The upper branch of $\alpha_{ST}$ lies off-scale in $\alpha$ and is not shown. (a) $\mu > 0$. (b) $\mu < 0$. .............................. 145

7.4 The dispersion relation $s(k^2)$. The red (green) curves indicate $\Re(s)$ ($\Im(s)$). .................................................. 146

7.5 Temporal regions in the $(\nu, R_2)$ plane. The red (orange) portion of the saddle-node conic section represents $R_{2+}^SN$ ($R_{2-}^SN$). The purple lines represent $R_{2+}^NP$. The two black dashed rays represent the Hopf line $R_2 = \mu/2$. The bold face letters label the temporal types. Parameters: (a) $\beta = 1$, $\mu = 1$. (b) $\beta = 3$, $\mu = 1$. (c) $\beta = 1$, $\mu = -1$. (d) $\beta = 3$, $\mu = -1$. ........................................ 147
7.6 The critical curves for spatial and temporal stability of equilibria in the $(\nu, R^2)$ plane at $\alpha = 4$, $\beta = 1$ and $\mu = 1$. The colors of these curves correspond with earlier definitions. Also shown are the existence curves $R^\nu_2(\nu)$ for primary fronts (PF) in dot-dashed blue, $R^{LP}_2(\nu)$ for primary localized phase-winding states (LPWS) in solid blue and $R^{LP}_2(\nu)$ for secondary LPWS in solid yellow.

7.7 Bifurcation diagram for $\alpha = 4$, $\beta = 1$, $\mu = 1$, $\nu = 0.4$, $L = 200$. (a) The $L^2$-norm $N \equiv \sqrt{L^{-1} \int_0^L |A|^2 \, dx}$ as a function of $\gamma$. The blue curve represents the branch followed by AUTO, while the black curve represents the uniform state(s). (b) The solution profiles at three points on the bifurcation diagram in (a). The left part is the plot of $U$ and $V$ as functions of $x$, with $U/V$ represented by the solid red/green line. The right part is the plot of the projection of the solution trajectory on the 2D $(U, V)$ phase plane, with the horizontal/vertical axis representing $U/V$. The blue curve shows the trajectory and the dot(s) show the uniform state(s), with multiple equilibria distinguished by different colors.

7.8 Closeup of the lower/upper (a)/(b) portion of the branch following from the Turing bifurcation ($\phi = 0/\phi = \pi$ branch (red/green), $\alpha = 4$, $\beta = 1$, $\mu = 1$, $\nu = 2$, $L = 400$), and the branch of symmetric periodic states containing $SP^{\ast}$ (blue).

7.9 The critical curves for spatial stability of equilibria on the $(\nu, R^2)$ plane at $\alpha = -1.5$, $\beta = 6$ and $\mu = -1$. The colors of these curves correspond with earlier definitions. Also shown are the existence curve for PF in dot-dashed blue, and the limits of defect-mediated snaking (DMS) region in dot-dashed black.

7.10 (a) Bifurcation diagram for $\alpha = -1.5$, $\beta = 6$, $\mu = -1$ and $\nu = 7$. The first half of the branch is shown as the blue curve; the second half is shown in grey for comparison. The branch leaves the DMS region $\Gamma$ on the right and enters the pinning region $\Gamma'$ in the middle. (b) Five profiles on the blue curve in (a). In contrast to Fig. 3.4, all profiles are shown with the core placed at $x = 0$.

7.11 The sequel to Fig. 7.10.

8.1 Snapshot of $V(x, y)$ at (b) $t = 100$, starting from a localized stripe pattern at (a) $t = 0$. The domain size is $[-100, 100] \times [-100, 100]$ throughout this chapter. Parameters: $\nu = 7$, $\gamma = 2.8972$.

8.2 Snapshots of $V(x, y)$ at (a) $t = 400$ and (b) $t = 1000$ showing the shrinkage of planar localized hexagons. Parameters: $\nu = 7$, $\gamma = 2.8955$. A video can be viewed at http://www.youtube.com/watch?v=5OkFRs1Sczw.

8.3 Snapshots of $V(x, y)$ at (a) $t = 1500$ and (b) $t = 4000$ showing the expansion of planar localized hexagons. Parameters: $\nu = 7$, $\gamma = 2.899$. A video can be viewed at http://www.youtube.com/watch?v=gfsXKo1AQ0g.
8.4 Snapshots of $V(x,y)$ at (a) $t = 10000$ and (b) $t = 30000$ showing the competition between shrinkage and expansion of planar localized hexagons. Parameters: $\nu = 7, \gamma = 2.8972$. A video can be viewed at http://www.youtube.com/watch?v=6A0_ZDc0c_8. ........................................ 166

8.5 (a) The branch of 2D radially symmetric steady states followed from the lower saddle-node at $\nu = 5$. A video can be viewed at http://www.youtube.com/watch?v=cSE0Hk2sOP4. (b) A sample solution profile $V(x,y)$. .......... 167

8.6 Snapshots of $V(x,y)$ at (a) $d\gamma = -0.001$, $t = 600$, and (b) $d\gamma = 0.001$, $t = 600$, starting from the steady circular front in Fig. 8.5(b). Parameters: $\nu = 7$. Videos can be viewed at http://www.youtube.com/watch?v=gXVrvd1lvyo, and http://www.youtube.com/watch?v=VuL-IlqJrdo. ............ 168

8.7 The sequel to Fig. 8.6(b) at (a) $t = 3240$ and (b) $t = 3600$. ...... 169

8.8 (a) The branch of 2D radially symmetric steady states followed from the lower saddle-node at $\nu = 7$. A video can be viewed at http://www.youtube.com/watch?v=Z_MGpmm0uHk. (b) A sample solution profile $V(x,y)$. .......... 169

8.9 The branch of $d$-dimensional radially symmetric steady states followed in $d$ from a 1D localized state ($d = 1$) on the DMS branch at $\nu = 7, \gamma = 2.897$. 170

8.10 Snapshots of $V(x,y)$ at (a) $t = 300$, (b) $t = 450$, (c) $t = 4080$, and (d) $t = 5000$ showing the expansion of circular localized hexagons. Parameters: $\nu = 7, \gamma = 2.8989$. A video can be viewed at http://www.youtube.com/watch?v=-9H_6-lQHAY. ........................................ 172

9.1 An example of STC produced from the Kuramoto phase instability ($\alpha = 4$, $\beta = 1$, $\mu = 1$, $\nu = 1.5$, $\gamma = 0.385$). (a) 1D time evolution shown as a space-time plot of $V(x,t)$. (b) 2D time evolution shown as a snapshot of $V(x,y)$ at $t = 1000$, where $(x,y) \in [-400, 400] \times [-400, 400]$. A video can be viewed at http://www.youtube.com/watch?v=TKSA9VNdwcY. ................ 178

9.2 (a) Space-time plot of $V(x,t)$ at $\nu = 2.68$ and $\gamma = 1.5731$ showing an almost steady front between $A^+$ and a stable UO. (b) Snapshot of $V(x,y)$ at $\nu = 2.7$ and $\gamma = 1.6$ showing a stable cross-shaped solution homoclinic to $A^+$. The domain size is $[-100, 100] \times [-100, 100]$. A video can be viewed at http://www.youtube.com/watch?v=kr-31OxzwP8. ................ 179

9.3 Space-time plots of $V(x,t)$ showing weak and strong LSTC. (a) $\nu = 2.65$, $\gamma = 1.5409$; (b) $\nu = 2.4$, $\gamma = 1.277$. 179

9.4 (a) The long-time evolution of Fig. 9.3(b). (b) Snapshot of $V(x,y)$ at $t = 1000$ showing a planar LSTC. The parameters are the same as in (a). The domain size is $[-200, 200] \times [-200, 200]$ A video of a domain-filling 2D STC can be viewed at http://www.youtube.com/watch?v=YnBkJVQqV4g. 180

9.5 Snapshots of $V(x,y)$ at $t = 1000$ showing (a) a circular LSTC; (b) a reciprocal circular LSTC. The parameters and the domain size are the same as in Fig. 9.4(b). ................................. 181

9.6 Space-time plots of $V(x,t)$ at $\beta = 4$ and (a) $\gamma = 0.18$; (b) $\gamma = 0.181$. 182

9.7 Space-time plots of $V(x,t)$ at $\beta = 6$ and (a) $\gamma = 0.152$; (b) $\gamma = 0.15742$. 183
10.1 The “zig-zag” instability of rolls as seen in Avsec’s experiments [13]. The arrow represents the prevailing wind that selects the rolls. To see how well this is reproduced by the S-H equation go to the web site of M.C. Cross http://www.cmp.caltech.edu/~mcc/Pat terns/D emo6_3.html. 

10.2 The curve of resonances. We plot $\sigma + \sigma_k + k_1 \hat{z}$ in red and $\gamma_k$ in blue, with their intersection implying resonances. Parameters: $k_1 = 1$, $k_2 = 2$, $\sigma = 0.1$, $\gamma = -5$. 

10.3 Plot of $\hat{U}_{w,0}^{(2)}$, $\hat{U}_{\theta,0}^{(2)}$, $\hat{U}_{w,-1}^{(2)}$ and $\hat{U}_{\theta,-1}^{(2)}$ as functions of $z$ and $c_{p,q}$. 

10.4 Plot of $L_-$, $L_0$ and $L_+$ as functions of $c_{q,r}$. 

10.5 Plot of (10.114) on the $\rho_0$ (horizontal) - $\rho_2$ (vertical) plane for the S-H equation (10.111). (10.100)/(10.107) defines the solid line/shaded region. Their intersection forms the segment of possible stable localized states. From left to right, $d$ varies from 1 to 5, whereas $d \geq 6$ is qualitatively the same as $d = 5$. 

List of Tables
Acknowledgments

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Chapter 1

Introduction

The awareness of patterns in both natural and man-made circumstances is a significant factor in the progress of a wide range of sciences. Patterns are seen on all length scales from nanometers to megaparsecs and in many temporally evolving processes. By now, several books relating to this subject have appeared such as [145], [66], [98], [136], [85], [133] on the technical side, and [15] in a qualitative vein, while professional reviews [67], [30] provide some in-depth discussion. Patterns are also readily discerned in publications not specifically dedicated to that subject; examples from biology [170] and astronomy [32] are especially germane. Among the plethora of experiments exhibiting patterns, the most relevant to the current work is the observation of oscillons in forced media, which is summarized below.

1.1 Oscillons: from experiment to theory

The first experimental observation of stable, two-dimensional (2D) localized excitations was made in granular materials – “highly dissipative media composed of macroscopic grains that interact by means of contact forces” [174]. In this experiment, a container with a granular layer in the bottom is driven vertically with displacement amplitude $A$ and driving frequency $f$. At certain choices of parameters, there exist localized circularly symmetric excitations referred to as oscillons (Figure 1.1(a)). An oscillon oscillates at frequency $f/2$: during one forcing period, it becomes a peak; during the next it becomes a crater. These oscillons can exist anywhere in the container and are long-lived, often persisting for more than half a million container oscillation periods. The interactions between oscillons are short-range, with oscillons of opposite (same) polarity attracting (repelling) each other. As a result, one oscillon can be surrounded by $n$ oscillons of opposite polarity where $n$ is known as the coordination number. As shown in Figure 1.1(b), experimentally one observes oscillons forming (chains of) dipoles ($n = 2$), triangles ($n = 3$) and squares ($n = 4$). Later, oscillons have also been discovered in vertically vibrated colloidal suspensions, a non-granular material [122].

In contrast to fluid dynamics where Navier-Stokes equations provide a universal description [115], there is no equation of motion for granular media to explain quantitatively the oscillon phenomenon [7]. Nonetheless, one can seek mathematical models that capture essential aspects of these experiments such as forcing. A phenomenological model of oscillons
has been recently proposed in Ref. [71]. On the other hand, the spatially extended system studied in detail in later chapters is close to the onset of a uniform oscillation even in the absence of forcing. For an isotropic and homogeneous system, the spatiotemporal dynamics can be studied in terms of Fourier modes $e^{ikx+st}$ where $x$ and $t$ are the space and time coordinates, $k$ is the wavenumber and $s$ is the growth rate. Near the onset $|k| \ll 1$ and the dispersion relation takes the form

$$s = \mu - \xi_2 k^2 + i(\omega_0 + \nu + \omega_2 k^2) + O(k^4),$$  

where $\omega_0 \sim O(1)$ denotes the frequency of uniform oscillations ($k = 0$) at onset ($\mu = 0$), and odd powers of $k$ are absent due to the reflection symmetry $x \rightarrow -x$. The terms apart from $\omega_0$ are leading order corrections, where $\mu$ denotes the distance from the onset of oscillations and $\nu$ denotes the correction to the oscillation frequency due to $\mu \neq 0$. For them to contribute equally we adopt the scaling $k \sim O(\epsilon)$, $\mu \sim \nu \sim O(\epsilon^2)$ and $\xi_2 \sim \omega_2 \sim O(1)$ where $0 < \epsilon \ll 1$. After factoring out the fast oscillation frequency $\omega_0$, the dynamics then happen at the slow space and time scales $X = \epsilon x$ and $T = \epsilon^2 t$ and can be described by an amplitude function $A(X,T)$. The scaling of the physical field $u$ (e.g. the vertical displacement of granular material in the oscillon experiments) cannot be inferred from linear theory alone, but balancing the nonlinear terms with the linear ones leads to $u \sim O(\epsilon)$. Therefore we can write the physical field $u$ as

$$u(x,t) = \epsilon A(X,T)e^{i\omega_0 t} + c.c. + h.o.t.,$$  

where c.c. denotes complex conjugate and h.o.t. denotes higher order terms. Inverting the Fourier transform and rescaling the constants properly, we get the linear part of the amplitude equation

$$\frac{\partial A}{\partial T} = (\mu + i\nu)A + (1 + i\alpha) \frac{\partial^2}{\partial X^2} A.$$  

Figure 1.1: Experimental observation of oscillons in vertically vibrated granular media. (a) Top and side views of an individual oscillon; the two profiles are separated by one forcing period. (b) Examples of patterns exhibited by collections of oscillons – (chains of) dipoles, triangles and squares. Reproduced from Figures 1 and 3 of [174].
The nonlinear terms can be derived by asymptotic methods, but the form of these terms can be inferred from symmetry considerations alone. The symmetry responsible for selecting the nonlinear terms is the same as the normal form symmetry for Hopf bifurcation in ordinary differential equations (ODEs), which will be reviewed in the next chapter. The lowest order nonlinear term is $|A|^2A$ in both cases, and the real part of its coefficient is chosen to be $-1$ to saturate the instability. Hence to leading order, the amplitude equation is

$$\frac{\partial A}{\partial T} = (\mu + i\nu)A - (1 + i\beta)|A|^2A + (1 + i\alpha)\frac{\partial^2}{\partial X^2}A,$$

where all the coefficients are real and $O(1)$. Applying the transformation $A \rightarrow Ae^{i\nu T}$, we finally get

$$\frac{\partial A}{\partial T} = \mu A - (1 + i\beta)|A|^2A + (1 + i\alpha)\frac{\partial^2}{\partial X^2}A.$$  \hspace{1cm} (1.5)

This partial differential equation (PDE) is known as the complex Ginzburg-Landau equation (CGLE) [6]. By replacing $\partial^2/\partial X^2$ by the Laplacian $\nabla^2$, one can easily generalize the CGLE to higher spatial dimensions (2D or 3D). Regarding the sign choices, whereas the positive sign of $\partial^2/\partial X^2$ is necessary for the PDE to be well-posed as a nonlinear diffusion equation, one can dispense with the negative sign of $|A|^2A$ and still find stable solutions with nontrivial spatial structures [34]. However we will not attempt to address this case in the following chapters.

Now we submit the above oscillatory transition in a spatially extended system to an external sinusoidal forcing at frequency

$$\omega_e = n(\omega_0 - \nu),$$

where $n$ is an integer and $\nu$ is assumed to be small. The physical field $u$ then obeys the forced evolution equation

$$u_t = M[u, \nabla] + f[u](e^{i\omega_e t} + c.c.),$$

where $M$ describes the unforced dynamics and $f$ denotes the forcing that may depend on $u$. Under the effect of forcing, the system oscillates at the phase-locked frequency $\omega_e/n = \omega_0 - \nu$ rather than the natural frequency $\omega_0$. This fast oscillation can again be factored out by writing

$$u = \epsilon Ae^{i\omega_e t/n} + c.c. + h.o.t..$$

Multiplying (1.7) by $e^{-i\omega_e t/n}$ and averaging over $n$ forcing periods $T_e \equiv 2\pi/\omega_e$, we get

$$\epsilon^2 A_T - i\nu A = \tilde{M}[A, \nabla] + \epsilon^{-1}(f[Ae^{i\omega_e t/n} + \bar{A}e^{-i\omega_e t/n}])\epsilon^{i(n-1)\omega_e t/n} + e^{-i(n+1)\omega_e t/n}),$$

where the slow time $T$ has been defined by $\partial_t + i\omega_0 \rightarrow \epsilon^2 \partial_T$ as before. In the absence of forcing, $\tilde{M}[A, \nabla]$ is simply $\epsilon^2$ times the right hand side of the CGLE. The first term in the expansion inside the angular brackets, in addition to the terms already present in the CGLE, is $\epsilon^{n-2}\gamma_n A^{n-1}$ where $\gamma_n$ can be taken as real. For $1 \leq n \leq 4$, this forcing term can always balance the remaining terms at finite forcing ($\gamma_n \leq O(\epsilon^0)$). Hence for these
choices of $n$, known as strong resonances [10, 84], the amplitude equation for the oscillation amplitude $A(X,T)$ takes the form

$$\frac{\partial A}{\partial T} = (\mu + i\nu)A - (1 + i\beta)|A|^2A + (1 + i\alpha)\nabla^2A + \gamma_n A^{n-1},$$

(1.10)

where the detuning $\nu$ and the (dimensionless measure of) forcing strength $\gamma_n$ have both been rescaled to be $O(1)$. Eq. (1.10) is known as the $n : 1$ forced complex Ginzburg-Landau equation (FCGLE) [57]. In the bulk of this dissertation we consider the case $n = 1$, sometimes referred to as harmonic forcing or 1 : 1 resonance, where the forcing term takes an additive form:

$$\frac{\partial A}{\partial t} = (\mu + i\nu)A - (1 + i\beta)|A|^2A + (1 + i\alpha)\nabla^2A + \gamma.$$

(1.11)

For later discussions we have replaced $(X,T)$ by $(x,t)$ such that Eq. (1.11) is a PDE for $A(x,t)$ in 1D and $A(x,y,t)$ in 2D.

The above derivation of the 1:1 FCGLE does not make reference to any specific physical system and provides a universal description (or normal form) of 1:1 resonantly forced Hopf bifurcations in spatially extended systems. Nonlinear optics provides yet another context to which this equation may apply. In a Kerr medium, the index of refraction depends on $|E|^2$ where $E$ denotes the transverse electric field. Light propagation in an infinite Kerr medium (Figure 1.1) is described by the nonlinear Schrödinger equation (NLS)

$$i\frac{\partial E}{\partial t} + \frac{1}{2}\frac{\partial^2 E}{\partial x^2} + |E|^2E = 0,$$

(1.12)

which is integrable and admits soliton solutions [1]. However, nonlinear optical media in real life have finite dimensions and solitons can rarely propagate for more than a few centimeters before escaping. A solution is to put mirrors around the medium, thus confining the soliton into a finite slab of material. However, mirrors and materials are both lossy, but we can compensate for the loss by feeding the caged soliton with an input electric field. This configuration (Figure 1.1) leads to a perturbed NLS equation

$$i\frac{\partial E}{\partial t} + \frac{1}{2}\frac{\partial^2 E}{\partial x^2} + |E|^2E = i\epsilon(-E - i\nu E + E_{in}),$$

(1.13)

where $-E$ accounts for linear damping, $E_{in}$ is the input electric field and $\nu$ is the cavity detuning. Setting $\epsilon = 1$ (equivalent to rescaling the variables), we get the Lugnato-Lefever equation (LLE) [127]. The LLE is a special case of the 1:1 FCGLE with $\alpha \to -\infty$ and $\beta \to \infty$. 
1.2 An overview of methods

In this section, we briefly cover the analytical and numerical methods employed in this dissertation.

1.2.1 Asymptotic methods

In pattern forming systems the spatiotemporal scales are usually well separated, so a multiple scales expansion provides an efficient method to derive pattern forming PDEs from the underlying physical laws, which are PDEs themselves. The procedure starts from an ansatz that expresses the physical field \( u \) as an asymptotic series in a small parameter \( \epsilon \) (

\[
 u = \sum_{i=0}^{\infty} u_i \epsilon^i,
\]

(1.14)

in which the coefficients \( u_i \) at each order in \( \epsilon \) are functions of multiple spatiotemporal scales. Near a nascent instability the scales of space and time variables that need to be included in the expansion can be inferred from linear theory. The requirement that the physical field \( u \) in the form of (1.14) solves the original PDE leads to an equation of the form

\[
 \sum_{i=0}^{\infty} w_i \epsilon^i = 0,
\]

(1.15)

where \( w_i \) denotes the terms collected at the \( i \)-th order in \( \epsilon \). Eq. (1.15) can be solved by the following limiting procedure. Taking \( \epsilon \to 0 \), one gets \( w_0 = 0 \). Dividing the remaining sum by \( \epsilon \) and again taking \( \epsilon \to 0 \), one gets \( w_1 = 0 \). Iterating this procedure, one gets the following hierarchy of conditions

\[
 \forall i \geq 0, \quad w_i = 0.
\]

(1.16)

The \( i \)-th condition \( w_i = 0 \) generally takes the form

\[
 \mathcal{L} u_i = f_i,
\]

(1.17)

where \( \mathcal{L} \) is a linear operator obtained from linearizing around the nascent instability in the original PDE, and \( f_i \) is a function of \( u_j \) (\( j < i \)) and their space and time derivatives. The key feature of (1.17) is that \( \mathcal{L} \) has a nontrivial null eigenspace formed by the marginally unstable eigenfunction(s) of \( \mathcal{L} \), denoted by \( g_\alpha \) such that

\[
 \mathcal{L} g_\alpha = 0,
\]

(1.18)

where \( \alpha \) is an index encoding the multiplicity of independent eigenfunctions. In such situations the adjoint homogeneous equation

\[
 \mathcal{L}^\dagger v = 0
\]

(1.19)

has nontrivial solution(s) \( v_\alpha \neq 0 \), where the adjoint operator \( \mathcal{L}^\dagger \) is defined by

\[
 \langle u, \mathcal{L}^\dagger v \rangle = \langle \mathcal{L} u, v \rangle,
\]

(1.20)
\langle \cdot, \cdot \rangle \) being a suitable inner product, and the boundary conditions on \( v \) are determined by requiring that the boundary term in Eq. (1.20), known as the bilinear concomitant, vanishes identically. The Fredholm alternative theorem (cf. §1.6 of [161]) then states that Eq. (1.17) has a solution if and only if
\[ \forall \alpha, \quad \langle f_i, v_\alpha \rangle = 0. \tag{1.21} \]
The \( i = 1 \) condition in (1.17) usually specifies that \( u_1 \) is a linear combination of the null eigenfunction(s)
\[ u_1 = \sum \alpha A_\alpha g_\alpha, \tag{1.22} \]
where the \( A_\alpha \) are known as amplitude function(s). In general, the \( i \)-th order condition (1.17) can be solved for \( u_i \) as long as Eq. (1.21) is satisfied, and one can proceed to the next order. Otherwise, Eq. (1.21) provides a solvability condition on \( u_j \ (j < i) \) that must be satisfied in order that a solution for \( u_i \) exists. The first nontrivial condition that arises in the iterative solution process is a nonlinear PDE for the amplitude(s) \( A_\alpha \).

One usually pauses the expansion at this lowest nontrivial order and studies the amplitude equation thus obtained. The asymptotic patterns expected in the original PDE can be constructed by substituting the numerical or analytical solution to the amplitude equation into the expansion (1.14). The validity of the amplitude equation can be established by comparing the constructed solution to numerical solutions to the original PDE or laboratory experiments. If fundamental differences are discovered one can continue the expansion to higher orders, while keeping in mind that adding these higher order terms does not in fact improve the approximation since asymptotic expansions usually do not converge [20]. The success of such amplitude equations in understanding natural patterns has been remarkable, the best known example of which might be Navier-Stokes equations which can be derived from Boltzmann equations via the Chapman-Enskog expansion [51].

1.2.2 Numerical continuation

An amplitude equation in pattern formation theory takes the form of an evolution equation equating the rate of change of the amplitudes to a function of these amplitudes and their spatial derivatives. The term with the highest spatial derivatives, which determines the nature of this PDE, comes from the dispersion relation expressing the growth rate \( s \) of a Fourier mode as a function of the wavenumber \( k \). In many pattern-forming systems the highest wavenumbers are damped, so the leading term in the dispersion relation must be an even power in \( k \) whose coefficient has a negative real part. Hence the amplitude equation behaves similar to the diffusion equation in that both generate an arrow of time, or a semi-group in the proper functional space. In this context the asymptotic properties of time evolution for an amplitude equation can be fruitfully studied using techniques from dynamical systems. Besides, dynamical systems techniques are particularly useful for understanding the coherent structures arising in later chapters, namely solutions to the amplitude equation that exhibit various (broken) symmetries.

Nonlinear differential equations generally have no analytical solutions, but topological insights pioneered by Henri Poincaré combined with modern numerical methods have yielded rather complete understanding of low-dimensional dynamical systems. The
discipline of nonlinear dynamics thus established roughly consists of bifurcation theory and chaos theory [180]. The former studies qualitative changes in phase space geometry as control parameters vary, while the latter explains deterministic yet unpredictable behaviors. The studies of coherent structures undertaken in this dissertation rely heavily on bifurcation theory; even the spatiotemporal chaotic solutions from Chapter 9 are largely understood in terms of the underlying coherent structures. In the recent few decades, numerical continuation software developed by various groups has provided a powerful tool to study bifurcation phenomena numerically. The basic ideas behind continuation algorithms are illustrated here for an algebraic problem; see Ref. [5] for a comprehensive introduction on this topic. We wish to compute a family of equilibrium solutions $x$ to a nonlinear equation

$$f(x, p) = 0, \quad x \in \mathbb{R}^n, \quad p \in \mathbb{R}, \quad f \in \mathbb{R}^n,$$

(1.23)

as the parameter $p$ varies, given an initial solution

$$f(x_0, p_0) = 0.$$  

(1.24)

The solution to this algebraic problem can be interpreted as the family of equilibria for the corresponding dynamical system. Since $x$ and $p$ are on equal footing, we denote $u \equiv (x, p)$ and $u_0 \equiv (x_0, p_0)$. If $u_0$ is a regular point, i.e. the Jacobian of $f$ has full rank at $u_0$, then by the implicit function theorem, the solution family is locally an isolated curve passing through $u_0$. Otherwise, the solution family near the singular point $u_0$ takes the form of branched curves and generally requires techniques from bifurcation theory to achieve branch switching. However, in the most common scenario where the solution curve only undergoes simple folds as $p$ varies, one should be able to compute the entire solution curve in the $u$ space without extra work. In practice, the next point on the solution curve is first predicted by lengthening the existing curve along the tangential direction (defined by the tangent vector $v$), and this prediction is then corrected using Newton iteration (with proper step size control). An additional condition

$$g(u) = 0$$

(1.25)

imposed in the correction step selects the actual solution that the Newton iterations converge to; different choices of $g$ yield different flavors of numerical continuation algorithms. In the simplest algorithm known as natural continuation, one component of $u$ is fixed, which is often chosen as the component that changes the most in the previous step to avoid problems at folds. The more elegant method of pseudo-arclength continuation restricts the Newton iterations to the hyperplane perpendicular to the tangent vector $v$.

The software AUTO-07P [78] (hereafter AUTO) continues solutions to boundary value problems discretized on a mesh using pseudo-arclength continuation. In addition, AUTO is able to detect and continue bifurcations, as well as to switch branches at bifurcation points. By construction AUTO is capable of continuing not only stable solutions but also unstable ones, which are otherwise difficult to obtain using direct numerical simulation. The stability properties can be continued along the solution branch by continuing at the same time the linearization of the problem and computing the leading eigenvalues using e.g. Arnoldi’s method [11]. A typical AUTO computation requires an equation file, a constants
file and an (optional) data file containing the initial solution, and can be controlled by a master script file to accomplish more complicated tasks. The equation file mainly contains the ODE whose solutions are to be continued, the boundary and integral conditions, and an (optional) analytical expression for the initial solution. For computational purposes, the ODE needs to be rescaled to the truncated time-interval $t \in [0, 1]$. The most common boundary conditions are Neumann boundary conditions (to impose reflection symmetry at a boundary) or periodic boundary conditions (to identify the left and right boundaries). The integral conditions select a unique solution out of a family of solutions related by phase symmetry. Ref. [49] provides an overview of numerical continuation techniques for coherent structures.

The roles played by various AUTO constants as specified in the constants file are explained in the user guide; here we only discuss those constants affecting computational accuracy. The number of mesh points $NTST$ ranges from $10^2$ to more than $10^3$ depending on the complexity of the solution being followed. The error tolerances $EPSL$, $EPSU$ and $EPSS$ are usually fixed at $10^{-7}$, $10^{-7}$ and $10^{-5}$. The minimum pseudo-arclength stepsize $DSMIN$ is fixed at $10^{-10}$, while the maximum stepsize $DSMAX$ varies around $10^{-3}$ to $10^{-2}$ depending on the complexity of the bifurcation curve. As a general rule, if there is discernible jump in the computed bifurcation diagram where the solution profile changes abruptly, then the continuation should be redone using higher resolution.

1.2.3 Numerical solution of PDE

Examples of coherent structures in an amplitude equation include steady states and traveling waves. The bifurcation structures of these solutions can indeed be elucidated by numerical continuation techniques, but their relevance to real experiments depends on their stability to perturbations. The linear or spectral stability of a traveling wave can be analyzed by linearizing the PDE around this wave and computing the eigenvalues of the resulting linear operator $L$. If the traveling wave is embedded in a background equilibrium $A_0$, its spectrum (the collection of eigenvalues of $L$) normally consists of an essential spectrum and a point spectrum. The essential spectrum is formed by the collection of continuously distributed eigenvalues and depends only on the linear stability of $A_0$. If $A_0$ is linearly stable, the essential spectrum necessarily has negative real parts and is thus uninteresting. In such cases, the spectral stability of the traveling wave depends on the point spectrum, which often needs to be determined numerically. To compute the point spectrum, we discretize $L$ using a finite difference scheme with proper boundary conditions to yield a sparse matrix $M$, whose eigenvalues and eigenvectors can then be computed using the MATLAB function `eigs`. The `eigs` function requires an initial guess for the eigenvalues, for which we choose a set of points faithfully representing the right half complex plane in order not to miss any unstable eigenvalue. The collection of eigenvalues of $M$ thus obtained should approach the unstable part of the point spectrum of $L$ as the number of mesh points increases. The general theory for the stability of traveling waves is reviewed in Ref. [152]. Although the possibility for the point spectrum to merge into or emerge from the essential spectrum introduces additional complications, recently numerical continuation algorithms have also been developed to compute spectral stability along bifurcation curves [146].

If a traveling wave is spectrally stable, any infinitesimal perturbation will decay
and the traveling wave continues to propagate forever when evolved in the PDE. Otherwise, the projections of the infinitesimal perturbation onto the unstable eigenfunctions of $\mathcal{L}$ (or the unstable modes) will grow exponentially for a short time with the growth rate of each mode determined by its eigenvalue. The subsequent dynamics cannot be predicted from spectral stability alone, and in such cases one must resort to direct numerical simulation (DNS) of the PDE. The most efficient algorithms to numerically solve a nonlinear diffusive PDE on a periodic domain are known as spectral methods [173]. To apply such methods, we perform a spatial Fourier transform on the PDE and convert it into a system of ODEs. This ODE system can then be split into a linear part and a nonlinear part, the latter of which can be evaluated by first transforming the current solution to physical space, computing the nonlinear terms at the mesh points, and then transforming back to Fourier space. Solving this ODE system formally by the integrating factor method, we arrive at an expression of the future solution as the sum of a linear term that can be exactly computed, and a nonlinear term that must be approximated. Different choices of the approximation scheme yield different numerical methods; we have adopted the ETD2 scheme in Ref. [63] for the 1:1 FCGLLE. The number of mesh points is chosen such that each sinusoidal variation in the spatial profile is represented by at least 10 points. The time step is usually fixed at $5 \times 10^{-2}$. As a general rule, higher spatial/temporal resolutions should be used when large spatial/temporal gradients are observed at certain points.

1.3 A brief summary of results

As pointed out in §1.1, the 1:1 FCGLLE (1.11) is a normal form for 1:1 resonantly forced Hopf bifurcations in spatially extended systems and thus applies whenever the system is sufficiently close to the onset of the Hopf bifurcation and the applied forcing is sufficiently weak. It is therefore invaluable to understand the solutions to Eq. (1.11) in as much detail as possible. As noted in Ref. [57], a key property of the $n:1$ FCGLLE (1.10) is that there are broad parameter regimes for the coexistence of multiple equilibria. This property leads naturally to the presence of spatially localized states studied in Chapters 3 to 9 of this dissertation. A typical localized state in these chapters consists of a patterned state surrounded by a flat background, where the flat background corresponds to one equilibrium and the patterned state bifurcates from a different equilibrium. Since equilibria in Eq. (1.11) correspond via Eq. (1.8) to phase-locked states in the original physical system, the flat background corresponds to spatially uniform oscillations while the localized state corresponds to spatially localized oscillations that resemble oscillons.

Localized states in 1D are studied in Chapters 3 to 7. In Chapter 3 (based on Ref. [129]) and Chapter 6 the bifurcation structures for two types of steady localized states are studied using AUTO. Examples of their profiles are shown in Figures 1.3(a) and 1.3(b) in terms of $U(x)$ and $V(x)$ where $U$ and $V$ are the real and imaginary parts of the oscillation amplitude $A$ in Eq. (1.11). In Chapter 4 the temporal dynamics associated with localized states in Chapter 3 are studied using DNS. A sample time evolution is shown in Figure 1.3(c) as a space-time plot of $V(x,t)$. In Chapter 5 (based on Ref. [48]) the bifurcation structures described in Chapter 3 are studied using dynamical systems methods. The phase portrait of the 2D map analyzed in this chapter is shown in Figure 1.3(d). In Chapter 7
the two types of localized states introduced in Chapters 3 and 6 are shown to fall into a classification program based on insights from elementary analytic geometry.

Localized states in 2D are studied in Chapters 8 to 9. Chapter 8 generalizes the 1D numerical studies in Chapters 3 and 4 to 2D. A snapshot during a 2D time evolution is shown in Figure 1.3(e) in terms of $V(x,y)$. Chapter 9 uses DNS to study localized states (in both 1D and 2D) built upon patterned states with nontrivial dependence on both space and time (see Figure 1.3(f) for a 2D snapshot of $V(x,y)$).

Finally Chapter 10 (based on Ref. [131]) introduces an analytic technique for the derivation of pattern equations from PDEs governing the original physical systems. This procedure reveals certain issues to be addressed in making quantitative predictions based on pattern-forming PDEs, but nonetheless PDEs such as Eq. (1.11) have garnered considerable attention in recent decades. An important reason is that certain solutions of physical interest, spatially localized states among them, can be fruitfully studied from the perspective of dynamical systems as reviewed in the next chapter.
Figure 1.3: (a-b) Examples of steady localized states in 1D studied in (a) Chapter 3 and (b) Chapter 6, shown in terms of $U(x)$ and $V(x)$ where $U$ and $V$ are the real and imaginary parts of $A$ in Eq. (1.11); (c) A sample time evolution of a 1D localized state studied in Chapter 4, shown in terms of a space-time plot of $V(x, t)$; (d) The phase portrait of the 2D map studied in Chapter 5; (e-f) Sample snapshots during time evolutions of 2D localized states studied in (e) Chapter 8 and (f) Chapter 9, shown in terms of $V(x, y)$ on square domains of size $[-L, L] \times [-L, L]$ where $L = 100$ in (e) and $L = 200$ in (f).
Chapter 2

Background

The study of pattern formation can be regarded as an outgrowth of bifurcation theory coupled with techniques from either group theory or asymptotic analysis, which lead respectively to the subfields of bifurcations with symmetry and amplitude equations in spatially extended systems. These topics are broadly covered in Ref. [98], while in this chapter we take a slightly different route, starting from elements of classical bifurcation theory and leading to recent studies on localized states in spatially extended systems.

The word “bifurcation” generally refers to topological changes of a geometric object as control parameters vary. The geometric objects of relevance here exist in the phase space of a dynamical system in either continuous time (differential equation or flow) or discrete time (difference equation or map). Bifurcation theory can be roughly divided into local bifurcation theory and global bifurcation theory. For our purposes, the former studies bifurcations from equilibria while the latter studies behavior near homoclinic/heteroclinic orbits. In addition, we mainly focus on codimension-1 bifurcations, namely those that only require the variation of a single parameter, and unless otherwise specified, we assume no special structure such as conserved quantities and symmetries in the dynamical system studied.

The general expression of a dynamical system is, in the case of a flow

$$\dot{x} = V(\mu, x), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R},$$

and in the case of a map

$$x_{j+1} = f(\mu, x_j), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R},$$

where $\mu$ is the control parameter in both cases.

2.1 Local bifurcations

Without loss of generality, we can assume that the origin is a fixed point

$$V(0,0) = 0 \quad \text{(for flow),} \quad f(0,0) = 0 \quad \text{(for map).}$$

For flows, the linear stability of the equilibrium at $x = 0$ is determined by the Jacobian matrix $DV$ at $x = 0$. In the generic situation, each eigenvalue $\lambda$ of $DV$ corresponds
to a unique eigenvector $v$. If $\lambda$ is real, then $v$ is also real and spans a 1D eigenspace. If $\lambda$ is complex, then its complex conjugate $\bar{\lambda}$ is also an eigenvalue, and their eigenvectors $v$ and $\bar{v}$ span a 2D eigenspace with real basis $(v + \bar{v})/2$ and $(v - \bar{v})/(2i)$. In the linearized dynamics, the amplitude $a(t)$ of an eigenvector $v$ depends on $t$ as $a(t) = a(0)e^{\lambda t}$. Therefore the linear stability in each eigenspace depends on the sign of $\Re(\lambda)$, according to which we can group the eigenvectors of $DV$ into three invariant subspaces, namely the unstable subspace $E^u$ ($\Re(\lambda) > 0$), the stable subspace $E^s$ ($\Re(\lambda) < 0$) and the center subspace $E^c$ ($\Re(\lambda) = 0$). The dimensions of $E^s$, $E^u$ and $E^c$ are denoted respectively by $n^s$, $n^u$ and $n^c$. If $n^c = 0$, i.e. the equilibrium is hyperbolic, then the Hartman-Grobman theorem (cf. §19.12a of Ref. [180]) states that in a neighborhood of this equilibrium, the nonlinear flow is topologically equivalent to the linear flow $\dot{x} = DV(0)x$. When the equilibrium loses hyperbolicity as the control parameter $\mu$ varies, the local nonlinear dynamics will undergo a qualitative change, which is referred to as a local bifurcation. The nature of this bifurcation depends on how the eigenvalue $\lambda$ passes through the imaginary axis $\Re(\lambda) = 0$. The case of a single real eigenvalue passing through $\lambda = 0$ is known as a steady-state bifurcation, while the case of a complex conjugate pair passing through $\Re(\lambda) = \Re(\bar{\lambda}) = 0$, $\Im(\lambda) \neq 0$ is known as a Hopf bifurcation.

In the phase space of a dynamical system, an invariant manifold is a manifold left invariant by the dynamics (cf. §3 of Ref. [180]). If a nontrivial invariant manifold exists, the effective dimension of the dynamical system can be reduced by considering the dynamics restricted to such a manifold. In particular, one can define the stable (resp. unstable, center) manifold $W^s$ (resp. $W^u$, $W^c$) to the equilibrium $x = 0$ as an $n^s$ (resp. $n^u$, $n^c$) dimensional invariant manifold that contains $x = 0$ and is tangent to the invariant subspace $E^s$ (resp. $E^u$, $E^c$) at $x = 0$. The stable and unstable manifolds are unique and play important roles as skeletons for the global dynamics of the system. The center manifold only exists at the bifurcation point and is non-unique, but it plays a key role in bifurcation theory because it can be shown to capture all local recurrence.

For maps, the local theory can be developed in a similar fashion. The linear stability of the equilibrium at $x = 0$ is determined by the Jacobian matrix $Df$ at $x = 0$. In the linearized dynamics, the amplitude $a_j$ of an eigenvector $v$ depends on the discrete time $j$ as $a_j = a_0\lambda^j$ where $\lambda$ is the corresponding eigenvalue. Therefore the stable, unstable and center subspaces ($E^s$, $E^u$ and $E^c$) are defined by $|\lambda| < 1$, $|\lambda| > 1$ and $|\lambda| = 1$. The nature of a local bifurcation depends on how the eigenvalue $\lambda$ passes through the unit circle $|\lambda| = 1$. Analogous to flows, the case of a single real eigenvalue passing through $\lambda = 1$ is known as a steady-state bifurcation, while the case of a complex conjugate pair passing through $|\lambda| = 1$ is known as a Neimark-Sacker bifurcation. The new case is a single real eigenvalue passing through $\lambda = -1$, known as a period-doubling bifurcation. The definitions and properties of the stable, unstable and center manifolds ($W^s$, $W^u$ and $W^c$) for maps parallel those for flows.

The studies of bifurcations for fixed points also capture the behavior near periodic orbits in the following connection. In the map case a periodic orbit with discrete period $n$ ($n \in \mathbb{Z}$) corresponds to a fixed point of the $n$-th iterated map $f^n$. In the flow case, an infinitesimal perturbation to a periodic orbit satisfies a linear ODE with periodic coefficients. The matrix representation of the linear transformation on the perturbation over a single
period is known as the Floquet matrix. The eigenvalues of this matrix, known as the Floquet multipliers, determine the linear stability of the periodic orbit. The reduction to a fixed point is achieved by choosing an \((n - 1)\)-manifold \(\Sigma\) (known as the Poincaré section) transverse to the periodic orbit, such that the dynamics of the original \(n\)-dimensional flow become equivalent to an \((n - 1)\)-dimensional Poincaré return map on \(\Sigma\). Via this procedure, the periodic orbit in the former corresponds to a fixed point in the latter, whereas the set of \(n\) Floquet multipliers in the former coincides with the \(n - 1\) eigenvalues of the Jacobian \(Df\) around the fixed point in the latter, apart from the trivial Floquet multiplier at 1 factored out by taking the Poincaré section.

To study the behavior of a dynamical system near a bifurcation point, one first performs center manifold reduction to reduce the dimensionality of the system to the number of marginally stable eigenvalues \(n_c\), and then transforms the reduced dynamics on the center manifold to a simplest form using normal form theory. These two steps, as outlined below, encode the dynamics near a given type of bifurcation into a series of normal form coefficients. It then only remains to analyze the normal form dynamics as a function of these coefficients, as summarized below for the above bifurcations in both flows and maps. All figures in this section are reproduced from the review article [64] by Crawford; the discussion also largely follows this article.

2.1.1 Center manifold reduction

An invariant manifold is in general a highly complicated geometric object and rarely admits an algebraic expression. However, at the bifurcation point \(\mu = 0\), we can obtain an approximate expression for the center manifold \(W^c\) near the equilibrium \(x = 0\) as follows. For flows, by a linear coordinate change we can rewrite (2.1) as

\[
\begin{align*}
\dot{x}_1 &= Ax_1 + N_1(x_1, x_2), \\
\dot{x}_2 &= Bx_2 + N_2(x_1, x_2),
\end{align*}
\]

where \(A\) is an \(n_c \times n_c\) matrix with eigenvalues on the imaginary axis, \(B\) is an \((n_s + n_u) \times (n_s + n_u)\) matrix with eigenvalues off the imaginary axis, and \(N_1\) and \(N_2\) are nonlinear terms. Note that although the stable and unstable directions should have been treated separately, we have combined them into the single coordinate \(x_2\) for computational convenience. The center manifold \(W^c\) is tangent to the center subspace \(E^c\) at \(x = 0\), so near \(x = 0\) we can express \(W^c\) as

\[
x_2 = h(x_1),
\]

where

\[
h(0) = 0, \quad D_{x_1}h(0) = 0.
\]

Equating the expression for \(\dot{x}_2\) obtained from (2.4) and (2.6) with the original expression (2.5), we arrive at the consistency condition

\[
Dh(x_1)[Ax_1 + N_1(x_1, h(x_1))] = Bh(x_1) + N_2(x_1, h(x_1)),
\]

which can be used to compute \(h(x_1)\) given (2.7). Eqs. (2.4) and (2.6) then determine the dynamics on the center manifold \(h(x_1)\) as

\[
\dot{x}_1 = Ax_1 + N_1(x_1, h(x_1)).
\]
In practice, we approximate \( h(x_1) \) by a formal power series in \( x_1 \) and solve Eq. (2.8) order by order. The resulting expression for \( h(x_1) \) is then inserted into (2.9), which leads to an \( n_c \)-dimensional ODE on the center manifold. Note that although the power series representation for \( h(x_1) \) can be computed to arbitrarily high order, the center manifold \( W^c \) itself is not necessarily unique. However, the possible lack of uniqueness is always beyond all algebraic orders and hence does not affect the bifurcation analysis carried out later, for which only the first few terms matter. The Shoshitaishvili theorem states that in a neighborhood of \( x = 0 \), the flow of (2.1) is topologically equivalent to the flow of the decoupled system consisting of (2.9) and

\[
\dot{x}_2 = Bx_2. 
\]

To extend the validity of the above reduction beyond the bifurcation point \( \mu = 0 \), we can supplement (2.4) and (2.5) with an additional equation

\[
\dot{\mu} = 0,
\]

and carry out the same reduction procedure on this new (or “suspended”) set of ODEs. The resulting \( (n_c + 1) \)-dimensional center manifold \( \tilde{W}^c \) coincides with \( W^c \) at \( \mu = 0 \), and can be shown to capture all local recurrence near \( \mu = 0 \).

The center manifold reduction for maps is analogous to that for flows.

### 2.1.2 Normal form theory

Before analyzing the dynamics on the center manifold in detail, it is essential to simplify (2.9) algebraically as much as possible. To this end, we rewrite (2.9) as

\[
\dot{x}_1 = Ax_1 + N(x_1) = Ax_1 + \sum_{i=2}^{\infty} N^{(i)}(x_1), 
\]

where \( N^{(i)} \) represents the collection of nonlinear terms of order \( i \) in \( x_1 \), and seek a near identity nonlinear coordinate change

\[
x = x_1 + \phi^{(k)}(x_1)
\]

to iteratively remove as many nonlinear terms in \( N^{(k)} \) as possible. For any given \( k \), Eq. (2.13) transforms Eq. (2.12) into

\[
\dot{x} = Ax + N(x) - L(\phi^{(k)}) + O(x^{k+1})
\]

where

\[
L(\phi^{(k)}) \equiv A\phi^{(k)} - D\phi^{(k)}Ax
\]

is a linear operator acting on \( \phi^{(k)} \in \mathcal{H}^{(k)} \), \( \mathcal{H}^{(k)} \) being a finite-dimensional vector space. The operator \( L \) depends only on \( A \), and if \( A \) has been transformed into diagonalized form with eigenvalues \( \sigma_j \), then the eigenvectors of \( L \) can be denoted by \( \phi_{j,\alpha}^{(k)} \), an \( n_c \)-vector whose
The $j$-th component is the monomial $x^\alpha$, $\alpha$ being a multi-index with $|\alpha| = k$, and the other components are all zero. The corresponding eigenvalue is

$$\lambda_{j,\alpha}^{(k)} = \sigma_j - \sum_{l=1}^{n_c} \alpha_l \sigma_l.$$  \hfill (2.16)

Among all the terms of $O(x^k)$ in Eq. (2.14), those in the range of $L$ denoted by $L(H^{(k)})$ can be removed, while those in the complement of $L(H^{(k)})$ are essential nonlinear terms and cannot be removed. The latter can be chosen to be the null eigenvectors of $L$, namely $\phi_{j,\alpha}^{(k)}$ that satisfy the resonance condition $\lambda_{j,\alpha}^{(k)} = 0$.

For a Hopf bifurcation ($n_c = 2$), the eigenvalues of $A$ at criticality are $\sigma_1 = -i\omega$ and $\sigma_2 = i\omega$, and the pair of complex conjugate components of $x$ are denoted by $z$ and $\bar{z}$.

At order $k$, the eigenvalues of $L$ are

$$\lambda_{1,\alpha}^{(k)} = i\omega(-1 + 2l - k), \quad \lambda_{2,\alpha}^{(k)} = i\omega(1 + 2l - k) \quad \text{where} \quad \alpha = (l, k - l), \quad 0 \leq l \leq k.$$  \hfill (2.17)

These eigenvalues can be identically zero only when $k$ is odd, so all the nonlinear terms with even $k$ can be removed. For odd $k$, the essential nonlinear terms appear at $l = (k + 1)/2$ when $\lambda_{1,\alpha}^{(k)} = 0$ and $l = (k - 1)/2$ when $\lambda_{2,\alpha}^{(k)} = 0$, and thus take the form

$$\phi_{1,\alpha}^{(k)} = \begin{pmatrix} z |z|^{k-1} \\ 0 \end{pmatrix}, \quad \phi_{2,\alpha}^{(k)} = \begin{pmatrix} 0 \\ \bar{z} |z|^{-k+1} \end{pmatrix}.$$  \hfill (2.18)

The resulting normal form for the Hopf bifurcation is

$$\begin{pmatrix} \dot{z} \\ \dot{\bar{z}} \end{pmatrix} = \begin{pmatrix} \gamma - i\omega & 0 \\ 0 & \gamma + i\omega \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \sum_{j=1}^{\infty} \begin{pmatrix} \beta_j z |z|^{2j} \\ \bar{\beta}_j \bar{z} |z|^{2j} \end{pmatrix},$$  \hfill (2.19)

where the dependence on the control parameter $\mu$ has been restored with the incorporation of the “unfolding” term $\gamma(\mu), \gamma'(\mu) > 0$, and the second equation is simply the complex conjugate of the first. Rewriting (2.19) in polar coordinates $z = re^{-i\theta}$, we have

$$\dot{r} = r \left( \gamma + \sum_{j=1}^{\infty} a_j r^{2j} \right), \quad \dot{\theta} = \omega + \sum_{j=1}^{\infty} b_j r^{2j},$$  \hfill (2.20)

where $a_j = \Re(\beta_j)$ and $b_j = -\Im(\beta_j)$. Compared to the unreduced dynamics, the Hopf normal form has the additional symmetry of invariance under rotations in $\theta$. This symmetry is a hidden symmetry for the unreduced dynamics with exponentially small error.

On a deeper level, the normal form symmetry can be deduced from the linearization matrix $A$ at criticality. Let

$$\Gamma = \overline{\{ \exp(s A^T) | s \in \mathbb{R} \}}$$  \hfill (2.21)

denote the closure of the one-parameter group generated by $A^T$, where $A^T$ is the transpose of $A$. It can be shown that the $k$-th order terms that cannot be removed by the normal form transformation (2.13) are those with $\Gamma$ symmetry, i.e. $N^{(k)}$ such that

$$N^{(k)}(g \cdot x) = g \cdot N^{(k)}(x) \quad \forall g \in \Gamma.$$  \hfill (2.22)
For Hopf bifurcations, $\Gamma$ is indeed the group $S^1$ of phase rotation matrices. For codimension-1 steady-state bifurcations, however, $\Gamma$ consists of the $1 \times 1$ identity matrix only, so normal form theory is not very useful in this case.

For maps, the dynamics on the center manifold take the form

$$x'_{j+1} = Ax'_j + \sum_{k=2}^{\infty} N^{(k)}(x'_j).$$  \hfill (2.23)

The goal of normal form theory is again to remove terms in $N^{(k)}$ as much as possible using the transformation $x = x' + \phi^{(k)}(x')$. The operator $L$ for maps is

$$L(\phi^{(k)}) = A\phi^{(k)} - \phi^{(k)}(Ax).$$  \hfill (2.24)

The eigenvectors of $L$ take the same form $\phi^{(k)}_{j,\alpha}$ as in the flow case, but the eigenvalues are

$$\lambda^{(k)}_{j,\alpha} = \sigma_j - \sigma^\alpha.$$  \hfill (2.25)

The essential nonlinear terms can then be shown to be those with the symmetry

$$\Gamma = \{(A^T)^n|n \in \mathbb{Z}\},$$  \hfill (2.26)

namely $N^{(k)}$ such that

$$N^{(k)}(A^T x) = A^T N^{(k)}(x).$$  \hfill (2.27)

For period-doubling bifurcations ($n_c = 1$), $A^T = -1$ so the essential nonlinear terms are the odd terms, and the period-doubling normal form is

$$x_{j+1} = \lambda x_j + \sum_{k=1}^{\infty} \beta_k x^{2k+1},$$  \hfill (2.28)

where $\lambda(\mu)$ depends on the control parameter $\mu$ as $\lambda(0) = -1$ and $\lambda'(0) < 0$. For Neimark-Sacker bifurcations ($n_c = 2$), $A^T = \text{diag}(e^{i\theta}, e^{-i\theta})$ $(0 < \theta < \pi)$ and the essential nonlinear terms depend on the rationality of $\theta/(2\pi)$. If $\theta/(2\pi)$ is irrational, $\Gamma = S^1$ so that the essential nonlinear terms are the same as for flows, which leads to the Neimark-Sacker normal form

$$z_{j+1} = z_j \left( \lambda + \sum_{k=1}^{\infty} \beta_k |z_j|^{2k} \right).$$  \hfill (2.29)

If $\theta/(2\pi)$ is rational, let $\theta/(2\pi) = p/q$ where $p$ and $q$ are relatively prime and $p/q < 1/2$ (hence $q \geq 3$). In this case the symmetry $\Gamma = Z_q$ is the discrete subgroup of $S^1$ generated by rotation through $2\pi/q$, so additional nonlinear terms will appear in the normal form. These are known as resonant terms, the lowest order ones being

$$\phi^{(q-1)}_{1,\alpha} = \begin{pmatrix} z^{q-1} \\ 0 \end{pmatrix}, \quad \phi^{(q-1)}_{2,\alpha} = \begin{pmatrix} 0 \\ z^{q-1} \end{pmatrix}.$$  \hfill (2.30)

For $q = 3$ and $q = 4$, these resonant terms enter at quadratic and cubic orders and compete with the leading cubic nonlinearity in the non-resonant terms. This situation is known
Figure 2.1: Bifurcation diagrams for the saddle-node bifurcation (2.33). (a) $\epsilon_1 = \epsilon_2 = 1$; (b) $\epsilon_1 = \epsilon_2 = -1$; (c) $-\epsilon_1 = \epsilon_2 = 1$; (d) $\epsilon_1 = -\epsilon_2 = 1$. Solid (dashed) branches are stable (unstable).

as strong resonance, but we will not review the associated normal form dynamics which are quite complicated [10]. For $q \geq 5$, the resonant terms are at least quartic and can be neglected in the truncated normal form. Note that the forcing term in Eq. (1.10) describing an $n : 1$ forced Hopf bifurcation (for flows) coincides with the first resonant term in Eq. (2.30), and strong resonances only occur for $n \leq 4$ owing to the same reason.

2.1.3 Local bifurcations in flows

For steady-state bifurcations in flows ($n_c = 1$), the dynamics

$$\dot{x} = V(\mu, x), \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R},$$

(2.31)
depend only on the scalar function $V(\mu, x)$, expressible as a power series in $\mu$ and $x$ near $(\mu, x) = (0, 0)$. The requirements $V(0, 0) = 0$ and $V_x(0, 0) = 0$ imply that

$$V(\mu, x) = \sum_{(i,j) \neq (0,0) \text{ or } (0,1)} \beta_{ij} \mu^i x^j, \quad \beta_{ij} \in \mathbb{R}.$$  

(2.32)

Generically all the $\beta$ coefficients are nonzero and the dominant balance is between the $O(\mu)$ and $O(x^2)$ terms. Therefore we can discard all the other terms and rescale $\mu$ and $x$ properly to arrive at the normal form

$$\dot{x} = \epsilon_1 \mu + \epsilon_2 x^2,$$

(2.33)

where $\epsilon_1$ and $\epsilon_2$ are the signs of $\beta_{10}$ and $\beta_{02}$. Figure 2.1 shows bifurcation diagrams that plot the existence and stability of important solutions $x(t)$ (equilibria in this case) as a function of the control parameter $\mu$ for all four combinations of $\epsilon_1$ and $\epsilon_2$. The stability at $\mu \neq 0$ of an equilibrium $x(\mu)$ has been determined by linearizing around it. We observe that Eq. (2.33) describes the creation or annihilation of a stable equilibrium and an unstable one as $\mu$ passes through 0. This type of steady-state bifurcation, where the solution branch undergoes a fold at the bifurcation point but otherwise remains unique, is the most common in applications. It is known as the saddle-node bifurcation since in 2D flows the two equilibria involved resemble a saddle and a node in terms of their ambient flows.

In a special situation, the equilibrium persists through a steady-state bifurcation. Making a $\mu$-dependent coordinate shift in $x$ to have the equilibrium always located at the origin regardless of $\mu$, we require in this case

$$V(\mu, 0) = 0 \quad \forall \mu.$$  

(2.34)
Figure 2.2: (a) Bifurcation diagrams for the transcritical bifurcation (2.35). (a) $\epsilon_1 = \epsilon_2 = 1$; (b) $\epsilon_1 = \epsilon_2 = -1$; (c) $-\epsilon_1 = \epsilon_2 = 1$; (d) $\epsilon_1 = -\epsilon_2 = 1$. Solid (dashed) branches are stable (unstable). (b) Perturbed transcritical bifurcation diagrams.

As a result $\beta_{i0} = 0$ for all $i$ and the dominant balance is between the $O(\mu x)$ and $O(x^2)$ terms. After truncation and rescaling, we arrive at the normal form

$$\dot{x} = x(\epsilon_1 \mu + \epsilon_2 x), \quad (2.35)$$

where $\epsilon_1$ and $\epsilon_2$ are the signs of $\beta_{11}$ and $\beta_{02}$. As seen from the bifurcation diagrams in Figure 2.2(a), Eq. (2.35) describes collision and exchange of stability between two equilibria as $\mu$ passes through 0, known as the transcritical bifurcation. If the generic terms ruled out by (2.34) are restored by an extra control parameter $\epsilon$, then for $\epsilon \neq 0$ the transcritical bifurcation diagram is perturbed in two possible ways as shown in Figure 2.2(b). A general theory for such universal unfoldings forms the subject of Chapter III of Ref. [88]. In applications, the transition between the diagrams for $\epsilon > 0$ and $\epsilon < 0$ is the most common mechanism for solution branches to reconnect.

In another special situation, the requirement (2.34) still holds but the coefficient $\beta_{02} = 0$. A typical context for this is when $V(\mu, x)$ has a reflection symmetry

$$-V(\mu, x) = V(\mu, -x). \quad (2.36)$$

The dominant balance is then between the $O(\mu x)$ and $O(x^3)$ terms, and after truncation and rescaling we get the normal form

$$\dot{x} = x(\epsilon_1 \mu + \epsilon_2 x^2), \quad (2.37)$$

where $\epsilon_1$ and $\epsilon_2$ are the signs of $\beta_{11}$ and $\beta_{03}$. As seen from the bifurcation diagrams in Figure 2.3(a), Eq. (2.37) generates a pitchfork of equilibria in the $(\mu, x)$ plane, hence the name pitchfork bifurcation. As shown in Figure 2.3(b), there are four possible perturbed pitchforks if the generic terms are restored, including the possibility of hysteresis. In contrast
Figure 2.3: (a) Bifurcation diagrams for the pitchfork bifurcation (2.37). (a) $\epsilon_1 = -\epsilon_2 = 1$; (b) $-\epsilon_1 = \epsilon_2 = 1$; (c) $\epsilon_1 = \epsilon_2 = -1$; (d) $\epsilon_1 = \epsilon_2 = 1$. Solid (dashed) branches are stable (unstable). (b) Perturbed pitchfork bifurcation diagrams.

Figure 2.4: Bifurcation diagrams for the Hopf bifurcation (2.38). (a) The supercritical case ($a_1 < 0$); (b) the subcritical case ($a_1 > 0$).

To the transcritical case, two unfolding parameters are required for the perturbed pitchfork as detailed in Chapter III.7 of Ref. [88].

For Hopf bifurcations in flows ($n_c = 2$), the truncation of the Hopf normal form (2.20) to lowest nontrivial order is

$$\dot{r} = r(\gamma + a_1 r^2), \quad \dot{\theta} = \omega + b_1 r^2,$$

where $\gamma(\mu)$ and $\omega(\mu)$ satisfy $\gamma(0) = 0$, $\gamma'(0) > 0$, and $\omega(0) \neq 0$. The $r$ equation happens to be identical to the normal form for pitchfork bifurcations (2.37) restricted to $x > 0$, while the $\theta$ equation simply describes phase rotation. Hence the bifurcation diagram for the Hopf bifurcation depends only on the sign of $a_1$ and can be readily drawn by rotating the pitchfork bifurcation diagrams in Figure 2.3(a) perpendicularly to the paper. As shown in Figure 2.4, as $\mu$ increases past 0, for $a_1 < 0$ (known as the supercritical case) a stable periodic orbit and an unstable equilibrium emerge from a stable equilibrium, while for $a_1 > 0$ (known as the subcritical case) an unstable periodic orbit and a stable equilibrium merges into an unstable equilibrium. A formal statement of these results constitutes the Poincaré-Andronov-Hopf bifurcation theorem (cf. §20.2 of Ref. [180]).
2.1.4 Local bifurcations in maps

For steady-state bifurcations in maps \((n_c = 1)\), the dynamics

\[
x_{j+1} = f(\mu, x_j), \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R},
\]

depend only on the scalar function \(f(\mu, x)\), which satisfies \(f(0, 0) = 0\) and \(f_x(0, 0) = 1\). If we define \(V(\mu, x) = f(\mu, x) - x\), then the study of existence of equilibria as determined by \(V(\mu, x) = 0\) is identical to flows. By considering the form of \(V(\mu, x)\) under different constraints, we can again deduce the saddle-node normal form

\[
x_{j+1} = x_j + \epsilon_1 \mu + \epsilon_2 x_j^2,
\]

the transcritical normal form

\[
x_{j+1} = x_j(1 + \epsilon_1 \mu + \epsilon_2 x_j),
\]

and the pitchfork normal form

\[
x_{j+1} = x_j(1 + \epsilon_1 \mu + \epsilon_2 x_j^2),
\]

where \(\epsilon_1\) and \(\epsilon_2\) are the signs of the corresponding normal form coefficients. It turns out that the dependence of stability of equilibria on \(\epsilon_1\), \(\epsilon_2\), and \(\mu\) also coincides with flows. The conclusion is that the bifurcation diagrams for steady-state bifurcations are exactly the same between maps and flows.

For period-doubling bifurcations \((n_c = 1)\), the period-doubling normal form (2.28) truncated to lowest nontrivial order is

\[
x_{j+1} = \lambda x_j + \beta_1 x_j^3 \equiv f(\mu, x_j),
\]

where \(\lambda(0) = -1\). A short calculation suggests that the map \(f^2(\mu, x) \equiv f(\mu, f(\mu, x))\) undergoes a pitchfork bifurcation at \(\mu = 0\) with the normal form

\[
x_{j+1} = x_j(1 + \epsilon_1 \mu + \epsilon_2 x_j^2),
\]

where \(\epsilon_1\) and \(\epsilon_2\) are the signs of \(-\lambda'(0)\) and \(-\beta_1(0)\). Among the equilibria of \(f^2\), the trivial equilibrium \(x = 0\) is also an equilibrium of \(f\). In addition, there are two nontrivial equilibria created in the pitchfork bifurcation, denoted by \(x_\pm\) with \(x_- = -x_+\). They are related via the original map \(f\) by

\[
x_- = f(\mu, x_+), \quad x_+ = f(\mu, x_-),
\]

so together they form a 2-cycle in \(f\). The stability of this 2-cycle in \(f\) is identical to the stability of either equilibrium \(x_\pm\) in \(f^2\).

For Neimark-Sacker bifurcations in maps \((n_c = 2)\), we denote the complex eigenvalue by \(\lambda = (1 + a)e^{iθ(1+b)}\) where \(0 < θ < π\), while \(a(\mu)\) and \(b(\mu)\) satisfy \(a(0) = b(0) = 0\) and \(a'(0) > 0\). If strong resonances do not occur \((qθ/(2π) \not\in \mathbb{Z}\) for \(q = 3\) or \(4\)), the truncation of the Neimark-Sacker normal form (2.29) to lowest nontrivial order can be written in the polar coordinates \((r, ψ)\) as

\[
r_{j+1} = (1 + a)r_j(1 + a_1 r_j^2), \quad ψ_{j+1} = ψ_j + θ(1 + b) + b_1 r_j^2.
\]
Figure 2.5: Bifurcations in a Poincaré return map. (a) Period-doubling; (b) Neimark-Sacker.

Analogous to the flow case, the \( r \) equation undergoes a pitchfork bifurcation at \( \mu = 0 \), and the positive nontrivial equilibrium \( r_H \) created in this bifurcation defines an invariant circle of radius \( r_H \) in the 2D normal form map. For \( a_1 < 0 \) (\( a_1 > 0 \)), the Neimark-Sacker bifurcation is supercritical (subcritical) and the invariant circle is attracting (repelling). The \( \psi \) equation defines a map on this invariant circle, but in contrast to the flow case, the uniform rotation independent of \( \psi \) as described by the truncated normal form is structurally unstable to \( \psi \)-dependent perturbations that could be present in the higher order terms. In general, such maps of the circle can exhibit considerable complexity (cf. §21.6 of Ref. [180]) and are outside the scope of this review.

In a Poincaré return map, the above bifurcations of equilibria correspond to bifurcations of periodic orbits in the original flow. Specifically, a saddle-node bifurcation corresponds to a merger of two periodic orbits, a period-doubling bifurcation corresponds to the creation of a new periodic orbit with approximately twice the original period (Figure 2.5(a)), and a Neimark-Sacker bifurcation corresponds to the creation of an invariant torus around the original periodic orbit (Figure 2.5(b)).

### 2.2 Global bifurcations

Many global phenomena in dynamical systems can be understood in terms of stable and unstable manifolds to equilibria, periodic orbits or other invariant sets. In this sense, a special orbit \( \Gamma \) that approaches a forward limit set \( X_+ \) and a backward limit set \( X_- \) plays a central role because it lies precisely in the intersection between the unstable manifold \( W^u(X_-) \) of \( X_- \) and the stable manifold \( W^s(X_+) \) of \( X_+ \). Such an orbit is known as a homoclinic orbit if \( X_- = X_+ \), and a heteroclinic orbit if \( X_- \neq X_+ \). The codimension in parameter space of a homoclinic/heteroclinic orbit can be determined by dimension counting as follows. Let \( n^u \) and \( n^s \) denote the dimensions of \( W^u(X_-) \) and \( W^s(X_+) \). Then the codimension \( n^c \) of \( \Gamma \) can be shown to be

\[
\begin{align*}
  n^c = n + 1 - n^s - n^u \quad & \text{(flows),} \\
  n^c = n - n^s - n^u \quad & \text{(maps),}
\end{align*}
\]

where \( n \) is the dimension of the phase space. For example, a homoclinic orbit to a saddle is of codimension-1 in a 2D flow, and of codimension-0 in a 2D map.
A central part of global bifurcation theory deals with behavior near such orbits. Indeed the last few decades have witnessed considerable progress in homoclinic bifurcation theory. This subject is extremely rich with strong connections to chaos theory and cannot be covered in a single chapter, so in this section we will only review an important category of homoclinic bifurcations in the flow setting. The dimension of the phase space is chosen as \( n = 3 \) (the lowest “nontrivial” dimension that supports chaos), and \( X_- = X_+ = 0 \) is assumed to be a saddle-type equilibrium at the origin. The two essentially different configurations for the three eigenvalues of the Jacobian matrix around 0 are

\[
\text{Saddle: } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \text{ with } \lambda_1, \lambda_2 < 0, \lambda_3 > 0; \quad (2.48)
\]

\[
\text{Saddle focus: } \rho \pm i\omega, \lambda \text{ with } \rho < 0, \lambda > 0. \quad (2.49)
\]

Thus the homoclinic orbit \( \Gamma \) is of codimension-1 by dimension counting, and is assumed to exist at \( \mu = 0 \) where \( \mu \) is the control parameter.

These two eigenvalue configurations will be separately addressed below, but the common strategy is to reduce the 3D flow to a 2D map. To study dynamics near the homoclinic orbit \( \Gamma \), it is natural to take two Poincaré sections \( \Pi_0 \) and \( \Pi_1 \) near 0 and transverse to the end and beginning of \( \Gamma \), and construct the following two maps. The first map \( P_0 : \Pi_0 \to \Pi_1 \) is constructed locally from the linearized flow near 0. The second map \( P_1 : \Pi_1 \to \Pi_0 \) needs to be constructed globally from linearization around \( \Gamma \), but the only fact used in this step is that a neighborhood around \( p_1 \equiv \Gamma \cap \Pi_1 \) is mapped linearly to a neighborhood around \( p_0 \equiv \Gamma \cap \Pi_0 \) to leading order. The composite map \( P \equiv P_1 \circ P_0 : \Pi_0 \to \Pi_0 \) can then be constructed from \( P_0 \) and \( P_1 \) and further analyzed.

All figures in this section are reproduced from §27 of Ref. [180]; the discussion also largely follows this book chapter.

### 2.2.1 The saddle case

Let us assume that the eigenvector with eigenvalue \((\lambda_1, \lambda_2, \lambda_3)\) lies in the \((x, y, z)\) direction and \( \Gamma \) breaks for \( \mu \neq 0 \) as shown in Figure 2.6. Also without loss of generality, we assume that \( \Gamma \) enters a neighborhood of 0 along a curve tangent to the \( y \)-axis (i.e. \( \lambda_2 > \lambda_1 \)). The Poincaré sections \( \Pi_0 \) and \( \Pi_1 \) are chosen as

\[
\Pi_0 = \{(x, y, z)|y = \epsilon, z > 0\}, \quad \Pi_1 = \{(x, y, z)|z = \epsilon\}, \quad 0 < \epsilon \ll 1. \quad (2.50)
\]

The local and global maps \( P_0 \) and \( P_1 \) then take the form

\[
P_0 : \begin{pmatrix} x \\ \epsilon \\ z \end{pmatrix} \to \begin{pmatrix} x \left( \frac{\epsilon}{z} \right)^{\lambda_1/\lambda_3} \\ \epsilon \left( \frac{\epsilon}{z} \right)^{\lambda_2/\lambda_3} \\ \epsilon \end{pmatrix}, \quad (2.51)
\]

\[
P_1 : \begin{pmatrix} x \\ y \\ \epsilon \end{pmatrix} \to \begin{pmatrix} 0 \\ \epsilon \\ 0 \end{pmatrix} + \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} e\mu \\ 0 \\ f\mu \end{pmatrix}, \quad (2.52)
\]

where \( a, b, c, d, e, \) and \( f \) are constants, and \( x \) and \( z \) are both sufficiently small.
Figure 2.6: Breakup of the homoclinic orbit Γ to a saddle as μ varies around 0.

Figure 2.7: Deformation of a rectangle D by the Poincaré map P. (a) The orientation reversing case. (b) The orientation preserving case.

The action of P can be seen from the deformation of a rectangle $D \subset \Pi_0$. Under $P_0$, D is mapped onto a wedge $T$ with the segment $D \cap W^s(0)$ mapped onto the single point $\Pi_1 \cap \Gamma$. Under $P_1$, $T$ is linearly transformed by $A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to $\Gamma$. For $\mu \neq 0$, $\Pi_0 \cap \Gamma$ differs from the origin by $v\mu$ where $v = \begin{pmatrix} e \\ f \end{pmatrix}$, and since $f > 0$ from Figure 2.6 we can set $f = 1$ by rescaling $\mu$. These actions are shown in Figure 2.7 for $\mu = 0$. The following two cases must be considered separately. The linear transformation $A$ can be either orientation reversing ($\det A < 0$) or orientation preserving ($\det A > 0$), or equivalently the stable manifold $W^s(0)$ in a tubular neighborhood of $\Gamma$ can be homeomorphic to either a Möbius band or a cylinder. In the former case, $T$ lies below $W^s(0)$ at $\mu = 0$ and $P$ only has a fixed point when $\mu > 0$. In the latter case, $T$ lies above $W^s(0)$ at $\mu = 0$ and the presence of fixed points depends on whether $T$ is compressed or dilated in the $z$ direction compared to $D$. In the case of compression ($|\lambda_2| > \lambda_3$), a fixed point of $P$ only exists when $z$ is increased by varying $\mu$, namely for $\mu > 0$. Similarly in the case of dilation ($|\lambda_2| < \lambda_3$), a fixed point of $P$ only exists for $\mu < 0$. These geometric arguments can be made precise by studying the explicit expression for $P$, and the stability of fixed points can also be analyzed, but we omit these details here. In the original 3D flow, the above analysis leads to existence and stability results for periodic orbits near the homoclinic orbit $\Gamma$.

Next we assume that there are two homoclinic orbits $\Gamma_r$ and $\Gamma_l$ to 0 at $\mu = 0$. These two orbits are assumed to lie in separate branches of $W^u(0)$. They can approach $W^s(0)$ either from the same direction or from opposite directions, but here we only consider the former case (Figure 2.8(a)). Note that while the coexistence of two homoclinic orbits is
Figure 2.8: (a) Poincaré sections for two homoclinic orbits $\Gamma_r$ and $\Gamma_l$ to a saddle. (b) Horizontal and vertical strips that generate horseshoe dynamics.

generally of codimension-2, such a configuration is of codimension-1 in the presence of the discrete symmetry $(x, y, z) \rightarrow (-x, y, -z)$ (which would be the case for the celebrated Lorenz equations). As shown in Figure 2.8(a), the following Poincaré sections can be separately constructed for $\Gamma_r$ and $\Gamma_l$

$$\Pi_r^0 = \{(x, y, z) | y = \epsilon, z > 0\}, \quad \Pi_l^0 = \{(x, y, z) | y = \epsilon, z < 0\}, \quad (2.53)$$

$$\Pi_r^1 = \{(x, y, z) | z = \epsilon, y > 0\}, \quad \Pi_l^1 = \{(x, y, z) | z = -\epsilon, y > 0\}, \quad (2.54)$$

where $0 < \epsilon \ll 1$. Denoting the composite Poincaré maps for $\Gamma_r$ and $\Gamma_l$ by $P_r$ and $P_l$, we proceed to analyze the situation where both $P_r$ and $P_l$ are orientation preserving. Assuming generically that $0 < -\lambda_2 < \lambda_3 < -\lambda_1$ and $d \neq 0$, the homoclinic orbits can be shown to break for $\mu < 0$ to yield the Poincaré map $P$ as shown in Figure 2.8(b). Note that the tips of the two wedges $P(\Pi_r^0)$ and $P(\Pi_l^0)$ are respectively the images of the right and left neighborhoods of the centerline under $P$ and hence can be identified with each other. Now we can choose $\mu_h$-horizontal strips in $\Pi_r^0$ and $\Pi_l^0$, which are mapped by $P$ to $\mu_v$-vertical strips, as also shown in Figure 2.8(b). Identifying the tips of the two vertical strips, we observe that the two horizontal strips are vertically contracted, horizontally expanded and finally folded into an S-shape by $P$. This series of actions are identical to the Smale horseshoe map except that the latter takes a $\subset$ or $\supset$ shape, but the dynamical features do not depend on this particular detail. In both cases, the map has an invariant Cantor set on which it is equivalent to a full shift on two symbols, or in simpler terms the system exhibits chaos. Remarkably, the horseshoe dynamics appear suddenly when $\mu$ becomes negative. This type of global bifurcation has been known as a homoclinic explosion.

### 2.2.2 The saddle-focus case

Let us assume that the eigenvectors with eigenvalues $\rho \pm i\omega$ and $\lambda$ are in the $(x, y)$ and $z$ direction and $\lambda > -\rho > 0$. At $\mu = 0$, a Poincaré section defined by $y = 0$ intersects $\Gamma$ infinitely many times near 0, with successive intersections differing by a ratio $e^{\rho T}$ where $T = 2\pi/\omega$ denotes the first return time. To make the Poincaré map diffeomorphic, we can
restrict it to the interval between two adjacent intersections and define the Poincaré sections \( \Pi_0 \) and \( \Pi_1 \) as

\[
\Pi_0 = \{(x, y, z) | y = 0, e^{2\pi \rho/\omega} \leq x \leq \epsilon, 0 \leq z \leq \epsilon\}, \quad \Pi_1 = \{(x, y, z) | z = \epsilon\}. \tag{2.55}
\]

The local and global maps \( P_0 \) and \( P_1 \) then take the form

\[
P_0 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \left(\frac{\epsilon}{z}\right)^{\rho/\lambda} \cos \left(\frac{\omega}{\rho} \log \frac{\epsilon}{z}\right) \\ y \\\frac{\epsilon}{z} \end{pmatrix}, \tag{2.56}
\]

\[
P_1 : \begin{pmatrix} x \\ y \\ \bar{x} \end{pmatrix} \rightarrow \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{x} \\ 0 \\ 0 \end{pmatrix}, \tag{2.57}
\]

where \((\bar{x}, 0, 0) = \Gamma \cap \Pi_0\). Now we divide the section \( \Pi_0 \) into infinitely many rectangles

\[
R_k = \{(x, y, z) | y = 0, e^{2\pi \rho/\omega} \leq x \leq \epsilon, e^{2\pi(k+1)\lambda/\omega} \leq z \leq e^{2\pi k \lambda/\omega}\}, \tag{2.58}
\]

such that \( \Pi_0 = \bigcup_{k=0}^{\infty} R_k \). Then each \( R_k \) is mapped by \( P_0 \) into one turn of a logarithmic spiral (Figure 2.9(a)). It can be shown that for \( k \) sufficiently large, \( R_k \) possesses an invariant Cantor set on which \( P \) is equivalent to a full shift on two symbols. Therefore in contrast to the saddle case, a countable infinity of horseshoes is created around a single homoclinic orbit, and the horseshoes in different \( R_k \) can interact (Figure 2.9(b)). This type of global bifurcation has been known as the Shilnikov phenomenon.

The above analysis applies to \( \mu = 0 \). As \( \mu \) varies around 0, the creation of the horseshoes as the homoclinic orbit \( \Gamma \) is created is hard to analyze, but a bifurcation analysis of the fixed points of \( P \) provides some insights. Recall that these fixed points correspond to periodic orbits near \( \Gamma \) in the 3D flow. Assuming that \( \Gamma \) breaks for \( \mu \neq 0 \) as shown in Figure 2.10, we can incorporate \( \mu \) dependence into \( P \) by adding a term linear in \( \mu \) as in the saddle case. The detailed calculations to determine existence and stability are omitted here, but the results show that the eigenvalue ratios \( \delta \equiv -\rho/\lambda \) and \( \xi \equiv -\omega/\lambda \) determine the structure of the bifurcation diagram. For \( \delta > 1 \) (Figure 2.11(a)), the branch of periodic orbits approaches \( \mu = 0 \) monotonically from \( \mu > 0 \) as the period increases, and the whole branch remains stable. For \( \delta < 1 \) (Figure 2.11(b)), the branch wiggles to approach \( \mu = 0 \) as the period increases, and period-doubling bifurcations occur between the saddle-nodes. The successive saddle-nodes \( \mu_i \ (i \in \mathbb{N}) \) along the branch approach 0 exponentially as

\[
\lim_{i \rightarrow \infty} \frac{\mu_{i+1}}{\mu_i} = -\exp \frac{\pi \delta}{\xi}. \tag{2.59}
\]

An alternative method to solve Shilnikov’s problems, known as Lin’s method, is based on choosing Poincaré section(s) far from the fixed point(s) in the spirit of Melnikov [120]. An application of this method to heteroclinic cycles in reversible systems provides an explanation for the collapsed snaking bifurcation diagram in Figure 3.2 [110].
Figure 2.9: Poincaré maps in the saddle-focus case. (a) The local map $P_0$. (b) The composite map $P = P_1 \circ P_0$.

Figure 2.10: Breakup of the homoclinic orbit $\Gamma$ to a saddle-focus as $\mu$ varies around 0.

Figure 2.11: Bifurcation diagrams of periodic orbits near the homoclinic orbit $\Gamma$ to a saddle-focus. (a) $\delta > 1$. (b) $\delta < 1$. 
2.3 Homoclinic orbits in reversible systems

Many pattern forming PDEs such as variants of the Swift-Hohenberg equation (SHE) and the complex Ginzburg-Landau equation (CGLE) are spatially isotropic. Hence in 1D, these PDEs are invariant under the spatial reflection $x \rightarrow -x$. Among the solutions of greatest interest in applications are steady patterns, which can be fruitfully studied in the framework of spatial dynamics [106]. In this framework, the steady state of the PDE is regarded as a reversible ODE in the spatial variable $x$; in particular homoclinic and heteroclinic orbits in the latter correspond to pulses and fronts in the former. In this regard, the theory of homoclinic orbits in reversible systems leads to important insights for understanding 1D localized states in pattern forming systems. Thus following Ref. [46], in this section we review relevant aspects of this theory.

Techniques of local bifurcation theory can be generalized to reversible systems. Specifically, a homoclinic center manifold theorem justifies reduction to a lower dimensional center manifold, and normal form theory leads to simplified dynamics on this manifold. As discussed in §2.2, homoclinic orbits play important roles as organizing centers for the global dynamics, possibly generating periodic orbits and chaos nearby. An important reason for studying homoclinic orbits in reversible systems is that such orbits are often of codimension-0 if they are themselves reversible, a property shared by Hamiltonian systems. Indeed many reversible systems are Hamiltonian and vice versa, and in fact most reversible systems considered in this section are both reversible and Hamiltonian.

2.3.1 Linear theory

Since an ODE invariant under reversibility in the form of temporal reflection $t \rightarrow -t$ (or spatial reflection $x \rightarrow -x$ in the context of spatial dynamics) is usually even-dimensional, we will focus on the following 4D reversible ODE that exemplifies homoclinic orbits found in the lowest nontrivial dimension:

$$u'''' - bu'' + au = f(u, u', u'', u'''), \quad a, b \in \mathbb{R}, \quad f \text{ is nonlinear}. \quad (2.60)$$

As before we are interested in orbits $\Gamma$ homoclinic to $0$, which lie in the intersection between the stable and unstable manifolds $W^s(0)$ and $W^u(0)$. The dimensions of these manifolds are determined by linear theory around $0$, or the characteristic equation $\lambda^4 - b\lambda^2 + a = 0$ for the eigenvalues $\lambda$. This equation is invariant under $\lambda \rightarrow -\lambda$ and $\lambda \rightarrow \bar{\lambda}$ and leads to eigenvalue configurations symmetric with respect to both axes as depicted in Figure 2.12. One can identify four qualitatively different eigenvalue configurations

configuration (1): a quartet of complex eigenvalues
configuration (2): the eigenvalues are all real
configuration (3): two eigenvalues are real and two are imaginary
configuration (4): the eigenvalues are all imaginary,

bounded by four codimension-1 curves on the $(b, a)$ plane

$$C_0 : a = 0, b > 0; \quad C_1 : a = 0, b < 0; \quad C_2 : b = 2\sqrt{a}; \quad C_3 : b = -2\sqrt{a}. \quad (2.61)$$
In the presence of the reversibility symmetry \( t \to -t \), \( \Gamma \) is formed by the transverse intersection between \( W^u(0) \) and the 2D symmetric section \( S = \{ (u, u', u'', u''') : u' = u''' = 0 \} \). If the equilibrium 0 is hyperbolic such that \( W^u(0) \) is also 2D, then \( \Gamma \) is of codimension-0 by dimension counting. In what follows, we will classify the homoclinic orbit \( \Gamma \) according to the eigenvalue configuration of the equilibrium 0. For each case, we address both the bifurcation of \( \Gamma \) as a control parameter \( \mu \) varies, and the general properties of \( \Gamma \).

In general there also exists a one-parameter family of periodic orbits that accumulate on each homoclinic orbit as the period approaches infinity. The linear stability of a reversible periodic orbit in a reversible system depends on the Floquet multipliers denoted by \( \Lambda \), which must be invariant under the inverse \( \Lambda \to \Lambda^{-1} \). Therefore in 4D, there are always two Floquet multipliers at \( \Lambda = 1 \), while the remaining two must lie in one of the following three configurations as shown in Figure 2.13:

- Configuration (I): both lie on the real axis, \( \Lambda > 0 \)
- Configuration (II): both lie on the unit circle
- Configuration (III): both lie on the real axis, \( \Lambda < 0 \).

### 2.3.2 The saddle case

Near \( C_0 \) (a saddle-node bifurcation), the truncated normal form is

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = \text{sgn}(\mu)x_1 - \frac{3}{2}x_1^2.
\]  

For \( \mu > 0 \), (2.62) possesses a single homoclinic orbit \( \Gamma : x_1(t) = \text{sech}^2(t/2) \). This solution profile has monotonic tails and is sometimes referred to as a KdV-type soliton because the normal form (2.62) follows from the Korteweg-de Vries (KdV) equation by seeking a
traveling wave solution and integrating the resulting ODE once. In the presence of an up-down symmetry $x_1 \rightarrow -x_1$, the truncated normal form is

$$
\dot{x}_1 = x_2, \quad \dot{x}_2 = \text{sgn}(\mu)x_1 - \beta x_1^3,
$$

(2.63)

where $\beta = \pm 1$. If $\beta = -1$, there is a pair of homoclinic orbits $\Gamma_{\pm}: x_1(t) = \pm \text{sech}(t)$ related by the up-down symmetry for $\mu > 0$. In all cases the equilibrium $0$ is a saddle because its eigenvalues are in configuration (2).

The homoclinic orbit $\Gamma$ to a saddle is generally unique and there exists a one-parameter family of hyperbolic periodic orbits accumulating on $\Gamma$, similar to a Hamiltonian system with one degree-of-freedom. On the other hand, codimension-1 global bifurcations can occur in four or higher dimensions, including in particular the orbit-flip bifurcation that occurs when $\Gamma$ is tangent to the strong stable eigenvector of $0$ as $t \rightarrow \infty$ [153]. This bifurcation causes the tangent vector to $\Gamma$ as $t \rightarrow \infty$ to flip between the two components of the weakly stable eigenvector of $0$, or equivalently the tangent bundle of the stable manifold around $\Gamma$ to switch between orientable and non-orientable, as depicted in Figure 2.14. There is a bifurcation to infinitely many homoclinic orbits on the non-orientable side. Another bifurcation that we will not address here is the inclination-flip bifurcation that occurs when the above switching happens in the adjoint linear problem. In the Hamiltonian case, when at least two homoclinic orbits exist that asymptote to $0$ in certain directions, multiplicity of homoclinic orbits and Smale horseshoe dynamics can be shown similarly to §2.2.1.

### 2.3.3 The saddle-focus case

Near $C_2$ (known as a reversible double-Hopf bifurcation with 1:1 resonance [100] or a Turing bifurcation in spatial dynamics), the normal form is a 4D completely integrable ODE for two complex variables $(z_1, z_2)$. Similar to Hopf bifurcations in flows, the Turing bifurcation can be either supercritical or subcritical. The normal form (named Iooss-Pérouème) and its dynamics in the supercritical case are reviewed in §3.6. In the subcritical case, there is a family of homoclinic orbits $\Gamma_{\phi}: z_1(t) = A\text{sech}(Bt)\exp(i\omega t + \phi)$ parametrized by arbitrary phase $\phi$. However, terms beyond all algebraic orders (i.e. absent in any truncated normal form) select generically two values $\phi = 0$ and $\pi$ from the circle $[0, 2\pi)$. The resulting solution profiles have oscillatory tails and are sometimes referred to
Figure 2.14: The orbit flip bifurcation in a 4D reversible ODE. Reproduced from Figure 2 of Ref. [46].

as NLS-type solitons or envelope solitons because they are special solutions to the nonlinear Schrödinger (NLS) equation with the phase and group velocities equal to each other. In the presence of an additional up-down symmetry, there are four possible values $\phi = 0, \pi/2, \pi$ or $3\pi/2$ selected. In all cases $\Gamma$ is homoclinic to a saddle-focus because the eigenvalues of the equilibrium 0 are in configuration (1). Near the codimension-2 transition between the supercritical and subcritical cases, there exist homoclinic orbits that wind arbitrarily many times around a (reversible) periodic orbit. These homoclinic orbits are organized into bifurcation curves called homoclinic snaking [182], which occupy a central place in recent studies of localized states and will be reviewed in the next section.

A Shilnikov-type analysis analogous to §2.2.2 shows that the existence of a primary homoclinic orbit implies that there are infinitely many $N$-pulse homoclinic orbits for each $N > 1$. As the family of periodic orbits approaches the homoclinic orbit, the Floquet multipliers have been shown to change between all three configurations infinitely often.

2.3.4 The saddle-center case

Near $C_1$ (a saddle-node bifurcation), the normal form (named Iooss-Kirchgässner) is again a 4D completely integrable ODE for two real variables $x_1$ and $x_2$ and one complex variable $z$. This normal form admits a sech$^2$ orbit $\Gamma$ homoclinic to a saddle-center whose eigenvalues are in configuration (3), but by dimension counting we expect $\Gamma$ to persist in the original ODE only at isolated values of $\mu$. Nonetheless, the normal form admits a large class of solutions homoclinic to periodic orbits (known as generalized solitary waves) with
a sech$^2$ core and exponentially small tails (Figure 2.15).

In 4D Hamiltonian systems, the isolated parameter values at which $N$-pulse homoclinic orbits exist have been shown to possess a cascade structure. Also in the Hamiltonian case, near a transverse homoclinic orbit to a periodic orbit, there exist a homoclinic tangle and hence infinitely many $N$-pulse homoclinic orbits to the periodic orbit.

### 2.3.5 Other homoclinic orbits

Near $C_3$ (named Belyakov-Devaney), the equilibrium 0 remains hyperbolic and only changes between a saddle and a saddle-focus. As a result, all finite-amplitude homoclinic orbits should persist across $C_3$. Nonetheless, infinitely many orbits homoclinic to the saddle-focus can be created by a nonlocal bifurcation at $C_3$.

If the equilibrium 0 is a focus (configuration (4)), then generally homoclinic orbits to 0 cannot exist due to the absence of strong stable/unstable manifolds. However, orbits homoclinic to a periodic orbit or equivalently orbits homoclinic to an equilibrium in a suitable Poincaré map are still expected. In particular, a (subcritical) transition between periodic orbits with Floquet multipliers in configurations (I) and (II) (referred to as an Eckhaus bifurcation) generates two branches of orbits homoclinic to a configuration (I) periodic orbit.

### 2.4 Homoclinic snaking and spatially localized states

Near the subcritical Turing bifurcation, a hyperbolic equilibrium (configuration (1)) together with a family of unstable periodic orbits bifurcate from an elliptic equilibrium (configuration (4)). An interesting phenomenon of bistability between an equilibrium and a family of periodic orbits occurs if the periodic orbits re-stabilize at larger amplitudes. In this situation, there can be homoclinic orbits in the form of heteroclinic cycles that start from
the equilibrium, wind around a periodic orbit arbitrarily many times, and end at the same equilibrium. In the spatial dynamics context, the heteroclinic connection corresponds to a front between a background equilibrium and a periodic pattern, and the homoclinic orbit corresponds to a spatially localized state of arbitrary length. Because both the equilibrium and the periodic orbit are hyperbolic, the unstable manifold of the equilibrium is 2D and the center-stable manifold of the periodic orbit is 3D. Hence by dimension counting, the heteroclinic connection is of codimension-0 and there exists a finite interval (called the snaking region) of the control parameter in which such localized states can be observed.

In this section, we first construct the bifurcation diagram of these localized states known as homoclinic snaking, taking the generalized Swift-Hohenberg equation (SHE) as an example. Next we address aspects of temporal dynamics in variational PDEs such as the SHE by calculating the speed of the front between two equilibria or between one equilibrium and one periodic pattern near the existence regions of the respective steady localized states. Finally we mention recent progress on localized states, some of which has motivated later chapters of this dissertation.

2.4.1 Standard homoclinic snaking

The geometric picture underlying homoclinic snaking is easiest to understand in a 4D Hamiltonian ODE for $u(x) \in \mathbb{R}^4$. A fundamental example is the steady state of the quadratic-cubic SHE (SH23)

$$u_t = ru - (\partial_{xx} + k_0^2)u + b_2u^2 - u^3, \quad b_2 \in \mathbb{R}, \quad r \in \mathbb{R},$$

where $u \equiv (u, u_x, u_{xx}, u_{xxx})$ and $k_0$ can be taken to be 1. A conserved Hamiltonian $H$ such that $dH/dx = 0$ is (cf. [86])

$$H = -\frac{1}{2}(r - k_0^4)u^2 + k_0^2u_x^2 - \frac{1}{2}u_{xx}^2 + u_xu_{xxx} - \frac{1}{3}b_2u^3 + \frac{1}{4}u^4.$$

The Turing bifurcation at $r = 0$ is supercritical (towards $r > 0$) if $b_2 < b_2^*$ and subcritical (towards $r < 0$) if $b_2 > b_2^*$, where $b_2^* = k_0^2\sqrt{27/38}$. Now we show the construction of all primary homoclinic orbits to the equilibrium 0 in the subcritical regime $b_2 > b_2^*$. Since $H(0) = 0$ for the equilibrium 0, it suffices to consider a 3D ODE on $H^{-1}(0)$. Taking a Poincaré map, one can further reduce this ODE to a 2D reversible map with reversibility denoted by $R$. In this map, the equilibrium 0 and the periodic orbit $\gamma(\varphi)$ have been reduced to two saddle-type equilibria on the symmetric section $S = \text{fix}(R)$. As shown in Figure 2.16, the snaking region is created by the codimension-0 transverse intersection between $W^u(0)$ and $W^s(\gamma(\varphi))$, known as a heteroclinic tangle. The left (right) limit of the snaking region, say $r = r_-$ ($r = r_+$), is the parameter value where $W^u(0)$ is tangent to $W^s(\gamma(\varphi))$ from outside (inside), known as an outer (inner) tangency.

As $\gamma(\varphi)$ is approached, $W^u(0)$ ($W^s(0)$) wiggles to intersect $W^s(\gamma(\varphi))$ ($W^u(\gamma(\varphi))$) infinitely many times. The successive intersections and their associated manifold segments can be labeled by integer indices $n^u$ ($n^s$) that increase to infinity towards $\gamma(\varphi)$ and satisfy $n^u = n^s$ for two intersections related by the reversibility $R$. (The notations $n^u$ and $n^s$ should not be confused with the dimensions of unstable and stable manifolds that they
denote in the earlier sections.) As shown in Figure 2.17(a), topologically $W^u(0)$ ($W^s(0)$) can be represented by horizontal (vertical) directed lines labeled by $n^u$ ($n^s$). In either family of parallel lines, adjacent lines have opposite directions and are connected to each other at infinity. In this setting each point in $W^u(0) \cap W^s(0)$ is represented by a lattice point $(n^u, n^s) \in \mathbb{Z}^2$, and the 2D map takes the form of the constant shift

$$g : (n^u, n^s) \to (n^u + 2, n^s - 2)$$

(2.66)
on the lattice $\mathbb{Z}^2$, which results from the geometry of the heteroclinic tangle. Since the equilibrium 0 is approached under forward (backward) iterations as $n^s \to -\infty$ ($n^u \to -\infty$), the orbit of $(n^u, n^s)$ under the group $G$ generated by $g$ corresponds to a primary homoclinic orbit to the equilibrium 0. Therefore the quotient set $L = \mathbb{Z}^2 / G$ is the complete collection of such homoclinic orbits, and it only remains to study the disappearance of points in $L$ at the outer (inner) tangency $r_-$ ($r_+$).

If we assume, without loss of generality, that every two segments $n^u = 2i$ and $n^u = 2i + 1$ ($i \in \mathbb{Z}$) annihilate at $r_-$, then it follows from reversibility that $n^s = 2i$ and $n^s = 2i + 1$ also annihilate at $r_-$. Hence the four points $2i + (0, 0)$, $2i + (1, 1)$, $2i + (0, 1)$ and $2i + (1, 0)$ annihilate at $r_-$. Similarly the four points $2i + (1, 1)$, $2i + (2, 2)$, $2i + (1, 2)$ and $2i + (2, 1)$ annihilate at $r_+$. These processes are represented in Figure 2.17(b) as a bifurcation diagram with the control parameter $r$ horizontal and the “norm” $N = (n^u + n^s)/2$ vertical via the following procedure. First a series of horizontal segments (red) between $r_-$ and $r_+$ are drawn corresponding to the diagonal points in the lattice. Every two segments $N = 2i$ and $N = 2i + 1$ are connected at $r = r_-$ and joined by two new segments $N = 2i + 1/2$ (black) corresponding to the points $2i + (0, 1)$ and $2i + (1, 0)$. Similarly, every two segments $N = 2i + 1$ and $N = 2i + 2$ are connected at $r = r_+$ and joined by two new segments $N = 2i + 3/2$ (black) corresponding to the points $2i + (1, 2)$ and $2i + (2, 1)$. In this way a snake-shaped bifurcation curve formed by horizontal segments with integer norms is constructed, and two horizontal segments with identical half-integer norms bifurcate from each fold of the snake. Applying the same argument to the series of four points two rows below (or equivalently
Figure 2.17: (a) The horizontal (vertical) lines labeled by $n^u$ ($n^s$) represent segments of $W^u(0)$ ($W^s(0)$). The red (blue) dots correspond to symmetric localized states with maxima (minima) at $x = 0$, while the black crosses correspond to asymmetric localized states. (b) The snakes-and-ladders bifurcation diagram where $r$ is the control parameter and $N = (n^u + n^s)/2$ is the norm. The solid red, blue and black segments represent respectively the red dots, blue dots and black crosses in (a).

Two rows above the diagonal, a second snake (blue) with two coinciding segments (black) bifurcating from each fold can be constructed, which happens to be the mirror image with respect to the vertical line $r = (r_- + r_+)/2$ of the first snake together with its bifurcating segments. Now since the annihilation of all the points in $L$ at both $r_-$ and $r_+$ has been considered, the bifurcation diagram can be completed by connecting every two segments with identical half-integer norms that bifurcate from the first and second snakes. These bifurcating segments are known as ladders and the complete bifurcation diagram is known as the snakes-and-ladders structure.

Localized states in the steady state of (2.64), in which snakes-and-ladders were originally discovered [42], can be reconstructed from homoclinic orbits in the 2D map as follows. The Poincaré section is chosen to be $\Sigma = \{(u, u_x, u_{xx}, u_{xxx}) | u_x = 0, u > 0\}$, which effectively tracks the successive maxima of $u(x)$. Though the autonomous dynamical systems studied here are invariant under translation in the analog of “time” in spatial dynamics, namely either the number of iterations $n$ or the spatial coordinate $x$, in view of (2.66) it is most natural to require that the lattice point $(n^u, n^s)$ exists at $n = (n^u - n^s)/4$ and the corresponding maximum exists at $x = n\lambda$ where $\lambda$ is the wavelength of the periodic pattern. Figure 2.18 shows the bifurcation diagram of localized states (to be compared with Figure 2.17(b)), plotted in terms of the $L^2$-norm of $u(x)$ as a function of the control parameter $r$, and the solution profiles $u(x)$ at representative points (to be compared with Figure 2.17(a)). The first and second snakes correspond to the $\phi = 0$ and $\phi = \pi$ branches $L_0$ and $L_\pi$ created from the subcritical Turing bifurcation at $r = 0$. On the $L_0$ branch (the red snake in Figure 2.17(b)), since the homoclinic orbit is the orbit of either the even point
2i + (0, 0) or the odd point 2i + (1, 1) (the red dots in Figure 2.17(a)) under $G$, the localized state is always symmetric with a maximum at $x = 0$ and the left and right bounding fronts are identical. However, since in the even and odd cases the equilibrium 0 is respectively approached along the even and odd segments of $W^u(0)$ and $W^s(0)$, the shapes of the fronts connecting the periodic pattern $\gamma(\varphi)$ to the background equilibrium 0 differ in these two cases. The even (odd) front, denoted by $\xi_0(x)$ ($\xi_1(x)$), persists from $r_+ \rightarrow r_-$ (from $r_-$ to $r_+$) as the $L_0$ branch is followed upwards. The parity of the front changes at each fold, which results in one extra wavelength being added for every two folds on either side of the localized state by continuous deformation of the corresponding front. The $L_\pi$ branch (the blue snake in Figure 2.17(b)) only differs from the $L_0$ branch for consisting of localized states with minima at $x = 0$ (the blue dots in Figure 2.17(a)). On the ladders (black segments in Figure 2.17(b)), a localized state is always asymmetric with its maxima and minima located at odd multiples of $\lambda/4$ (black crosses in Figure 2.17(a)), but in practice one usually chooses the “center of gravity” of such a localized state to be $x = 0$ as done in Figure 2.18. Because the left and right fronts have opposite parities, each point on the ladders represents two asymmetric solutions related by reflection with respect to $x = 0$.

The above discussion applies sufficiently high up the snakes-and-ladders diagram, but major features in the lower portion can also be explained by the geometry of the 2D map as done in parts of Chapter 5. For example, the $n$-th fold $r_n$ on the snakes approaches $r_\pm$ exponentially from larger $r$ as a function of $n$ with an accumulation rate calculated at the end of §5.4. Though not explicitly addressed in Chapter 5, the bifurcation points of the ladders do not coincide with but exponentially approach the folds on the snakes as $n \rightarrow \infty$.

### 2.4.2 Depinning in variational systems

Now that steady localized states are characterized, we can proceed to study their temporal dynamics in the original PDE. The PDE dynamics is relatively simple because of the variational structure of (2.64), namely there exists a Lyapunov functional

$$\mathcal{F} = \int \mathcal{L} dx \quad \text{where} \quad \mathcal{L} = -\frac{1}{2} ru^2 + \frac{1}{2}[(\partial_{xx} + k_0^2)u]^2 - \frac{1}{3} b_2 u^3 + \frac{1}{4} u^4$$

(2.67)

such that (2.64) can be written as $u_t = -\delta \mathcal{F}/\delta u$ where $\delta \mathcal{F}/\delta u$ denotes functional derivative with respect to $u$. As a result, the PDE (2.64) behaves like a gradient flow that always evolves towards a local minimum of $\mathcal{F}$. Thus for any variational PDE, if a front separates a left pattern $P_-$ with free energy density $\mathcal{L}_-$ and a right pattern $P_+$ with free energy density $\mathcal{L}_+$, the front should move with speed $c > 0$ (to the right) if $\mathcal{L}_- < \mathcal{L}_+$, and vice versa. The value of the control parameter $r$ such that $\mathcal{L}_- = \mathcal{L}_+$ is known as the Maxwell point $r = r_M$.

This expectation is indeed correct if $P_-$ and $P_+$ are both equilibria, but for localized states on the homoclinic snaking branches where $P_- = 0$ is an equilibrium but $P_+ = \gamma(\varphi)$ is a periodic pattern, steady states ($c = 0$) exist in the finite interval $r \in [r_-, r_+]$ as suggested by spatial dynamics. This apparent paradox has been resolved by a physical mechanism called pinning [144], which roughly states that the front becomes pinned to the periodic pattern and fails to propagate for a finite interval of parameter values around the Maxwell point. Outside this interval, the difference in free energy density becomes sufficiently large to overcome the pinning potential, leading to a dynamical process known as depinning.
Figure 2.18: The snakes-and-ladders bifurcation diagram and selected profiles of localized states in the nonlinear optical model \((1 + \partial_{xx})^2 f + \epsilon^4 f + 3\epsilon E_{hg} f^2 + \epsilon^2 f^3 = 0\) which is a rescaled version of the steady state of SH23 (2.64). The bifurcation diagram has been computed at \(\epsilon = 0.55\) for the optical model with control parameter \(E_{hg}\), but qualitatively the same bifurcation diagram with control parameter \(r\) is present in the steady state of (2.64) at suitable \(b_2\) (e.g. \(b_2 = 1.8\)). The two snakes are shown in black while the ladders are shown in green. Reproduced from Figure 2 of Ref. [52].
As mentioned in §1.2.3, the first step towards understanding temporal dynamics of coherent structures (e.g. depinning) is spectral stability analysis. For sufficiently wide localized states on the snakes-and-ladders diagram, since both the background equilibrium 0 and the periodic pattern \( \gamma(\varphi) \) are stable, the overall stability depends only on the bounding fronts. Numerically one finds that the odd front \( \xi_1(x) \) (the even front \( \xi_0(x) \)) is stable (unstable) and the eigenfunction \( \xi(x) \) is concentrated at the location of the front. Hence the symmetric localized states on the segments of either snake extending from \( r_- \) to \( r_+ \) (from \( r_+ \) to \( r_- \)) are stable (unstable), while the asymmetric localized states on the ladders are always unstable because both \( \xi_0(x) \) and \( \xi_1(x) \) are present. The two fronts \( \xi_1(x) \) and \( \xi_0(x) \) annihilate in a saddle-node bifurcation at either snaking limit \( r_\pm \). Let \( \xi(x) \) denote the common profile of \( \xi_1(x) \) and \( \xi_0(x) \) at \( r_\pm \). To leading order, the odd and even fronts at distance \( 0 < \delta r \ll 1 \) inside the snaking region then take the form

\[
\xi_1(x) = \tilde{\xi}(x) - C(\delta r)^{1/2}\tilde{\xi}(x), \quad \xi_0(x) = \tilde{\xi}(x) + C(\delta r)^{1/2}\tilde{\xi}(x),
\]

where \( C \) is a constant. It follows that a localized state formed by two bounding fronts can also be expressed as the localized state at \( r_\pm \) superimposed with a suitable eigenfunction.

A symmetric localized state formed by two \( \xi_1 \) fronts or two \( \xi_0 \) fronts requires an even eigenfunction called the amplitude mode that takes the form \( \tilde{u}_a(x) = \tilde{\xi}(x) + \tilde{\xi}(-x) \), while an asymmetric localized state formed by one \( \xi_1 \) front and one \( \xi_0 \) front requires an odd eigenfunction called the phase mode that takes the form \( \tilde{u}_p(x) = \tilde{\xi}(x) - \tilde{\xi}(-x) \). Though we have only shown the existence of \( \tilde{u}_a(x) \) and \( \tilde{u}_p(x) \) near \( r_\pm \), both modes actually exist everywhere along the two snakes with identical growth rates to \( \tilde{\xi}(x) \). In addition, there is a Goldstone (or translational) mode \( \tilde{u}_G(x) \propto u_x \) with growth rate identically 0.

For \( 0 < -\delta r \ll 1 \) (outside the snaking region), the space-time plots for the depinning processes near \( r_- \) and \( r_+ \) are shown in the left panels of Figure 2.19. The depinning period \( T \), defined as the time taken to annihilate (near \( r_- \)) or create (near \( r_+ \)) one wavelength of periodic pattern on either side of the localized state, can be predicted using center manifold reduction [42]. Let \( \tilde{u}(x) \) denote the localized state at the saddle-node (i.e. at \( \delta r = 0 \)). Near this bifurcation all three modes \( \tilde{u}_a(x) \), \( \tilde{u}_p(x) \) and \( \tilde{u}_G(x) \) are marginally stable and need to be included in the center manifold, but the latter two are both odd and must have zero amplitudes because the evolving localized state remains even. The truncated ODE on the 1D center manifold is the familiar saddle-node normal form

\[
\alpha_1 \frac{da}{dt} = |\delta r|^{1/2}(\alpha_2 \text{sgn}(\delta r) + \alpha_3 a^2),
\]

where \( a(t) \) is the amplitude of \( \tilde{u}_a(x) \) and the coefficients \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are integrals involving \( \tilde{u}(x) \) and \( \tilde{u}_a(x) \). Eq. (2.69) applies to both \( \delta r > 0 \) and \( \delta r < 0 \). In the former case, there are two fixed points that correspond to symmetric localized states assembled from (2.68). In the latter case, there is no fixed point and \( a(t) \) goes from \( -\infty \) to \( \infty \) in a finite time

\[
T = \frac{\pi \alpha_1}{(\alpha_2 \alpha_3 \delta r)^{1/2}}.
\]

As shown in the right panels of Figure 2.19, Eq. (2.70) compares well with the numerics. It follows that the average front speed \( c = \lambda/T \) scales as \( c \sim |\delta r|^{1/2} \).
Figure 2.19: Space time plots showing the depinning of a localized state in SH23 (2.64) (a) $|\delta r| = 0.0005$ to the left of $r_-$ and (b) $|\delta r| = 0.0005$ to the right of $r_+$. Parameters: $b_2 = 1.8$. Also shown is the depinning period $T$ as a function of $r$ with numerical values in circles and predicted values in a solid curve. Reproduced from Figures 3.10 and 3.11 of Ref. [35].
Now we consider a variational PDE where the two patterns $P_{\pm}$ are both equilibria, denoted by $u_{\pm}$. The front then takes the form of a traveling wave $u(x - ct)$ with boundary conditions $u(\pm \infty) = u_{\pm}$. We again denote the control parameter by $r$ and define $\delta r \equiv r - r_M$ where $r_M$ is the Maxwell point. In terms of the free energy densities $\mathcal{L}_{\pm}$, we have

$$\frac{dF}{dt} = c(\mathcal{L}_- - \mathcal{L}_+). \quad (2.71)$$

On the other hand, using the variational structure we get

$$\frac{dF}{dt} = \int \frac{\delta F}{\delta u} u_t dx = - \int u_t^2 dx = -c^2 \int u_x^2 dx. \quad (2.72)$$

Equating Eqs. (2.71) and (2.72), we arrive at the following expression for the front speed

$$c = (\mathcal{L}_+ - \mathcal{L}_-)/\left(\int u_x^2 dx\right). \quad (2.73)$$

Eq. (2.73), which holds for arbitrary $\delta r$, states that the front speed equals the difference in free energy density divided by the $L^2$-norm of the Goldstone mode $u_x$. For $|\delta r| \ll 1$, since $\mathcal{L}_+ - \mathcal{L}_- \sim \delta r$ and $\int u_x^2 dx \sim 1$, the front speed scales as $c \sim \delta r$. This relation derived for traveling fronts between two equilibria is different from the relation $c \sim |\delta r|^{1/2}$ previously derived for depinning dynamics near the snaking region. However, since the steady localized states in these two cases are not known to be related by any bifurcation, a gradual change in the scaling exponent probably cannot be observed in variational PDEs such as SH23.

### 2.4.3 Recent work on localized states

Inspired by the discovery of snakes-and-ladders in SH23, 1D localized states have been further studied in the past five years. The following (incomplete list of) numerical results reveal additional features of homoclinic snaking in other PDEs. In the cubic-quintic SHE (SH35) which has an additional up-down symmetry, both even and odd localized states form snakes-and-ladders as studied by Burke and Knobloch [39]. In the Lugiato-Lefever equation (LLE) which is nonvariational, the study of Gomila et al suggests that in addition to those on the snaking branches, there are other localized states that can be found by directly computing the invariant manifolds to the background equilibrium and the periodic pattern [90]. A more recent study of a nonvariational extension of SH35 by Houghton and Knobloch stresses that asymmetric localized states on the ladders are expected to travel in nonvariational systems [97]. In the presence of a large-scale neutral mode which is typical in pattern forming systems with a conservation law, homoclinic snaking becomes slanted and the existence region of localized states broadens as studied by Dawes [70]. Two chapters in this dissertation contain numerical results on homoclinic snaking of 1D localized states in the 1:1 FCGLE (1.11). In Chapter 3, we will describe a new mechanism of homoclinic snaking called defect-mediated snaking (DMS) that appears to have little in common with snakes-and-ladders. In Chapter 4, we will study the depinning of localized states on the DMS branch, which happens by successive phase slips and exhibits both scaling exponents identified in §2.4.2.
Meanwhile, theories of homoclinic snaking are being developed. On the dynamical systems side, Beck et al formulated a set of hypotheses about the connecting orbit between the background equilibrium and the periodic pattern that guarantee snaking [18]. On the analytical side, Kozyreff and Chapman constructed the snakes-and-ladders bifurcation diagram near the onset of homoclinic snaking at $b_2 = b_2^*$ using a special technique in asymptotic analysis known as exponential asymptotics [52]. Subsequently, a considerably simpler construction using a variational approximation was proposed by Susanto and Matthews [164]. In Chapter 5, we will formulate a theory of interaction between homoclinic snaking and a saddle-center bifurcation of the periodic orbit leading to a new snaking scenario referred to as snakes mating, and use this theory to predict the rate of fold accumulation in DMS. A numerical example of snakes mating in a variational extension of the SHE is also presented.

A necessary condition for homoclinic snaking is that the periodic pattern inside the localized states respects spatial reversibility. In Chapter 6, we will introduce a new category of localized states referred to as localized phase-winding states (LPWS), which possess an asymmetric periodic pattern and hence do not undergo homoclinic snaking. In Chapter 7, we will propose a scheme to classify 1D localized states in the 1:1 FCGLE as a function of the control parameters, including but not limited to fronts between two equilibria, DMS and LPWS. A classification program in a similar vein has been carried out in the 2:1 FCGLE by Burke et al [41], but their study focused on various fronts between two equilibria that may arise from the up-down symmetry of the PDE.

Although many questions about 1D localized states remain open, localized states in higher spatial dimensions (2D or 3D) have already attracted much attention. A typical pattern forming PDE is isotropic and thus admits radially symmetric localized solutions, which are quasi-1D because they solve a nonautonomous ODE in the radial direction. In the multi-dimensional SHE, such solutions were discovered analytically by Lloyd and Sandstede [123], and bifurcation curves that exhibit homoclinic snaking were later computed by McCalla and Sandstede [135]. Inherently 2D localized states in the SHE have also been studied using advanced numerical continuation techniques. In particular, localized hexagons were studied by Lloyd et al [124], and other solutions such as localized stripes and localized squares were studied by Avitabile et al [12]. In Chapter 8, we will numerically investigate quasi-1D and 2D localized states in the 1:1 FCGLE, including circular traveling fronts, localized ring patterns, and planar and circular localized hexagons. In 3D, a notable example of homoclinic snaking has been found by Schneider et al in plane Couette flow described by the Navier-Stokes equations [156]. The authors suggested that these localized states, which result from bistability between a laminar state and an exact periodic solution called the Nagata state, can be a useful step towards understanding transitional turbulence in spatially extended flows from the dynamical systems viewpoint.

The laminar-turbulent transition in shear flows as the Reynolds number $Re$ (ratio of inertial force to viscous force) increases is an old problem in fluid mechanics that saw revived interest in recent years [16]. In the transitional regime, the laminar and turbulent states coexist to create turbulent spots that could either shrink or expand in time. In Chapter 9, we will present solutions to the 1:1 FCGLE referred to as localized spatiotemporal chaos (LSTC), that in many ways behave similarly to turbulent spots. Another classical problem in fluid mechanics that figured prominently in pattern formation is Rayleigh-Bénard
convection [148]. Here the main parameter controlling convective activity is the Rayleigh number $Ra$ (ratio of buoyancy to diffusion). As $Ra$ increases, experimentally observed convection patterns range from stripes, spirals, hexagons and squares near onset to spiral-defect chaos and turbulence far from onset [30]. In Chapter 10, we will derive an integro-differential equation from the governing hydrodynamic equations near the onset of convection. Such derivations reveal certain issues that may arise in using pattern-forming PDEs (e.g. the 2D SHE) to model pattern dynamics in physical problems. Although localized states are not the main focus of this chapter, there is a theorem providing necessary conditions for the existence of stable localized states in variational PDEs.
Chapter 3

Defect-mediated snaking

3.1 Introduction

Spatially localized structures in one spatial dimension correspond to homoclinic orbits of the governing equations written as a dynamical system in space. When this system is reversible in space these structures may be reflection-symmetric and hence stationary. Equilibrium structures of this type have been extensively studied in recent years both theoretically [38, 46] and numerically [17, 24], and are found in bistable regimes, i.e., in parameter regimes with two coexisting equilibria, $A$ and $B$, say. Two cases are of particular interest: (i) $A$ and $B$ are both spatially homogeneous, (ii) $A$ is spatially homogeneous while $B$ is spatially periodic. The associated localized structures are organized by heteroclinic cycles, i.e., connections of the form $A \to B \to A$ or vice versa. Since the return connection is guaranteed by the reversibility of the vector field a heteroclinic cycle between two homogeneous equilibria in four dimensions is of codimension one while a heteroclinic cycle between a homogeneous and a periodic equilibrium is of codimension zero. In the former case the cycle exists at a particular parameter value; in the latter there is an interval in parameter space filled with such cycles.

The detailed behavior near such cycles depends on the spatial eigenvalues of the homogeneous states and the spatial Floquet multipliers of the periodic states. If the eigenvalues of both homogeneous states are real the fronts connecting state $A$ to state $B$ and back again will be monotonic and a single localized state is present whose length diverges as the heteroclinic cycle is approached. If the eigenvalues of one of the states involved are complex, the fronts will no longer be monotonic, and there will be a finite multiplicity of localized states on either side of the parameter value corresponding to the formation of the heteroclinic cycle. We refer to this behavior as collapsed snaking to distinguish it from homoclinic snaking that accompanies heteroclinic cycles between homogeneous and periodic equilibria. In this case localized structures are found in an interval of parameter values that define the so-called snaking [46] or pinning [144] region. This type of behavior is well understood in the context of variational systems such as the Swift-Hohenberg equation [42, 39, 38]. This equation possesses a trivial homogeneous state and the localized structures bifurcate from this state simultaneously with the branch of spatially periodic states, provided these bifurcate subcritically. At small amplitude the localized structures
are broad but as the branch enters the pinning region their width and amplitude become comparable to the wavelength and amplitude of the competing periodic pattern. As one continues to follow localized states towards larger amplitude one finds that the structure broadens by nucleating additional rolls symmetrically on either side. Structures of this type therefore grow from the outside. Associated with this growth are back-and-forth oscillations of the solution branch across the pinning region [39, 182]. Generically there are two types of reflection-symmetric localized states, distinguished by having maxima or minima at the point of symmetry. In addition, asymmetric localized states are also present, and in variational systems these also correspond to steady solutions. Such states connect the two branches of symmetric localized states in a characteristic snakes-and-ladders structure [38]. In addition, the snaking or pinning region contains a multitude of multipulse states, i.e., bound states of localized structures. These may fall on additional snaking branches or be organized in stacks of isolas [40].

In variational systems such as the Swift-Hohenberg equation the origin of this remarkably complex structure can be related to the presence of a Maxwell point at which the homogeneous and periodic equilibria have the same energy. In this case the periodic structure within the localized structure pins the fronts at either end that connect it to the homogeneous state preventing their motion even when the energies differ [144]. The rich structure of the resulting pinning region is a direct consequence of this effect and may be thought of as the result of a broadening of the Maxwell point owing to the heterogeneity of one of the competing equilibria.

In the systems studied thus far the wavelength of the periodic state within the localized structure appears to be well-defined and is found to be a monotonic function of the bifurcation parameter. In variational systems this behavior is easy to understand using energy considerations [38]. However, a quantitative understanding of the wavelength selection mechanism is only possible in the presence of a Hamiltonian that is conserved by the time-independent system written as a dynamical system in space [42].

Recent work by Beck et al [18] shows that the snakes-and-ladders structure persists when nonvariational terms are added, although the asymmetric states now correspond to drifting localized structures [36]. Studies of natural doubly-diffusive convection [23, 24] and of binary fluid convection [17] confirm this expectation. These results, taken together, give the impression that all localized structures in a snaking or pinning region grow qualitatively as described by the Swift-Hohenberg equation. We show in this chapter that this is not the case. Specifically, we exhibit an example where the localized structures grow by roll insertion in the center of the structure, accompanied by outward displacement of the existing rolls. The center corresponds to a defect that is responsible for the "cell division" required to grow the structure. The central defect thus plays the role of a steady state "pacemaker".

This new growth mechanism has been discovered in the forced complex Ginzburg-Landau equation describing parametrically driven dissipative media in appropriate parameter regimes [57, 79]. Recent experimental studies of systems driven by spatially homogeneous periodic forcing with frequency near twice the natural oscillation frequency of the system have revealed the presence of different types of localized states, including oscillons [8, 122, 126, 174] and reciprocal oscillons [28, 142, 185]. The former are localized time-
dependent oscillations that are embedded in a stationary background and oscillate with half the forcing frequency [122, 174]. The latter are localized oscillations embedded in an oscillating background, both of which oscillate at half the forcing frequency. Bound states of oscillons have also been observed. Spatially localized oscillations are also found in systems driven additively near their natural oscillation frequency [90, 167, 168]. In the following we refer to these two cases as 2:1 and 1:1 resonances, respectively.

This chapter is organized as follows. In Section 3.2 we introduce the forced complex Ginzburg-Landau equation used to describe near-resonant forcing of oscillatory systems and summarize the key properties of this equation. In Section 3.3 we analyze in detail the new types of behavior present in the 1:1 resonance. This section relies heavily on numerical continuation using the software AUTO [78]. In Section 3.4 we describe the results of time integration to examine both the stability of the localized structures we identify and their evolution when they prove to be unstable. The chapter concludes with brief remarks in Section 3.5.

### 3.2 The forced complex Ginzburg-Landau equation

We consider a continuous system in one spatial dimension near a bifurcation to spatially homogeneous oscillations with natural frequency $\omega$ in the presence of spatially homogeneous forcing with frequency $\Omega$. We focus on the behavior near strong resonances of the form $\Omega : \omega = n : 1$, where $n = 1, 2$. In both cases the forcing leads to an Arnol’d tongue containing spatially homogeneous states that are phase-locked to the forcing frequency $\Omega$. When $n = 1$ the amplitude of the phase-locked states may take the form of an S-shaped curve as the forcing increases, leading to a region with multiple coexisting phase-locked states. Likewise, when $n = 2$ the phase-locked states may bifurcate subcritically on one side of the Arnol’d tongue, again leading to bistability, this time between the trivial background state and a pair of finite amplitude phase-locked states that are $180^\circ$ out of phase.

As reviewed in §1.1, under appropriate conditions [57, 79] the oscillation amplitude $A(x, t)$ obeys the forced complex Ginzburg-Landau equation (FCGLE)

$$A_t = (\mu + i\nu)A - (1 + i\beta)|A|^2A + (1 + i\alpha)A_{xx} + \gamma\bar{A}^{n-1},$$  \hspace{1cm} (3.1)

where $\mu$ represents the distance from onset of the oscillatory instability, $\nu$ is the detuning from the unforced frequency, and $\alpha$, $\beta$ and $\gamma > 0$ represent dispersion, nonlinear frequency correction and the forcing amplitude, respectively. We distinguish two regimes, the regime $\mu < 0$ in which spatially homogeneous oscillations decay in the absence of forcing and $\mu > 0$ in which they grow. We refer to the former as the damped regime and the latter as the self-exciting regime. For numerical exploration it will be useful to write Eq. (3.1) in terms of real amplitudes $U(x, t), V(x, t)$ such that $A \equiv U + iV, \bar{A} \equiv U - iV$.

Though we will focus on the case of 1:1 resonance, in this section we study properties of Eq. (3.1) for arbitrary $n$. This equation has a large variety of nontrivial solutions, both homogeneous in space and spatially inhomogeneous, for all values of $n \geq 1$; when $n > 1$ it also possesses the trivial solution $A = 0$. These solutions either respect or break the two symmetries of Eq. (3.1), equivariance under $(x, A) \rightarrow (x, Ae^{2\pi i/n})$, and reversibility.
in space: \((x, A) \rightarrow (-x, A)\). Of interest in the following are localized structures corresponding to solutions that are homoclinic to the same homogeneous state as \(|x| \rightarrow \infty\) and fronts corresponding to heteroclinic solutions connecting two distinct spatially homogeneous states. In addition, there are connections between homogeneous states and spatially periodic states created in Turing bifurcations. Finally, we are also interested in homoclinic orbits connecting a periodic orbit to itself. The presence of reversibility renders certain of these connections structurally stable, as discussed further below.

Equation (3.1) possesses nontrivial stationary spatially homogeneous solutions of the form \(A = R \exp i \phi\), where
\[
(1 + \beta^2)R^4 - 2R^2(\mu + \nu \beta) + \mu^2 + \nu^2 = \gamma^2 R^{2n-4}
\] (3.2)
and \(\phi\) solves
\[
\sin n \phi = \frac{\nu - \beta R^2}{\gamma R^{n-2}}, \quad \cos n \phi = \frac{R^2 - \mu}{\gamma R^{n-2}}.
\] (3.3)
For each positive solution of Eq. (3.2) these equations have \(n\) solutions \(\phi\) in \([0, 2\pi)\). This is a consequence of the equivariance of Eq. (3.1) under \((x, A) \rightarrow (x, Ae^{2\pi i/n})\). The temporal stability of these solutions with respect to spatially homogeneous perturbations is determined by writing \(A = R \exp(i \phi)(1 + a(t))\), linearizing in \(a(t)\), and looking for solutions of the form \(a(t) = a_+ \exp st + a_- \exp st\). The (complex) growth rate \(s\) obeys the quadratic equation
\[
s^2 + 2[R^2 + \gamma R^{n-2} \cos n \phi]s + 2n\gamma R^n(\cos n \phi - \beta \sin n \phi) - n(n - 2)\gamma^2 R^{2n-4} = 0.
\] (3.4)
This growth rate vanishes at
\[
\cos n \phi - \beta \sin n \phi - \frac{1}{2}(n - 2)\gamma R^{n-4} = 0,
\] (3.5)
i.e., at saddle-nodes. This equation is equivalent to
\[
(1 + \beta^2)R^2 = \mu + \beta \nu + \frac{1}{2}(n - 2)\gamma^2 R^{2n-6}.
\] (3.6)
These results hold for all integers \(n \geq 1\). A similar calculation shows that the growth rate of perturbations with wavenumber \(k\) satisfies the equation
\[
s^2 + 2[R^2 + \gamma R^{n-2} \cos n \phi + k^2]s + 2n\gamma R^n(\cos n \phi - \beta \sin n \phi) - n(n - 2)\gamma^2 R^{2n-4}
\[
+ 2[(1 + \alpha \beta)R^2 + \gamma R^{n-2}(\cos n \phi - \alpha \sin n \phi)]k^2 + (1 + \alpha^2)k^4 = 0,
\] (3.7)
or equivalently
\[
s^2 - 2[\mu - k^2 - 2R^2]s + (1 + \alpha^2)k^4 + 2[2(1 + \alpha \beta)R^2 - \mu - \alpha \nu]k^2
\]
\[
-2nR^2[\mu + \beta \nu - (1 + \beta^2)R^2] - n(n - 2)\gamma^2 R^{2n-4} = 0.
\] (3.8)
A Turing bifurcation occurs at the extremum of the neutral stability curve \(s = 0\) in the \((k, \gamma)\) plane. At this point \(\gamma = \gamma^T\), \(R = R_T\) and the solution \(k = k_T\) has multiplicity two. Thus
\[
[2(1 + \alpha \beta)R^2_T - \mu - \alpha \nu]^2 = -n(1 + \alpha^2)R^2_T[2(\mu + \beta \nu) - 2(1 + \beta^2)R^2_T + (n - 2)(\gamma^T)^2 R^{2n-6}_T].
\] (3.9)
The corresponding Turing wavenumber is given by

$$k_T^2 = \frac{\mu + \alpha \nu - 2(1 + \alpha \beta)R_T^2}{1 + \alpha^2}. \quad (3.10)$$

Equation (3.9) is to be solved simultaneously with Eq. (3.2) for the strength $\gamma_T$ of the forcing at which this bifurcation sets in. The corresponding amplitude $R_T$ then determines the wavelength $2\pi/k_T$ of the resulting structure via Eq. (3.10). In both cases Hopf bifurcations may arise but these lead to temporally periodic states not considered here.

In the following we study steady spatially localized structures that approach a homogeneous state as $x \to -\infty$ or $x \to +\infty$ or both. Necessary conditions for exponential approach or departure from such a state are provided by the spatial eigenvalues of these states. These are obtained by writing $A = R \exp(i\phi)(1 + a(x))$, linearizing in $a(x)$, and looking for solutions of the form $a(x) = a_+ \exp(\lambda x + a_- \exp \lambda x$. The spatial eigenvalues $\lambda$ obey the quadratic equation

$$(1 + \alpha^2)\lambda^4 - 2[(1 + \alpha \beta)R^2 + \gamma R^{n-2}(\cos n\phi - \alpha \sin n\phi)]\lambda^2$$

$$+ 2n\gamma R^n(\cos n\phi - \beta \sin n\phi) - n(n - 2)\gamma^2 R^{2n-4} = 0, \quad (3.11)$$

or equivalently

$$
(1 + \alpha^2)\lambda^4 + 2[\mu + \alpha \nu - 2(1 + \alpha \beta)R^2]\lambda^2$$

$$- 2nR^2[\mu + \beta \nu - (1 + \beta^2)R^2] - n(n - 2)((\mu - R^2)^2 + (\nu - \beta R^2)^2] = 0. \quad (3.12)
$$

The structure of this equation is a consequence of the fact that the time-independent version of Eq. (3.1), written as a four-dimensional dynamical system in $x$, is equivariant with respect to the reversibility symmetry mentioned above, $(x, A) \to (-x, A)$. This condition translates into the requirement that if $\lambda$ is a spatial eigenvalue so are $\bar{\lambda}$ and $-\lambda$. As a result the spatial eigenvalues fall into one of the four generic configurations in §2.3.1. The reversibility symmetry also constrains the spatial Floquet multipliers to lie in the three generic configurations in §2.3.1.

As reviewed in §2.3, the spatial eigenvalues provide necessary conditions for the presence of homoclinic and heteroclinic orbits between homogeneous equilibria, and it is possible to establish the bifurcation of such states from the homogeneous state. The existence of these states can be established either using multiple scale methods near the corresponding bifurcations [41], or using normal form theory for reversible systems [100]. In the following we explore the behavior of these states as a function of the parameters focusing on situations where new types of behavior arise. We mention that all of the above solutions are invariant under reflection with respect to a suitable origin $x_0$ (i.e., $A(2x_0 - x) = A(x)$, localized states) or under the combined action of reflection followed by the operation $(x, A) \to (x, Ae^{2\pi i/n})$ (fronts). Solutions that break reflection symmetry ($A(2x_0 - x) \neq A(x)$, all $x_0$) generally drift and will not be considered.

### 3.3 The 1:1 resonance

In this section we examine steady solutions to Eq. (3.1) in the case $n = 1$, i.e., in the 1:1 resonance. In this case the phase symmetry $(x, A) \to (x, Ae^{2\pi i/n})$ is trivial. We
Figure 3.1: (a) The critical curves and limits of snaking regions in the \((\nu, \gamma)\) plane. The lines \(\gamma_{SN}^\pm\) delimit the region with three homogeneous equilibria while the line \(\gamma^T\) corresponds to Turing bifurcations on \(A^+\). Belyakov-Devaney points on \(A^-\) occur along the line \(\gamma^{BD}\). The transition from collapsed snaking at \(\gamma^{CS}\) to defect-mediated snaking between \(\gamma_{DMS}^1\) and \(\gamma_{DMS}^2\) occurs at the codimension-two point \((\nu^*, \gamma^*) \approx (5.983, 2.339)\), marked with an open circle. (b) At larger \(\nu\), defect-mediated snaking has an additional limit \(\gamma_{DMS}^3\).

Adopt the parameter values \(\alpha = -1.5, \beta = 6\) and \(\mu = -1\) corresponding to Region \(V^-\) in the classification scheme of Chapter 7, and explore several values of the detuning \(\nu\). In addition to exhibiting bifurcation diagrams for stationary states we also examine their temporal stability.

Localized states exist in the region of bistability, in this case between two distinct phase-locked states. Thus we focus on the parameter regime corresponding to the presence of three homogeneous equilibria (Fig. 3.1, region between lines \(\gamma_{SN}^\pm\)). The equilibrium with the smallest/middle/greatest \(L^2\)-norm will be denoted by \(A^-/A^0/A^+\). In Region \(V^-\) the equilibrium \(A^-\) is temporally stable (cf. Chapter 7) implying that all localized states considered here are embedded in a stable background. This is a consequence of the fact that \(\mu = -1\).

The loci of bifurcation points of the equilibria in the \((\nu, \gamma)\) plane are shown in Fig. 3.1, together with the limits of the snaking region. These include the lines \(\gamma_{SN}^\pm\) delimiting the region with three homogeneous equilibria and the lines \(\gamma^T\) and \(\gamma^{BD}\) corresponding respectively to Turing bifurcations on \(A^+\) and Belyakov-Devaney points on \(A^-\). In what follows, we study the branch of localized states embedded in an \(A^-\) background, hereafter \(L_0\). As pointed out in §2.3.2 these bifurcate from the lower saddle-node at \(\gamma_{SN}^-\). For the chosen values of \(\nu\), there is always a supercritical Turing bifurcation on \(A^+\) at \(\gamma^T\).
### 3.3.1 Collapsed snaking

In this section we describe the behavior of the branches identified above when these are followed in parameter space. In general we use $\gamma$ as the bifurcation parameter and show the resulting solution branches for several fixed values of $\nu$. Figure 3.2 shows the bifurcation diagram of stationary solutions at $\nu = 5 < \nu^*$, plotted in terms of the $L^2$-norm $N$ defined as

$$N = \sqrt{\frac{1}{\ell} \int_{-\ell/2}^{\ell/2} \{|A(x)|^2 + |A_x(x)|^2\} \, dx}, \tag{3.13}$$

where $\ell$ is the large but finite domain used in the computation. The $L_0$ branch undergoes an infinite sequence of saddle-node bifurcations far from onset. However, in contrast to homoclinic snaking introduced in §2.4.1, in this case the branch exhibits collapsed snaking as the separation in $\gamma$ between consecutive saddle-nodes decreases exponentially much as in Figure 2.11(b), and both the left and right saddle-nodes asymptote to the same limiting value of $\gamma = \gamma^{CS} > \gamma^T$ [41, 110]. The profiles high up the collapsed snaking branch spend considerable ‘time’ in a neighborhood of the homogeneous state $A^+$, and resemble a heteroclinic cycle from $A^-$ to $A^+$ and back again. The limit $\gamma = \gamma^{CS}$ corresponds to the (codimension-one) point at which a heteroclinic orbit connecting the fixed points $A^-$ and $A^+$ is present. The back-and-forth approach of the collapsed snaking branch to this limiting value is a direct consequence of the fact that the spatial eigenvalues of $A^+$ at $\gamma^{CS}$ are complex (configuration (1)), as discussed in [110]. These eigenvalues are visible in the form of decaying oscillations in $U(x)$ and $V(x)$ as $x \to 0^-$ (and the growing oscillations for $x > 0$). Elsewhere in the parameter space of Eq. (3.1) there are regions where the spatial eigenvalues of the $A^+$ state at the limit point $\gamma^{CS}$ are real (configuration (2)); in this case the profiles $U(x)$ and $V(x)$ lack spatially oscillating tails and the branch $L_0$ collapses monotonically in $\gamma$ without any (asymptotic) saddle-node bifurcations (cf. Chapter 7).

The localized states near $\gamma_{SN}^\pm$ are temporally unstable. However, temporal stability changes at every saddle-node along the collapsed snaking branch. As a result, there is an infinite number of branch segments with stable localized states, located between the $(2i-1)$-th and $2i$-th saddle-nodes, $i \in \mathbb{N}$.

### 3.3.2 Two-limit defect-mediated snaking

Figure 3.3 shows the bifurcation diagram at $\nu = 7 > \nu^*$. A snaking branch $L_0$ of spatially localized states emerges from the lower saddle-node much as in regular homoclinic snaking discussed in §2.4.1 but this time the partner branch $L_\pi$ is absent. Instead the $L_0$ branch combines elements from both the $L_0$ and $L_\pi$ branches in Figure 2.18 and it does so via a distinct growth mechanism which we call defect-mediated snaking (DMS). When describing this mechanism we label the left and right limit values of the saddle-node bifurcations on either side of $L_0$ by $\gamma_1^{DMS}$ and $\gamma_2^{DMS}$, respectively, and continue to refer to $\gamma_1^{DMS} \leq \gamma \leq \gamma_2^{DMS}$ as a pinning or snaking interval.

The growth of the localized states along the DMS branch is illustrated in Fig. 3.4. This branch contains two distinct families of states. The first of these are the uniform amplitude segments, where the profiles consist of slugs of a uniform amplitude spatially periodic pattern connected to the $A^-$ background state by a pair of fronts, and therefore
Figure 3.2: (a) Bifurcation diagram corresponding to the $\nu = 5$ slice of Fig. 3.1, where the branch of localized states undergoes collapsed snaking towards $\gamma = \gamma_{CS} \approx 1.842$. The branch of localized states is computed on a domain of half-length $\ell/2 = 50$. Stable (unstable) segments are shown in solid (dashed) lines. For clarity, only the first three unstable segments are shown. On the homogeneous branches, solid (dotted) lines correspond to stable (unstable) solutions, and the labels indicate the spatial eigenvalue configurations. (b) A sample solution high up the $L_0$ branch.
Figure 3.3: (a) Bifurcation diagram corresponding to the $\nu = 7$ slice of Fig. 3.1, where the branch of localized states undergoes defect-mediated snaking between $\gamma_1^{DMS} \approx 2.8949$ and $\gamma_2^{DMS} \approx 2.8970$. Stability on the homogeneous branches is as in Fig. 3.2, but stability on $L_0$ is not indicated. (b) A sample solution high up the $L_0$ branch.

resemble the localized states found in regular homoclinic snaking. These segments extend from $\gamma_1^{DMS}$ to $\gamma_2^{DMS}$ as one moves up the DMS branch, and are shown in Fig. 3.4 using solid lines. Figure 3.4 also shows that the DMS branch alternates between two types of uniform amplitude segments: those where $V(x)$ has a minimum at $x = 0$ and those where $V(x)$ has a maximum at $x = 0$. The corresponding profiles are labeled by their spatial phase $\Phi$ (see below), $\Phi = 0$ for the former and $\Phi = \pi$ for the latter. The single DMS branch in Fig. 3.4 therefore combines elements of both the $L_0$ and $L_\pi$ branches in Fig. 2.18.

Accompanying the supercritical Turing bifurcation at $\gamma^T$ is a continuum of spatially periodic branches with wavenumbers $k$ in the neighborhood of $k_T$ that also bifurcate from $A^+$ into $\gamma < \gamma^T$. When both $\gamma$ and $k$ are viewed as bifurcation parameters, the continuum of fixed-$k$ branches forms a surface of spatially periodic states. We use the notation $A_p(x; \gamma, k)$ to refer to solutions on this surface. This surface of periodic states is shown in Fig. 3.5. The boundary of the surface is formed by the neutral stability curve of the $A^+$ state. Shading is used to indicate the configuration of Floquet multipliers at each $(\gamma, k)$. The patterns within the localized states always come from the Eckhaus-stable band, where the Floquet multipliers have configuration (I). In defect-mediated snaking, the curve of selected wavenumbers $k(\gamma)$ is ‘S’-shaped and spans the entire width of the Eckhaus band, terminating at the boundaries where the Floquet multipliers transition to configuration (II). Note that the selected wavenumber does not include the wavenumber $k_T$ selected at $\gamma^T$ because the DMS region is too far from $\gamma^T$ at this value of $\nu$.

The spatial dynamics explanation of the uniform amplitude segments of the DMS branch is similar to the previous explanation of regular homoclinic snaking. The localized
Figure 3.4: (a) Detail of the $L_0$ snaking branch in Fig. 3.3. Temporally stable (unstable) segments are shown as solid (dashed) lines and coincide with uniform (defect) segments. (b) Five sample profiles at $\gamma = 2.896$. The spatial phase $\Phi$ is indicated for each profile on a uniform amplitude segment.

Figure 3.5: Section of the surface of spatially periodic states for the parameters used in Fig. 3.3. The surface is bounded by the neutral stability curve of $A^+$. The configuration of the Floquet multipliers in each shaded region is indicated. The curve $C$ shows the wavenumber $k(\gamma)$ of the patterns included in defect-mediated snaking at this value of the parameters and spans the width of the Eckhaus band (inset).
states correspond to reversible homoclinic orbits which resemble heteroclinic cycles from $A^-$ to $A_P(x; \gamma, k)$ and back again. These are ultimately a consequence of the structurally stable intersection of the two-dimensional unstable manifold of the $A^-$ fixed point and the three-dimensional center-stable manifold of the periodic orbit (two center directions plus one strong stable direction), combined with the reversibility symmetry of the system. In the case of defect-mediated snaking, the extent in $\gamma$ of the heteroclinic connection is not determined by the first and last tangencies of these manifolds, but instead by a change in the dimensionality of one of the manifolds. In particular, the limits $\gamma_{1,2}^{DMS}$ lie on the Eckhaus boundary where the periodic orbit loses its one strong stable direction so the heteroclinic orbit from $A^-$ to $A_P(x; \gamma, k)$ must terminate.

The remaining segments in Fig. 3.4, plotted using dashed lines, are the defect segments. The profiles along these segments again correspond to reversible homoclinic orbits which resemble heteroclinic cycles to the $A^-$ state. But here each cycle can be decomposed into three pieces: a heteroclinic connection from $A^-$ to a reversible orbit $A_P(x; \gamma, k)$, a homoclinic connection from this orbit back to itself, and a heteroclinic connection from $A_P(x; \gamma, k)$ back to $A^-$, cf. [12]. The spatially periodic states $A_P(x; \gamma, k(\gamma))$ associated with the defect segments are identical to those associated with the uniform amplitude segments at the same $\gamma$. Thus the plot of $k(\gamma)$ in Fig. 3.5 also applies along the defect segments. Likewise, the heteroclinic connections between $A^-$ and $A_P(x; \gamma, k(\gamma))$ are the same along both the uniform amplitude and the defect segments (see Fig. 3.4). The only new feature along the defect segments is therefore the defect itself. The influence of this defect can be quantified by its spatial phase shift $\Phi$. To define $\Phi(\gamma)$ we fix the spatial phase of the periodic orbits $A_P(x; \gamma, k)$ so that $V_P(x; \gamma, k)$ has a local minimum at $x = 0$ and define $A_D(x; \gamma)$ to be the orbit with a defect centered at $x = 0$ which is homoclinic to $A_P(\cdot; \gamma, k(\gamma))$. Then $\Phi(\gamma)$ satisfies

$$
\lim_{x \to \infty} A_D(x; \gamma) = A_P \left( x - \frac{\Phi(\gamma)}{k(\gamma)} \cdot \gamma, k(\gamma) \right) .
$$

(3.14)

With this definition, a uniform profile with local minimum (maximum) in $V_P(x; \gamma, k)$ at $x = 0$ has phase $\Phi = 0$ ($\Phi = \pi$), and the total phase shift from $x \to -\infty$ to $x \to +\infty$ of a pattern with a defect relative to one without a defect is $2\Phi$. The phase $\Phi(\gamma)$ of the solution along two complete turns of the DMS branch is plotted in Fig. 3.6. The phase is constant along the uniform amplitude segments (either $\Phi = 0$ or $\Phi = \pi$) and increases by $\pi$ across each defect segment, indicating the insertion of one extra wavelength into the pattern. The variation in phase across the pinning region is slightly different for defect segments which increase $\Phi$ from 0 to $\pi$ and for those which increase $\Phi$ from $\pi$ to $2\pi$. We mention that the phase shown in the figure is determined empirically by examining the broad localized states found far up the defect-mediated snaking branch, rather than by computing the actual homoclinic orbit $A_D(\cdot; \gamma)$. The defect is typically only a few wavelengths wide so the limit in Eq. (3.14) converges rapidly and $\Phi(\gamma)$ is well approximated by our method.

Figure 3.7 shows in detail the variations of the profile along each type of defect segment. In each case, as $\gamma$ decreases across the pinning region the defect forces the central extremum to undergo “tip-splitting”, thereby squeezing in an extra half-wavelength on either side of $x = 0$ (i.e., one wavelength overall). As a result the localized state grows from
Figure 3.6: The phase $\Phi(\gamma)$ of the solution across two complete turns of the snaking branch in Fig. 3.4a. The labels mark the phases of the profiles shown in Fig. 3.4b.

The behavior along the defect segments can be understood from the perspective of spatial dynamics as follows. Each defect corresponds to a reversible homoclinic orbit to a periodic orbit. Within the Eckhaus band the center-stable manifold of a periodic orbit is three-dimensional. The symmetric section of the four-dimensional phase-space is two-dimensional. Hence, there is a continuous family of defects at each $\gamma$ parameterized by the wavenumber $k$ of the background periodic orbit. However, the wavenumber $k(\gamma)$ of the periodic orbits relevant in DMS is already selected by the front that connects it to $A^-$ as $|x| \to \infty$. Thus the defect structure varies continuously as $\gamma$ varies across the snaking region. In the Appendix we study the normal form for the supercritical Turing bifurcation at $\gamma = \gamma_T$ in order to understand the homoclinic-to-periodic solutions born there in greater detail. We conjecture that these homoclinics become involved in the defect-mediated snaking we observe farther from $\gamma = \gamma_T$ but have not studied this process in detail.

Within the snaking region, the uniform amplitude segments are temporally stable, while all the defect segments are temporally unstable. This is as expected since the uniform amplitude segments lie between the $(2i - 1)$-th and $2i$-th saddle-nodes, $i \in \mathbb{N}$, and these are precisely the segments that are stable in the collapsed snaking shown in Fig. 3.2.

### 3.3.3 Three-limit defect-mediated snaking

Figure 3.8 shows the solution behavior when $\nu = 36$. As in Fig. 3.3 the branch undergoes defect-mediated snaking, but this time the snaking region has three asymptotic limits, $\gamma_{1,2,3}^{DMS}$. The detailed growth mechanism of the localized states in this region is shown in Fig. 3.9 which shows one complete turn of the solution branch. As one follows the branch upwards, the uniform amplitude segments extend from $\gamma_1^{DMS}$ through $\gamma_3^{DMS}$ to $\gamma_2^{DMS}$ (profiles (i) and (ii)), and the defect segments extend from $\gamma_2^{DMS}$ through $\gamma_3^{DMS}$ to
Figure 3.7: Profiles from the defect segments across the snaking region of Fig. 3.4. The plot range is truncated to focus on the changes in $V(x)$ near the center of the domain at $x = 0$. The direction of decreasing $\gamma$ is indicated by an arrow, with the upper (lower) frames taken at the right (left) boundary of the snaking region.
Figure 3.8: (a) Bifurcation diagram corresponding to the $\nu = 36$ slice of Fig. 3.1, where the branch of localized states undergoes three-limit defect-mediated snaking. Stability on the homogeneous branches is as in Fig. 3.2, but stability on $L_0$ is not indicated. (b) A sample solution high up the $L_0$ branch.

As before, the defect segment is responsible for inserting an extra wavelength at the center of the localized state (compare (i) with (v)). Moreover, as in two-limit defect-mediated snaking, the limit points $\gamma_{1}^{DMS}$ and $\gamma_{2}^{DMS}$ define transition points between the uniform amplitude and defect segments. The additional snaking limit $\gamma_{3}^{DMS}$ is also associated with a qualitative change in the periodic state (compare the $U(x)$ profiles in (i) with (ii)). The details of this transition are shown in Fig. 3.10. The figure shows that this transition also occurs via “tip-splitting” but this time the tip-splitting occurs at $x = 0$ in $U(x)$ whereas the transition from profile (ii) to profile (v) takes place at $x = 0$ in $V(x)$ (Fig. 3.9). Neither transition is evident in the conjugate variable. Similar transitions take place on the next complete turn of the snaking curve, this time starting with state (v) and changing the minimum at $x = 0$ in $U(x)$ to a maximum when $\gamma \approx \gamma_{3}^{DMS}$, followed by a change of the maximum at $x = 0$ in $V(x)$ into a minimum along the corresponding defect segment. Figure 3.12 shows the resulting variation in the phase $\Phi(\gamma)$ of the solution across two complete turns of the snaking branch. Despite the additional fold $\gamma_{3}^{DMS}$, $\Phi$ again changes monotonically along the defect segments while remaining constant on the uniform amplitude segments.

The reason for the presence of the third limit point $\gamma_{3}^{DMS}$ can be explained with the assistance of Fig. 3.11. This figure shows a section of the surface of spatially periodic states. Near $(\gamma^T, k_T)$ the surface is smooth and single-valued as in Fig. 3.5. The limit points $\gamma_{1,2}^{DMS}$ lie on the boundaries of the Eckhaus-stable band, where the Floquet multipliers transition from configuration (I) to (II). As $\nu$ increases far above $\nu^*$ the curve of selected wavenumbers slides farther from the tip of the Eckhaus region at $(\gamma^T, k_T)$ and eventually encounters a
Figure 3.9: (a) One complete turn of the snaking branch $L_0$ in Fig. 3.8. Solid (dashed) lines show uniform (defect) segments of the branch. Temporal stability is not indicated. The limiting values for the saddle-node bifurcations are $\gamma_{DMS}^1 \approx 30.872$, $\gamma_{DMS}^2 \approx 30.695$ and $\gamma_{DMS}^3 \approx 30.285$. (b) Five sample solutions at $\gamma = 30.5$. The spatial phase $\Phi$ is indicated for profiles on the uniform amplitude segments. As $\Phi$ is defined in terms of $V(x)$, the change in the shape of $U(x)$ between (i) and (ii) does not change $\Phi$. 
section of the surface of spatially periodic states where this surface is folded. Figure 3.11 shows a section of this surface with two folds, one plotted as a solid line and the other as a dotted line. These folds meet in a cusp, marked by an open circle. The Floquet multipliers within each shaded region are indicated, although between the folds the Floquet multipliers are only indicated for the section of the surface with the largest $L^2$-norm, thus showing what one would see looking “down” on the surface from above. As for smaller values of $\nu$, the curve $k(\gamma)$ of wavenumbers included in DMS extends between the boundaries of the Eckhaus band, but in this case the curve must trace around both folds and hence include patterns from the upper, middle, and lower sections of the folded surface; the presence of the third limit point $\gamma_3^{DMS}$ in three-limit DMS is therefore a consequence of a global feature of the surface of spatially periodic states that is absent in the two-limit case. It follows that the spatial dynamics explanation of three-limit DMS is identical to the two-limit case. However, the temporal stability properties of the uniform amplitude segments are, inevitably, more complicated in the three-limit case than in the two-limit case where these states are stable, as discussed next.

### 3.4 Time evolution

In this section, we describe the temporal evolution of the different unstable states associated with both two-limit and three-limit snaking.

#### 3.4.1 The two-limit case: depinning

As discussed above, at $\nu = 7$ the localized states on the uniform amplitude segments in Fig. 3.4 are temporally stable for $\gamma_1^{DMS} < \gamma < \gamma_2^{DMS}$. The pair of fronts between the uniform and periodic states remains stationary in time, a phenomenon commonly re-
Figure 3.11: A section of the surface of spatially periodic states for the parameters used in Fig. 3.8. Refer to the text for notation.

Figure 3.12: The phase $\Phi(\gamma)$ of the solution across two complete turns of the snaking branch in Fig. 3.9a. The labels mark the phases of the profiles shown in Fig. 3.9b.
ferred to as pinning. However, outside the pinning region the fronts depin and begin to move. For $\gamma < \gamma_1^{DMS}$ Fig. 3.13a shows that the width of the localized state gradually decreases as the two fronts drift symmetrically inwards, ultimately leading to a final state consisting of a stable two-peak single-pulse state. In contrast, when $\gamma > \gamma_2^{DMS}$ the fronts drift apart allowing the structured state to invade the domain (Fig. 3.13b). This behavior is well-known from both variational systems such as the Swift-Hohenberg equation [42] where it admits a natural explanation based on the free energies of the uniform and periodic states, and a variety of nonvariational systems such as binary mixture convection [17, 24]. However, the evolution shown in Figs. 3.13a,b differs profoundly from that familiar from the Swift-Hohenberg equation and the convection systems cited above. In the latter the front represents a nucleation front, i.e., the front advances by nucleating new structures at a steady rate or retreats by annihilating structures also at a steady rate. Throughout the evolution the cells away from the front remain stationary and are “switched on or off” only by the passage of the front. This is because the eigenfunction responsible for the depinning of the front is strongly localized at the front or just outside it. In contrast, Fig. 3.13 reveals that in the present case the motion of the front is associated with a continuous drift of several cells near the location of the front without either nucleation of new cells in the front region or their destruction. Instead one finds that cells are progressively destroyed in an intermediate region between the front and the center of the structure, with the destruction zone gradually moving to the center once the structure becomes sufficiently narrow (Fig. 3.13a). Thus the behavior of the system takes the form of repeated phase slips generated by the Eckhaus instability [98], with the region where the phase slips take place drifting inward with approximately the speed of the front. This behavior is reversed in Fig. 3.13b where the fronts drift outward instead of inward. In this figure the phase slips occur initially in the center of the structure, at $x = 0$, but once the structure becomes broad enough the location of the phase slips splits into two locations which drift outwards more-or-less with the speed $v$ of the outer fronts.

Figure 3.13a shows that as the fronts move inward the pattern is more-or-less uniformly compressed resulting in an average wavenumber $k$ that increases with time. This progressive change in wavenumber triggers Eckhaus instability that is responsible for the elimination of a pair of rolls (one wavelength). After each phase slip the pattern relaxes again into a more-or-less uniform wavenumber state, but the continued motion of the fronts compresses the rolls again and triggers another phase slip. Evidently the speed of the fronts, $v$, is related to the interval $T$ between successive phase slips by the relation $vT \approx 2\pi/k$, where $k \approx k(\gamma_1^{DMS})$. Since $v$ varies as $|\gamma - \gamma_1^{DMS}|^{1/2}$ the phase slips occur with frequency $T^{-1} \approx |\gamma - \gamma_1^{DMS}|^{1/2}$. This relation is verified in Fig. 3.14a. A similar relation, with $\gamma_1$ replaced by $\gamma_2$, holds for fronts that are moving apart (Fig. 3.14b). The nonzero intercepts on the horizontal axis are most likely due to the finite spatial extent of the wave train. A detailed study of this depinning process is presented in Chapter 4.

We have also examined the time evolution starting from the unstable defect branches. Here we find that the defect heals and the solution either grows into a state with an extra half wavelength or it contracts into a state that is shorter by half a wavelength (on either side). The outcome appears to depend on whether the defect state is “closer” to the longer or the shorter uniform amplitude state, or equivalently whether $\gamma$ lies in the right half of
Figure 3.13: Space-time plot of $V(x,t)$ for $\nu = 7$ showing the depinning of the fronts bounding a localized state at (a) $10^{-3}$ to the left of $\gamma_{DMS}^1$ and (b) $10^{-4}$ to the right of $\gamma_{DMS}^2$. The final state in (a) is a steady localized state consisting of two wavelengths; in (b) the expanding state eventually fills the whole domain.

Figure 3.14: Plots of $1/T^2$, where $T$ is the time between the first two phase slips, as a function of the distance $|d\gamma|$ from the saddle-node for (a) $\gamma < \gamma_{DMS}^1$ and (b) $\gamma > \gamma_{DMS}^2$. The initial conditions are taken at the 69-th and 70-th saddle-nodes, respectively, and the $|d\gamma|$'s are measured relative to the location of the 67-th and 72-nd saddle-nodes. The solid lines show the best fit lines (a) $T^{-2} = 4.533 \times 10^{-4}|d\gamma| - 5.612 \times 10^{-10}$, and (b) $T^{-2} = 6.766 \times 10^{-4}|d\gamma| - 3.606 \times 10^{-10}$. Parameters: $\alpha = -1.5$, $\beta = 6$, $\mu = -1$, $\nu = 7$. 
the pinning region or the left half, although it depends sensitively on the amplitude of the initial perturbation as well.

### 3.4.2 The three-limit case

When $\nu = 36$ there are three additional critical values of $\gamma$ denoted by $\gamma_{A,B,C}$ separating the behaviors triggered by small amplitude perturbations of the localized states on a uniform amplitude segment. Figure 3.15a indicates that for $\gamma_{DMS}^2 < \gamma < \gamma_{DMS}^1$ (downward-pointing triangles), the fronts drift outward much like Fig. 3.13b and the domain is ultimately filled with a periodic state. This spatially periodic end state is time-independent and stable for $\gamma_{DMS}^2 < \gamma < \gamma_A$ (filled triangles) but oscillates in time for $\gamma_A < \gamma < \gamma_{DMS}^1$ (unfilled triangles). In the interval $\gamma_{DMS}^3 < \gamma < \gamma_{DMS}^2$ the branch of localized states has two parts, an upper part above the saddle-node at $\gamma_{DMS}^3$ and a lower part below it. In this interval the localized states on the upper branch evolve rapidly into those on the lower branch (e.g., Fig. 3.15b). Consequently, in the following we focus only on the time evolution of localized states on the lower branch. For these states we find that the fronts remain pinned when $\gamma_C < \gamma < \gamma_{DMS}^2$ (circles) but depin and move inward when $\gamma_{DMS}^3 < \gamma < \gamma_C$ (upward-pointing triangles). Simulations show that in the former case the localized state is either time-independent and stable ($\gamma_B < \gamma < \gamma_{DMS}^2$, filled circles) or that it oscillates in time ($\gamma_C < \gamma < \gamma_B$, unfilled circles). An example of such an oscillatory localized structure is shown in Fig. 3.16. This oscillatory state, which may have arbitrarily large spatial extent, will be called an extended breather and resembles breathers identified in other forced dissipative systems [74]. In the case $\gamma_{DMS}^3 < \gamma < \gamma_C$ both the inner wave train and the fronts are unstable to oscillations and these oscillations lead to depinning and ultimate collapse of the structure (unfilled triangles, e.g., Fig. 3.17). This collapsing state, with occasional interludes of meta-stability, will be called a Hanoi tower.

We note that the pinning region, $\gamma_C < \gamma < \gamma_{DMS}^2$, is narrower than the snaking region containing the localized states. A similar reduction in the interval of stable localized states is observed in the Swift-Hohenberg equation when two-dimensional perturbations of localized stripes are admitted [38]. The reason is simple: depending on parameters the two-dimensional perturbations may lead to the depinning of the fronts that would otherwise remain pinned, just as here the temporal oscillation of the fronts also leads to depinning. In both cases the result is a reduced interval of parameter values in which the fronts remain pinned, and hence a reduced parameter interval containing stable spatially localized states.

#### Extended breather

As already mentioned, localized states like (ii) in Fig. 3.15a are subject to an oscillatory instability, which is absent from variational systems. In this case, the only linearly unstable modes are a pair of odd parity Hopf modes $A_H(x)$ and $\dot{A}_H(x)$ with growth rates $\sigma \pm i\omega$ ($\sigma > 0, \omega > 0$). The amplitude of these modes peaks in the center of the localized state and decreases towards the fronts on either side indicating that the oscillatory mode is a body mode. Decaying oscillations of this type are also visible in Fig. 3.15b, between $t \approx 2$ and $t \approx 5$. Figure 3.16a shows the solution profiles at maximum amplitude, half an oscillation period apart, once the oscillation is fully developed. As expected, the profiles
Figure 3.15: (a) A branch of uniform amplitude localized states when $\nu = 36$ with different symbols indicating different final states reached from small amplitude perturbations (see text). The solution profiles at locations (a)–(f) straddling the fold at $\gamma^{DMS}_3$ are shown in Fig. 3.10. (b) Space-time plot of $V(x, t)$ showing the fast transition from the localized state (i) on the upper branch to the corresponding state on the lower branch.

are related to one another by the spatial reflection $x \rightarrow -x$.

**Hanoi tower**

As shown in Fig. 3.17, localized states like (iii) in Fig. 3.15a are subject to both oscillatory and depinning instabilities. The linearly unstable modes with the largest growth rates come in several complex conjugate pairs. The most unstable mode is the odd parity Hopf mode already encountered in the extended breather. The second and third most unstable modes are also body modes, this time an even parity Hopf mode and a second odd parity Hopf mode. The fourth most unstable mode is an oscillatory wall mode with support around the fronts. This pair may be either even or odd, and is directly responsible for the collapse of the structure shown in Fig. 3.17, i.e., this is the mode that is responsible for the depinning of the fronts.

The presence of multiple instabilities often leads to complicated temporal dynamics. Indeed, we find that the shape of the Hanoi tower depends sensitively on $\gamma$ as well as on the initial condition. An example can be seen in Fig. 3.17, where small amplitude but asymmetric noise in the initial condition is amplified by the growing instability leading to a considerable displacement of the final stable pulse from the center at $x = 0$.

### 3.5 Discussion

In this chapter we have identified a new mechanism for the growth of spatially localized structures as these structures are followed in parameter space. In contrast to the behavior familiar from existing studies of the Swift-Hohenberg equation and of localized
Figure 3.16: Time evolution of the localized state (ii) in Fig. 3.15a. (a) Snapshots of the solution profile at two instants half an oscillation period apart ($t_1 - t_0 \approx 0.3$), corresponding to maxima of the oscillation amplitude. (b) The time series $V(x_0, t)$ at fixed $x_0 \approx 6.05$.

Figure 3.17: Space-time plot showing the evolution of $V(x, t)$ corresponding to the localized state (iii) in Fig. 3.15a into a Hanoi tower. The final state is a stable spot at $x \neq 0$. 
states in different types of convection in which the structures grow by nucleating new cells at the location of the bounding fronts, the states we have found in the forced complex Ginzburg-Landau equation grow by the fission of the cell in the center of the structure, followed by outward displacement of all existing cells. We have called this mechanism defect-mediated snaking to contrast it with regular homoclinic snaking described in §2.4.1, and think of the mechanism as a steady-state analog of the familiar target in which a wave is emitted from a central pacemaker symmetrically in both directions.

We have found DMS in the FCGLE with both 2:1 and 1:1 resonance, although only the latter example is presented here. In the 2:1 case this mechanism appears to be present only in the self-exciting case ($\mu > 0$) and hence the localized states created by this mechanism are unstable. However, owing to the “reciprocity” between localized states asymptotic to a homogeneous background state and holes in a periodic state [59, 108] we conjecture that stable reciprocal versions of these states may be present in the 2:1 DMS region. In contrast, with 1:1 resonance localized states are present for $\mu < 0$ and hence are stable. We have identified a DMS region in this equation and found that unstable defect states connect stable, defect-free portions of the branch (Fig. 3.4).

We have observed differences in the motion of the bounding fronts outside the pinning region as well. In the regular scenario described in §2.4.2, the fronts depin above the pinning region and propagate outwards as a result of sequential nucleation of cells at the location of the fronts. Thus the passage of the front replaces the homogeneous state by a cellular state, but no motion of existing cells takes place. In contrast, in the 1:1 resonance case we have seen a different mechanism. The fronts still drift outwards with a speed proportional to the square root of the distance from the upper boundary of the DMS region, as in the regular case, but this time no nucleation of new cells at the location of the fronts takes place. Instead, the bounding cells are continually pushed outwards as phase slips repeatedly insert new roll pairs either symmetrically between the outer front and the center of the localized structure, or in the center. We have conjectured that these phase slips are triggered by the Eckhaus instability as the dilating structure is stretched by the moving fronts and its wavenumber pushed outside the Eckhaus-stable band. The reverse process occurs below the DMS region.

We leave a number of questions to future work. Among these we list the determination of the Eckhaus stability boundary for localized structures, the prediction of the speed of the fronts and the confirmation of its conjectured relation to the frequency of the observed phase slips. The transition from collapsed snaking to defect-mediated snaking remains incompletely understood. In addition, since the FCGLE reduces to the real Swift-Hohenberg equation in appropriate regimes [28] DMS may turn into regular homoclinic snaking as parameters are varied. How this might happen is also unclear.

### 3.6 Appendix: The supercritical Turing bifurcation

The Turing bifurcation at $\gamma = \gamma^T$ produces spatially periodic states with wavenumber $k_T$. Near this point we can write $\gamma = \gamma^T - \epsilon^2 \mu_2$, where $\epsilon \ll 1$ and $\mu_2 \sim O(1)$. With $A \equiv U + iV$, a multiscale expansion involving both the short spatial scale $x$ and the long
Thus

\[ \mathbf{U} = \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} + \epsilon \begin{bmatrix} \xi \\ 1 \end{bmatrix} \left\{ A e^{ik_T x} + \overline{A} e^{-ik_T x} \right\} + O(\epsilon^2). \]

Here the first term represents the equilibrium \( A^+ \). In the second term \( A \) is a complex function of \( X \) that satisfies the Ginzburg-Landau equation with real coefficients

\[ A_{XX} = -q_1 \mu_2 A + q_2 |A|^2 A + O(\epsilon) \]

(3.15)

The coefficients in this equation are computed in Ref. [41] for the 2:1 resonance, and in Ref. [130] for the 1:1 resonance. These calculations show that \( q_1 > 0, q_2 > 0 \). Consequently the periodic states bifurcate supercritically, i.e., towards smaller \( \gamma (\mu_2 > 0) \).

The Turing bifurcation at \( \gamma = \gamma^T \) corresponds to a reversible Hopf bifurcation with 1:1 resonance in space. Existing analysis of the corresponding normal form sheds light on the origin of homoclinic orbits to periodic wavetrains that are of interest here in connection with the defect states that play such a prominent role in defect-mediated snaking. The normal form for this bifurcation is [100]

\[ \begin{align*}
A' &= ik_T A + B + iA P(\mu; y, w) \\
B' &= ik_T B + iB P(\mu; y, w) + A Q(\mu; y, w),
\end{align*} \]

(3.16)

where \( y \equiv |A|^2, w \equiv \frac{1}{2}(\overline{A} \overline{B} - \overline{\overline{A}B}), \) the overbar refers to complex conjugation, and in the context of spatial dynamics the prime denotes differentiation with respect to \( x \); \( \mu \) is an unfolding parameter analogous to \( \gamma^T - \gamma \). The functions \( A \) and \( B \) transform under spatial reflection as \( (A, B) \rightarrow (\overline{A}, -\overline{B}) \), and \( P \) and \( Q \) are polynomials with real coefficients:

\[ P(\mu; y, w) = p_1 \mu + p_2 y + p_3 w + \ldots, \quad Q(\mu; y, w) = -q_1 \mu + q_2 y + q_3 w + \ldots. \]

(3.17)

The 1:1 Hopf bifurcation from the trivial state \( A = B = 0 \) occurs at \( \mu = 0 \); by convention \( q_1 > 0 \) so that this state is hyperbolic in the region \( \mu < 0 \) and elliptic in \( \mu > 0 \). As shown in [100] the normal form (3.16) is an integrable two degree of freedom Hamiltonian system. This is true regardless of the system from which it is derived, and in particular it applies to systems that are not variational in time or Hamiltonian in space. Analysis shows that small amplitude orbits homoclinic to the trivial state are present for \( \mu < 0, |\mu| \ll 1 \), provided \( q_2 < 0 \) (the subcritical case). Here we consider the case \( q_2 > 0 \) (the supercritical case).

We write \( (A, B) = (\epsilon A(X), \epsilon^2 B(X)) e^{ik_T x} \), where \( X \equiv \epsilon x \), and write \( \mu = \epsilon^2 \mu_2 \). Thus \( P \) and \( Q \) in Eq. (3.17) become

\[ P = \epsilon^2 (p_1 \mu_2 + p_2 |A|^2) + O(\epsilon^3), \quad Q = \epsilon^2 (q_1 \mu_2 |A|^2) + O(\epsilon^3). \]

(3.18)

It follows from Eq. (3.16) that \( B = A' + O(\epsilon) \) and hence that Eq. (3.16b) reduces to Eq. (3.15).

Equation (3.15) has the constant solution \(|A|^2 = q_1 \mu_2 / q_2 > 0\) corresponding to a Turing state with wavenumber \( k_T \). The equation also possesses a front solution connecting two out-of-phase states with wavenumber \( k_T \). To find this solution we write \( A(X) = R(X) \exp i\phi \), where \( \phi \) is a constant, and integrate Eq. (3.15) once, obtaining

\[ \frac{1}{2} R'^2 + V_0(R) = E. \]

(3.19)
Here $V_0 = \frac{1}{4} q_1 \mu_2 R^2 - \frac{1}{4} q_2 R^4$ and $E$ is a constant of integration. The front corresponds to a heteroclinic orbit between $A = \pm R_0 \exp i \phi$, where $R_0^2 \equiv q_1 \mu_2 / q_2 > 0$ and is present when $E = \frac{1}{4} q_1 \mu_2^2 / q_2^2$. Thus $R(X)$ satisfies

$$\frac{1}{2} R'^2 = \frac{1}{4} q_2 (R_0^2 - R^2)^2$$

(3.20)

and hence

$$R(X) = R_0 \tanh(\sqrt{\frac{q_1 \mu_2}{2}} X).$$

(3.21)

This solution asymptotes to the uniform wavetrains $\pm R_0$ as $|X| \to \infty$, i.e., it describes a $\pi$ change in the spatial phase of the wavetrain between $X = -\infty$ and $X = \infty$.

We now suppose that the spatial phase $\phi$ is not constant. In this case Eq. (3.19) becomes

$$\frac{1}{2} R'^2 + V_L(R) = E,$$

(3.22)

where

$$V_L \equiv \frac{1}{2} q_1 \mu_2 R^2 - \frac{1}{4} q_2 R^4 + \frac{L^2}{2R^2}$$

(3.23)

and $R^2 \phi' = L$. The solutions are thus specified by the two constants $E$ and $L$, hereafter the energy and angular momentum, respectively.

In this case we define $R_\pm$ as the (larger) solution of $q_2 R^6 - q_1 \mu_2 R^4 + L^2 = 0$ and choose $E = q_1 \mu_2 R_\pm^2 - \frac{3}{4} q_2 R_\pm^4$. In this case $R(X)$ satisfies the equation

$$\frac{1}{2} R'^2 = \frac{q_2}{4R^2} (R_\pm^2 - R^2)^2 [R^2 - 2(R_0^2 - R_\pm^2)],$$

(3.24)

where $R_0^2 - R_\pm^2 > 0$. This equation can also be solved in closed form:

$$R^2 = 2(R_0^2 - R_+^2) + (3R_+^2 - 2R_0^2) \tanh^2 \sqrt{\frac{q_2}{2} (3R_+^2 - 2R_0^2)} X.$$

(3.25)

As $|X| \to \infty$ this solution approaches $R_\pm^2$. We think of $\phi_X \equiv k(X)$ as the (local) wavenumber of the solution. Thus $k(X) \to L/R_\pm^2$, as $|X| \to \infty$.

In addition to the homoclinic connection to periodic states with the wavenumber $L/R_\pm^2$, there are also solutions in the form of constant amplitude wavetrains. For these solutions $k$ is a constant, $k = k_\pm$, and the amplitude satisfies

$$R^2 = R_0^2 - \frac{k_\pm^2}{q_2} \equiv R_\pm^2.$$

(3.26)

Since for these solutions $k_\pm = L/R_\pm^2$ it follows that $L^2 = q_2 R_\pm^4 (R_0^2 - R_\pm^2)$. Thus $R_0^2 > R_\pm^2$ and moreover $R_0^2(R_\pm^2 + R_+^2) = R_\pm^4 + R_+^4 R_\pm^2 + R_\pm^4$. It follows that the two roots degenerate into the same root $R_\pm^2 = R_\pm^2 \equiv R_+^2$, where

$$R_+^2 = \frac{2}{3} R_0^2.$$

(3.27)
This occurs at a critical value of the constants \( E \) and \( L \) given by

\[
E_c = \frac{1}{3} q_2 R_0^4, \quad L_c^2 = \frac{4}{27} q_2 R_0^6 \tag{3.28}
\]

and corresponds to a critical wavenumber \( k_c \) given by

\[
k_c^2 = \frac{1}{3} q_2 R_0^2. \tag{3.29}
\]

The value \( L = L_c \) represents the maximum permitted value of \( L \). At this value the potential \( V_L(R) \) has an inflection point; for \( L > L_c \) the potential is a monotonic function of \( R \) and there are no equilibria or bound orbits.

We now examine the connection between the above analysis which follows closely Ref. [100] and the standard discussion of the Eckhaus or modulational instability. In the classical picture of this instability [98] we look for solutions of

\[
A_t = q_1 \mu_2 A - q_2 |A|^2 A + A_{XX} \tag{3.30}
\]

of the form \( A = R \exp \imath kX \). Steady solutions of this form satisfy \( R^2 = R_0^2 - \frac{k^2}{q_2} \). These states are marginally unstable with respect to long wavelength perturbations along the line \( R_0^2 = \frac{3k^2}{q_2} \) within the existence region \( k^2 \leq q_2 R_0^2 \). This line is the threshold for the Eckhaus instability which occurs in the region \( \frac{1}{3} q_2 R_0^2 \leq k^2 \leq q_2 R_0^2 \). Thus the amplitude of the wavetrain that becomes Eckhaus unstable is given by \( R^2 = \frac{2}{3} R_0^2 \) and its wavenumber is given by \( k^2 = \frac{1}{3} q_2 R_0^2 \), i.e., the classical Eckhaus instability corresponds precisely to the point \((E, L) = (E_c, L_c)\), as noted already by Iooss and Pérouème [100]. In other words, the classical Eckhaus instability corresponds to the formation of a degenerate homoclinic orbit within the spatial dynamics approach. However, the spatial dynamics approach predicts a variety of additional quasiperiodic states corresponding to nonlinear oscillations about \( R = R_- \). These can be written down explicitly in terms of elliptic functions and exist for \( E_-(L) < E \leq E_+(L) \), where \( E_\pm(L) \) are the two roots of the equation

\[
x(4x^2 - 3x - 6y) + y(y + 4) = 0 \tag{3.31}
\]

and \( x \equiv E/E_c < 1, \ y \equiv L^2/L_c^2 < 1 \). Thus

\[
y = 3x - 2 \pm 2(1 - x)^{3/2}. \tag{3.32}
\]

These conditions define the region computed in Ref. [100] (Fig. 3.18). We do not compute these states explicitly except to note that both the amplitude and wavenumber of these states oscillate periodically in space.

Some of the solutions discussed in this Appendix are also encountered in §7 of [154].
Figure 3.18: The condition (3.31) in the \((E/E_c, L/L_c)\) plane. The red dots indicate the Eckhaus points. Quasiperiodic solutions exist in the shaded region, consistent with [100].
Chapter 4

Depinning

4.1 Introduction

In recent years, there has been growing interest in spatially localized structures (LS) in forced dissipative systems described by partial differential equations in one or more spatial dimensions. Such states include spot-like structures found in reaction-diffusion systems [60], as well as states referred to as convectons that arise in various types of convection [17, 24, 29]. Related structures are present in subcritical shear flows [156]. The LS in these systems are typically time-independent although they can undergo both spontaneous and forced translation as explored recently in models based on the Swift-Hohenberg equation [37, 97]. In other systems, such as Faraday waves, the LS oscillate in time, either periodically or with a more complex time-dependence forming structures referred to as oscillons [121, 147]. This is also the case for optical oscillons [4]. In many of these systems the use of envelope equations removes the (fast) time-dependence and maps such time-dependent structures onto equilibria of the envelope equations.

The structures mentioned above are all examples of “dissipative solitons” in which energy loss through dissipation is balanced by energy input through spatially homogeneous forcing, and are located in regions of parameter space in which a spatially homogeneous state A coexists with a second state B, either a homogeneous state or a spatially periodic state. In the former case the LS consists of an inclusion of a homogeneous state B in a background of state A (Type-I LS); in the latter, the LS consists of an inclusion of a periodic state B in a background of state A (Type-II LS). Extended LS of these types can be viewed as bound states of fronts between the states A and B. Extended Type-I LS are of codimension one and so exist only at isolated parameter values. In contrast, extended Type-II LS are of codimension zero and so are found within an interval of parameter values. This fact may be ascribed to the phenomenon of self-pinning, whereby the fronts pin to the spatial oscillations between them [144]. Outside of the resulting pinning region the fronts depin, allowing either state A or the periodic state B to invade the domain.

These phenomena are simplest to understand physically in systems with gradient structure for which a free energy may be derived. This is the case, for example, for the Swift-Hohenberg equation [42, 38]. In this chapter, we seek to understand the process of depinning when gradient structure is absent, focusing on Eq. (1.11), the 1:1 forced complex
Ginzburg-Landau equation (FCGLE) for a complex scalar field \( A(x,t) \) [57, 129].

In the following we shall find it useful to rewrite Eq. (1.11) in terms of \( U \) and \( V \), the real and imaginary parts of \( A = U + iV \):

\[
\begin{bmatrix} U_t \\ V_t \end{bmatrix} = \left( \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix} + \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix} \partial_{xx} - (U^2 + V^2) \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix} \right) \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} \gamma \\ 0 \end{bmatrix}. \tag{4.1}
\]

In what follows, we will use these two representations interchangeably.

The analysis of temporal stability for the homogeneous equilibria leads to Eq. (3.7), which contains the following two categories of instabilities. The first has critical wavenumber \( k = 0 \) and corresponds to saddle-node and Hopf bifurcations in the temporal ODE obtained by setting \( A_{xx} = 0 \) in (1.11). The second has critical wavenumber \( k \neq 0 \) and is known as the Turing instability. The spatial stability of the homogeneous equilibria has been reviewed in §2.3.1 and computed in §3.2. The spatial eigenvalue configuration has consequences for temporal stability. In the absence of self-excitation (\( \mu < 0 \)) the hyperbolic equilibria (configurations (1) and (2)) are temporally stable, while the elliptic equilibria (configurations (3) and (4)) are Turing-unstable [130].

We use the term “periodic states” to refer to steady solutions of (1.11) which are periodic in \( x \) and respect reversibility (i.e., solutions invariant under \( x \rightarrow -x \)). The spatial stability of the periodic states has been reviewed in §2.3.1 and is likewise related to temporal stability. A hyperbolic periodic state (configuration (I)) is temporally stable, while the elliptic state (configuration (II)) is not. In an unstable wave train formed by a periodic state in configuration (II), the periodic state rapidly readjusts its wavelength into configuration (I) through phase slip(s). This phenomenon is known as the Eckhaus instability [98].

This chapter is organized as follows. §4.2 focuses on the process of depinning near the boundaries of the pinning region containing Type-I LS, while §4.3 tackles the more challenging case of Type-II LS. In both cases we seek to understand quantitatively how the speed \( c \) of the depinned front depends on the distance from the boundary of the pinning region and the details of the nucleation process leading to front advance. We conclude with some open questions in §4.4.

### 4.2 Depinning of Type-I LS

In this section, we fix \( \nu = 5 \) as in §3.3.1. The collapsing point is \( \gamma^{CS} = 1.8419 \), the bifurcation points for equilibria are \( \gamma^{+SN} = 1.5694 \), \( \gamma^T = 1.7501 \) and \( \gamma^{-SN} = 1.8958 \), and we define \( d\gamma = \gamma - \gamma^{CS} \). We study the solution behavior in the bistable region \( \gamma^{+SN} < \gamma < \gamma^{-SN} \), where a Type-I LS consists of a plateau of the upper equilibrium \( A^+ \) embedded in a background of the lower equilibrium \( A^- \).

#### 4.2.1 Time evolution

To study depinning, we take a Type-I LS at \( \gamma^{CS} \) as the initial condition and solve Eq. (1.11) at \( \gamma \neq \gamma^{CS} \) using the ETD2 scheme (cf. §1.2.3). As shown in Figures 4.1(a) and 4.1(b), the plateau of \( A^+ \) expands for \( \gamma > \gamma^{CS} \) and shrinks for \( \gamma < \gamma^{CS} \). In the former
Figure 4.1: Space-time plots of \( V(x,t) \) showing Type-I LS depinning at (a) \( d\gamma = 0.04 \); (b) \( d\gamma = -0.04 \); (c) \( d\gamma = -0.24 \).

case, the domain is eventually filled with \( A^+ \). In the latter case, the final state is either a stable pulse with background \( A^- \) (if it exists) or \( A^- \) everywhere (otherwise).

The region \( \gamma < \gamma^T \) is particularly interesting. In this case, \( A^+ \) is Turing-unstable and tends to evolve into a periodic state. If this happens before the depinning ends, the subsequent evolution will be altered. Indeed, as shown in Figure 4.1(c), the periodic state may evolve more rapidly into \( A^- \) thereby shortening the overall depinning process.

The speed \( c \) of the left front is plotted as a function of \( \gamma \) in Figure 4.2 (open circles); \( c \) depends smoothly on \( d\gamma \) with \( c \sim d\gamma \) as \( d\gamma \to 0 \). Thus \( c > 0 \) corresponds to a shrinking LS while \( c < 0 \) corresponds to an expanding LS.
Figure 4.2: Plot of the front speed $c$ as a function of $\gamma$ at $\nu = 5$. Open circles: DNS of (1.11). Solid line: continuation of heteroclinic orbits in (4.2). Vertical dashed lines: SN bifurcations (left/right: $\gamma^{\text{SN}}_{\pm}$). Horizontal dot-dashed line: RB bifurcations on $A^\mp$. The insets show the eigenvalue configuration of $A^\mp$ in (4.2) for different signs of $c$.

4.2.2 Theoretical explanation

The above results can be understood in terms of front motion. Each front of a broad LS is well approximated by a heteroclinic connection between $A^-$ and $A^+$, i.e., by a solution of the spatial dynamics problem on the real line. A moving front can likewise be approximated by the corresponding solution on the real line. It follows that a left front moving with speed $c$ solves the equation

$$(1 + i\alpha)A\xi + cA\xi + \gamma + (\mu + i\nu)A - (1 + i\beta)|A|^2A = 0,$$

where $\xi \equiv x - ct$, subject to the boundary conditions $A(\pm\infty) = A^\pm$, and similarly for a right front. Such moving fronts, or equivalently heteroclinic orbits $H$ between $A^\pm$ can then be superposed back to back to approximate the expanding or shrinking states shown in Figure 4.1.

The states $A^\pm$ represent equilibria of the 4-D ODE (4.2). If both have two stable and two unstable (spatial) eigenvalues, $H$ is of codimension-1. We may follow $H$ numerically with AUTO by imposing Neumann boundary conditions on a domain of length $\ell \gg 1$ and an integral constraint that factors out the translational invariance with respect to $\xi$. The result, shown in the $(\gamma, c)$-plane in Figure 4.2 (solid line), is a smooth curve that passes through the data points from direct numerical simulation (DNS).

Within Eq. (4.2), viewed as a dynamical system in $\xi$, an equilibrium may exhibit two types of bifurcations, the saddle-node (SN) bifurcation at $\gamma = \gamma^{\text{SN}}$ already encountered, and a bifurcation we refer to as a reversibility-breaking (RB) bifurcation that occurs when the equilibrium is type (3) or type (4) and $c$ passes through zero. In the type (3) case the RB bifurcation is equivalent to a (spatial) Hopf bifurcation, while in the type (4) case the eigenvalue movement shown in Figure 4.2 corresponds to a 1:1 splitting bifurcation in the Hamiltonian context [114]. The condition for the eigenvalues to leave the imaginary
axis in opposite directions (as in Figure 4.2) rather than in the same direction is derived in Appendix 4.5. From the loci of the SN and RB bifurcations indicated in Figure 4.2, we observe that the locus of $H$ straddles the loci of SN± and never intersects the locus of RB. Hence along the locus of $H$, neither $A^+$ nor $A^-$ bifurcates and $H$ exists in the entire bistable region.

As reviewed in §1.2.3, the spectrum of the front between $A^\pm$, namely its temporal eigenvalues in the comoving frame, inherits the essential spectrum of $A^+$. The latter is obtained from Eq. (1.11) by going into the comoving frame $(\xi, \tau) = (x - ct, t)$, writing $A = R \exp i\phi (1 + a(\xi, \tau))$, linearizing in $a(\xi, \tau)$ and looking for solutions of the form $a(\xi, \tau) = a_+ \exp (ik\xi + s\tau) + \bar{a}_- \exp (-ik\xi + \bar{s}\tau)$. The set of $s$ admitted by this dispersion relation, defined as the essential spectrum $\Sigma_{\text{ess}}$, can be calculated to be $\Sigma_{\text{ess}}(c = 0) + ikc$ where $\Sigma_{\text{ess}}(c = 0) = \{s : s \leq s_{\text{max}} \in \mathbb{R}\}$ with $s_{\text{max}} > 0$ due to the Turing instability. Thus the Turing instability ($0 < s \in \mathbb{R}$) for $c = 0$ manifests itself as a convective instability ($\Re(s) > 0, \Im(s) \neq 0$) for $c \neq 0$.

### 4.3 Depinning of Type-II LS

Before studying the depinning of Type-II LS, we briefly recall the wavenumber selection process associated with defect-mediated snaking. As in conventional snaking [42] the fronts at either end of the LS select a unique wavenumber $k_{\text{DMS}}(\gamma)$ of the periodic state between them. This wavenumber lies on a curve $C$ in the $(\gamma, k)$ plane, shown in Figure 3.5, and is the same on both the uniform and defect segments of the $L_0$ branch (solid blue). The figure also shows the family of periodic states parameterized by $(\gamma, k)$ and bifurcating from $(\gamma_T, k_T)$. The upper and lower boundaries of the region of existence of this family of periodic states, hereafter $k^\pm$ (solid red, Figure 3.5), represent the two marginally stable wavenumbers associated with the upper equilibrium $A^+(\gamma)$. The upper (lower) boundary for the Eckhaus stable region will be called $k_{E}^+(\gamma)$ ($k_{E}^-(\gamma)$) (dashed black, Figure 3.5). A key feature of DMS is that the snaking limits $\gamma_{DMS}^{1,2}$ lie on these Eckhaus boundaries. Interpreted within the framework of spatial dynamics, these are precisely the places, where the periodic state loses one strong stable direction so that the heteroclinic orbit from $A^-$ to the periodic state must terminate.

In this section, we fix $\nu = 7$ as in §3.3.2. The snaking limits are $\gamma_{DMS}^{1} = 2.8949$ and $\gamma_{DMS}^{2} = 2.8970$, the bifurcation points for equilibria are $\gamma_{SN}^{+} = 2.2197$ and $\gamma_{SN}^{-} = 3.0463$ ($\gamma_T$ lies outside the bistable region), and we define $d\gamma = \gamma - \gamma_{DMS}^{1}$ for $\gamma < \gamma_{DMS}^{1}$ and $d\gamma = \gamma - \gamma_{DMS}^{2}$ for $\gamma > \gamma_{DMS}^{2}$.

In contrast to SH23 (2.64), in the 1:1 FCGL equation depinning of Type-II LS occurs through successive phase slips as shown in Figure 3.13. At each phase slip a Type-II LS gains or loses either one or two roll(s). These two ways of depinning are respectively referred to as slow depinning and fast depinning. The distance by which either front drifts between successive phase slips is denoted by $\Lambda$. The time between successive phase slips, defined as the depinning period $T$, and (the time average of) the front speed $c$ are then related by $c = \Lambda/T$. In each depinning period $T$, the size of either half-domain changes by $\Lambda$ while the number of rolls therein changes by 1 for fast depinning and 1/2 for slow depinning. Thus $\Lambda$ (2$\Lambda$) can be regarded as the wavelength left behind by either bounding
front for fast (slow) depinning.

4.3.1 Time evolution

To study depinning for $\gamma < \gamma_{DMS}^1$ ($\gamma > \gamma_{DMS}^2$), we take a Type-II LS at $\gamma_{DMS}^1$ ($\gamma_{DMS}^2$) as the initial condition and solve Eq. (1.11). In the former case, as shown in Figure 3.13(a), the periodic pattern shrinks and the final state is either a stable Type-II LS with fewer rolls (if it exists) or everywhere $A^-$ (otherwise). In the latter case, as shown in Figure 3.13(b), the periodic pattern expands and eventually fills the domain (with appropriate wavelength adjustment in the final stages).

Figure 3.13 also shows that for small $|d\gamma|$ the depinning period $T$ depends both on $d\gamma$ and the index $N$ of the nearest saddle-node. (For a left/right saddle-node ($\gamma_{DMS}^1$/$\gamma_{DMS}^2$), $N$ is odd/even). This is due in part to the following difference between slow and fast depinning. For small $N$ and small $|d\gamma|$, the phase slips take place in the center of the structure. For large $N$ and large $|d\gamma|$, the central phase slip is replaced by a pair of phase slips symmetrically located on either side of the center. To quantify this qualitative change in behavior, we let $X$ denote the distance between the phase slip and the front (on either half-domain). The quantities $T$, $\Lambda$, and $X$ are indicated in Figure 4.3(b), while $N$ is approximately twice the number of rolls since on the DMS branch one extra roll is added through every two folds. In what follows we focus on $N = 67$ and $N = 72$ to exemplify small $N$ behavior, and choose $N = 175$ and $N = 162$ to illustrate large $N$ behavior. In each case we vary $d\gamma$ and plot the results on a logarithmic scale.

The plots of $T$, $\Lambda$, and $X$ as functions of $|d\gamma|$ are shown in Figs. 4.4(a), 4.4(c) and 4.4(e) (Figs. 4.4(b), 4.4(d) and 4.4(f)) for $\gamma_{DMS}^1$ ($\gamma_{DMS}^2$). The results are similar in both cases. The depinning period $T$ scales approximately as $|d\gamma|^{-1/2}$ in the on-center case and $|d\gamma|^{-1}$ in the off-center case. The front displacement $\Lambda$ equals $\pi/k(\gamma_{DMS}^i)$, $i = 1, 2$, in the on-center case, doubles to $2\pi/k(\gamma_{DMS}^i)$ at the transition point to the off-center case, and varies continuously with further increase in $|d\gamma|$. The distance $X$ between the phase slip and the front is half the width of the LS in the on-center case, but decreases sharply at the transition point to the off-center case and approaches 0 for large $|d\gamma|$.

Figure 4.4 shows that while on-center phase slips persist to larger $|d\gamma|$ when $N$ is small (Fig. 4.4(c,d)), the properties of the depinning become independent of $N$ for large $|d\gamma|$. Indeed, in such cases the depinning state on either half-domain approaches a pulsating front state that connects the equilibrium $A^-$ to the periodic state. The pulsating front is a special type of defect in oscillatory media as classified by Sandstede and Scheel [154], but we prefer not to adopt this terminology to avoid confusion with the defects responsible for defect-mediated snaking. The asymptotic regime is reached earlier for $X$ than for $\Lambda$ because $\Lambda$ approaches its limiting value only when the wave train defined as the portion between the pair of off-center phase slips consists of a large number of rolls.

In the asymptotic regime $N \to \infty$, the wavelength $\Lambda$ left behind by the bounding front depends only on the wavelength of the wave train $\lambda \equiv 2\pi/k$. If $\Lambda = \lambda$ then the wavelength of the wave train remains unchanged by the passage of the bounding front. This particular value of $\lambda$, denoted by $\lambda_0$, is therefore the dynamically selected wavelength that is a function of $\gamma$ alone. As $t \to \infty$, all pulsating fronts converge to an asymptotic state that connects $A^-$ to a periodic state with this wavelength. In practice, the value of
Figure 4.3: The first two phase slips during the depinning of Type-II LS, plotted on the left half-domain for $N = 67$. (a) Slow depinning ($d\gamma = -2 \times 10^{-5}$): phase slips take place at the center $x = 0$. (b) Fast depinning ($d\gamma = -4 \times 10^{-3}$): phase slips take place at a constant distance from the moving front. (c) Intermediate case ($d\gamma = -1 \times 10^{-3}$): phase slips gradually move towards the front. Figure (b) introduces the quantities $T$, $\Lambda$ and $X$ defined in the text.
Figure 4.4: Plots of $T$, $\Lambda$ and $X$ as functions of $|d\gamma|$ for (a,c,e) $\gamma_{1}^{DMS}$ (cross: $N = 67$, circle: $N = 175$); (b,d,f) $\gamma_{2}^{DMS}$ (cross: $N = 72$, circle: $N = 162$). The horizontal scale for $|d\gamma|$ is logarithmic. The vertical scale is logarithmic for $T$ but linear for $\Lambda$ and $X$. 
\(\lambda_0\) at fixed \(\gamma\) can be computed by a binary (or half-interval) search. On the other hand, the initial wavelength \(\lambda\) is more Eckhaus unstable than \(\lambda_0\) in our study of depinning (Figure 4.4). As a result \(\Lambda\) must be less than \(\lambda_0\) as observed in Figures 4.4(c) & 4.4(d) in order to neutralize the instability. In general for \(\lambda \neq \lambda_0\), the effective extent of the wave train on either half-domain changes by \(\lambda - \Lambda\) in each depinning period \(T\). This leads to a uniform wavelength variation in the wave train, so that for \(d\gamma < 0\) \((d\gamma > 0)\) the pulsating front bounds a slowly expanding (shrinking) wave train with edge speed \((\lambda - \Lambda)/T\).

We now examine the depinning process for long structures in a reference frame that is comoving with the front. Figure 4.5(a) shows a space-time plot corresponding to a structure that is contracting while Figure 4.5(b) shows the corresponding plot for an expanding structure. In both cases only the region close to the front is shown. In both cases repeated phase slips take place at a fixed location separated from a sharp front by several more or less stationary rolls, and separating the standing structure near the front from an incoming wave (Fig. 4.5(a)) or an outgoing wave (Fig. 4.5(b)). In the former (latter) the result is a stationary but time-periodic sink (source) a fixed distance from the front. As a result the structure shown in Figure 4.5(a) resembles an interaction between a fixed boundary and an incident finite amplitude traveling wave in a system with a preferred direction of propagation, i.e., in a system with broken reflection symmetry. In such systems reflected waves are necessarily evanescent, and a standing structure therefore only forms near the boundary, requiring the presence of phase slips to eliminate incoming phase. Similar behavior governs the source of phase as shown in Figure 4.5(b). However, phase may be removed in another way as well, and that is via a soft front (Figure 4.5(c)), in which the amplitude of the incoming wave drops gradually and smoothly to zero where the phase becomes undefined. In this case phase slips are no longer required. Figure 4.5(d) shows an example of the left portion of an expanding structure in the Swift-Hohenberg equation with competing quadratic and cubic nonlinearities (SH23), again in a frame moving with the expanding front. Here the amplitude of the traveling wave builds up gradually with increasing \(x\), leading to a source of waves whose structure can be understood in terms of a spatially growing front. However, in contrast to similar structures present, for example, in the complex Ginzburg-Landau equation with drift [172], in the present case the front always propagates into a stable state (such a front is pushed in the terminology of [175]) and hence its properties cannot be described in terms of the transition to absolute instability in the moving frame [37].

4.3.2 General analysis of depinning

In the present parameter regime Type-II LS coexist with Type-I LS and their speeds are comparable. Type-I LS correspond to a codimension-1 family of heteroclinic orbits \(H\) in Eq. (4.2) connecting \(A^\pm\), whose speed will be called \(c^I(\gamma)\). The locus of \(H\) in the \((\gamma, c)\) plane can be computed separately for \(c^I > 0\) (denoted by \(H^+\)) and \(c^I < 0\) (denoted by \(H^-\)) (Figure 4.6(a), solid line). This time, in contrast to §4.2.2, these two branches no longer connect at \(c^I = 0\). Instead we find that \(H^+\) and \(H^-\) respectively approach \((\gamma_1^{DMS}, 0)\) and \((\gamma_2^{DMS}, 0)\), such that the front between \(A^\pm\) does not exist in the DMS region. To see why, we consider the change in the front profile as \(c^I \to 0^+\) \((c^I \to 0^-)\) along \(H^+\) \((H^-)\). According to Appendix 4.5, the real part of the spatial eigenvalues on \(A^+\) also approaches
Figure 4.5: Fast depinning of a Type-II LS as a pulsating front. In each case the space-time plot is drawn in a frame comoving with the front. (a) Sink in the 1:1 FCGLE (1.11) at $d\gamma = -4 \times 10^{-3}$; (b) source in (1.11) at $d\gamma = 1 \times 10^{-2}$; (c) sink in SH23; (d) source in SH23, where SH23 refers to the PDE $u_t = \left[r - (1 + \partial_{xx})^2\right]u + b_2 u^2 - u^3$. Depinning is shown for (c) $r = -0.3410$ and (d) $r = -0.2580$ at fixed $b_2 = 1.8$. For comparison, the snaking region of steady Type-II LS is $r \in [-0.3390, -0.2593]$. 
Figure 4.6: (a) Plot of the front speed $c$ as a function of $\gamma$ at $\nu = 7$. Crosses: DNS of (1.11) for depinning of Type-II LS. Solid line: continuation of heteroclinic orbits in (4.2). Dashed: SN bifurcations (left/right: SN$^+$/SN$^-$). Dot-dashed: RB bifurcations on $A^+$. (b) Front profiles at (i) $c = -0.01$; (ii) $c = 0.01$. The insets show that the tail wavenumber in (ii) is larger than (i), consistent with the stable eigenvalues of $A^+$ depicted in (a).

0, so $A^-$ connects to a wave train that slowly decays to $A^+$ (Figure 4.6(b)). In the limit $c^I \to 0$, this wave train is formed by a conjunction of periodic orbits in the neighborhood of $k^+$ ($k^-$) in the existence region in Figure 3.5. In order for the wave train to decay to $A^+$ towards the right end, the Floquet multipliers of the periodic orbit anywhere in the wave train must be the same as $A^+$ (configuration (II)). On the other hand, towards the left end of this wave train, the connection between $A^-$ and a periodic orbit with effectively constant amplitude can be identified with the left half of a steady Type-II LS. These together imply that the termination point of $\mathbb{H}^+$ ($\mathbb{H}^-$) must lie precisely at $\gamma_{DMS}^{1,2}$. As a corollary we can show that DMS cannot occur in any variational PDE for a real scalar field. Indeed Eq. (2.73) implies that in such systems the front between $A^\pm$ remains stationary only at the Maxwell point. Therefore if the DMS region existed, we would get the contradiction $\gamma_{DMS}^{1,2}$. Since the Type-II LS in the DMS region contains a (stable) periodic state $P(x,k)$ with hyperbolic Floquet multipliers (configuration I) instead of the uniform state $A^+$ the front motion outside the DMS region will differ from that associated with Type-I depinning. We refer to the resulting depinning as Type-II depinning and the resulting front speed as $c^{II} = c^{II}(\gamma)$. Figure 4.6 shows that despite this difference the computed speed $c^{II}$ tracks well the speed $c^I(\gamma)$ corresponding to Type-I depinning. In fact $c^{II} = c^{II}(\gamma,k)$, where $k(\gamma)$ represents the wavenumber of the wavetrain in the interior of the LS and $c^{II} = 0$ at the boundaries $\gamma = \gamma_{DMS}^{1,2}$ of the DMS region, where $k = k^{DMS} \equiv k(\gamma_{DMS}^{1,2})$ (endpoints of curve C in Figure 3.5). In particular outside the pinning region the wavenumber $k(\gamma)$ lies in the Eckhaus unstable regime thereby triggering the phase slips responsible for the motion of the front.

Since the front speed $c^{II}(\gamma,k)$ is identically zero at $\gamma_{DMS}^{1,2}$ for $k^+$ and $k^E$ ($k^-$ and $k^E$), for all intermediate $k$ we expect $c^{II}(\gamma_{DMS}^{1,2},k) = 0$ and $c^{II} > 0$ (
0) for \( \gamma < \gamma_{1,DMS} \) \((\gamma > \gamma_{2,DMS})\). In the frame comoving with the front, for any fixed \( \gamma \not\in [\gamma_{1,DMS}, \gamma_{2,DMS}] \) these correspond to a family of pulsating fronts asymptotic to traveling waves with wavenumber \( k \) that interpolate between the front between \( A^+ \) in Figure 4.6(b) and the sink/source pulsating front in Figure 4.5(a)/4.5(b). Note that \( c^{II} \) denotes the average speed rather than the instantaneous speed; the latter varies within a depinning period (the front is not strictly stationary in Figure 4.5(a)/4.5(b)), especially for \( \gamma \) close to \( \gamma_{\{1,2\},DMS} \). In the time evolution of a Type-II LS with an arbitrary initial wavenumber \( k \), the pulsating front should be approached initially, though the phase of entry in the depinning period (hence the time to the first phase slip) depends on the detailed shape of the bounding front in the initial condition and cannot be predicted from the initial wavenumber \( k \) alone.

As discussed in §4.3.1, the pulsating front with wavenumber \( k_0 = 2\pi/\lambda_0 \) is approached asymptotically in time as long as the wave train is weakly Eckhaus unstable; otherwise the Eckhaus instability will develop rapidly as shown in Figure 4.7(a). The initial evolution of this Type-II LS formed by a strongly Eckhaus unstable periodic state involves gradual invasion of the unstable wave train by a stable one. This process is mediated by a series of phase slips that start from near the pair of bounding fronts and migrate towards the center until the unstable wave train is completely replaced. The pair of invasion fronts formed by these phase slips is reminiscent of a pair of pulled fronts [175], through which a stable periodic state invades a Turing unstable equilibrium (an example can be seen in Figure 4.7(b)). The speed of the invasion front should be calculable using a linear theory similar to the pulled front case, but the details are peripheral to this chapter and will be reported elsewhere.

We may relate Type-II LS to Type-I LS by considering the proximity of the periodic state to \( A^+ \) in function space, based on the observation that the upper Eckhaus unstable region is much wider than the lower one. The time evolutions exhibited in §4.3.1 are essentially done on the two slices \((\gamma, k_+^E(\gamma_{DMS})) \) for \( \gamma < \gamma_{1,DMS} \) and \((\gamma, k_-^E(\gamma_{DMS})) \) for \( \gamma > \gamma_{2,DMS} \). Hence on the \((\gamma, k)\)-plane, the former stays much farther from \( k^+(\gamma) \) than the latter from \( k^-(\gamma) \). This explains why in Figure 4.6, \( c^{II} \) differs notably from \( c^I \) to the left of \( \gamma_{1,DMS} \), but almost coincides with \( c^I \) to the right of \( \gamma_{2,DMS} \). As an illustration of the former effect, we have studied via DNS the depinning of a Type-I LS at \( \gamma = 2.87 < \gamma_{1,DMS} \) (Figure 4.7(b)). In this case, the rate at which the inner structure shrinks increases as the Turing instability drives \( A^+ \) to a periodic state.

### 4.3.3 Dynamics near the codimension-2 points

In this section we study the depinning process following similar work on depinning in the Swift-Hohenberg equation [42, 37]. Following this work suppose that the time \( T \) taken to travel from the vicinity of one saddle-node to the one immediately above (expanding front) or below (contracting front) scales as \( T \sim |d\gamma|^{-1/2} \) where \( |d\gamma| \) is the distance from the boundary of the DMS pinning region. In this regime we can predict the constant of proportionality \( \tau \equiv T/|d\gamma|^{-1/2} \) from weakly nonlinear theory. We let \( A_0(x) \equiv (U_0(x), V_0(x)) \) be a particular even parity stationary LS at \( \gamma_{i,DMS} \), \( i = 1, 2 \), and define \( |d\gamma| \equiv |\gamma - \gamma_{i,DMS}|. \)
Figure 4.7: Space-time plot of $V(x, t)$ showing the time evolution of (a) a Type-II LS at $(\gamma, k) = (2.8947, 1.85)$; (b) a Type-I LS at $\gamma = 2.87$.

For $|d\gamma| \ll 1$ we have

$$
\begin{bmatrix}
U(x, t) \\ V(x, t)
\end{bmatrix}
= \begin{bmatrix}
U_0(x) \\ V_0(x)
\end{bmatrix}
+ |d\gamma|^{1/2} \begin{bmatrix}
U_1(x, t) \\ V_1(x, t)
\end{bmatrix}
+ |d\gamma| \begin{bmatrix}
U_2(x, t) \\ V_2(x, t)
\end{bmatrix}
+ O(|d\gamma|^{3/2}).
$$

(4.3)

The time scale on which perturbations evolve is $|d\gamma|^{-1/2}$. Hence it follows from (4.1) that

$$
\mathcal{L}\left(\begin{bmatrix}
U_1 \\ V_1
\end{bmatrix}
+ |d\gamma|^{1/2} \begin{bmatrix}
U_2 \\ V_2
\end{bmatrix}\right)
= \frac{\partial}{\partial t} \begin{bmatrix}
U_1 \\ V_1
\end{bmatrix}
+ |d\gamma| \begin{bmatrix}
3U_0^2 + V_0^2 \\ 2U_0V_0 + 3V_0^2
\end{bmatrix}
- \begin{bmatrix}
\text{sgn}(d\gamma) \\ 0
\end{bmatrix} + O(d\gamma),
$$

(4.4)

where $\mathcal{L}$ is the linearized FCGL operator evaluated at $\gamma_i^{DMS},$

$$
\mathcal{L} = \begin{bmatrix}
\mu & -\nu \\ \nu & \mu
\end{bmatrix}
+ \begin{bmatrix}
1 & -\alpha \\ \beta & 1
\end{bmatrix}
\partial_{xx}
- \begin{bmatrix}
1 & -\beta \\ \beta & 1
\end{bmatrix}
\begin{bmatrix}
3U_0^2 + V_0^2 \\ 2U_0V_0 + 3V_0^2
\end{bmatrix},
$$

(4.5)

and the first term on the right side of Eq. (4.4) is formally of order $|d\gamma|^{1/2}$. We require that the perturbations decay to 0 as $x \to \pm \infty$.

At leading order in $|d\gamma|^{1/2}$ we solve

$$
\mathcal{L} \begin{bmatrix}
U_1 \\ V_1
\end{bmatrix} = 0,
$$

(4.6)

subject to the requirement $(U_1, V_1) \to 0$ as $x \to \pm \infty$. The solution of this problem determines the null eigenvector(s) of $\mathcal{L}$ and these completely determine the dynamics near the saddle-node. These eigenvectors are either odd or even, with the odd mode corresponding to the neutrally stable Goldstone mode while the even mode corresponds to an amplitude
mode whose growth rate passes through zero at the saddle-node. Since the latter mode is involved in the depinning process it is important to understand the behavior of this mode near the saddle-node. Figures 4.8(a) & 4.8(b) show this mode on the uniform segment of the DMS branch near the saddle-node. In contrast to the amplitude mode involved in front depinning in the Swift-Hohenberg equation the amplitude mode in the present case is spatially extended. This is a consequence of the fact that the saddle-nodes on the DMS branch are associated with Eckhaus instability of the periodic state: DMS is only possible in the interval \([\gamma_1^{DMS}, \gamma_2^{DMS}]\) in which the periodic state with wavenumber \(k(\gamma)\) is hyperbolic (Figure 3.5), and the Eckhaus instability forming the boundary of this region is associated with a spatially extended eigenfunction. In the following we refer to this mode as the Eckhaus mode \(A_E(x)\); its growth rate \(\sigma_E < 0\) depends on \(d\gamma\) as \(\sigma_E = f_E|d\gamma|^{1/2}\) (Figures 4.9(a) & 4.9(b)), where \(f_E < 0\) is a constant that depends on the index \(N\) of the particular saddle-node.

Near the saddle-node on the defect segment, the amplitude mode takes a very different form. This is a consequence of the fact that on this branch the LS no longer consists of a uniform amplitude wavetrain. As a result the near-marginal eigenfunction becomes localized near the defect in the center of the LS. We refer to the resulting mode as the defect mode \(A_d(x)\). Its growth rate \(\sigma_d > 0\) depends on \(d\gamma\) approximately as \(\sigma_d = f_d|d\gamma|\) (Figures 4.9(c) & 4.9(d)), where \(f_d > 0\) is a constant almost independent of \(N\). This is evidently a consequence of the fact that this mode is dominated by the central defect rather than the proximity to the saddle-node and the associated Eckhaus instability. Nonetheless, we anticipate that the expected square-root behavior of the growth rate \(\sigma_d\) is restored very close to the saddle-node. Indeed, the \(A_E(x)\) and \(A_d(x)\) modes turn into one another at the saddle-node despite their very different appearance on either side of it. The marginally stable even parity mode at this location is shown in Figures 4.8(e) & 4.8(f) and will be called a mixed mode \(A_m(x)\). Its growth rate is \(\sigma_m = 0\) by construction.

The profile of the Eckhaus mode \(A_E(x)\) can be constructed as follows. Introducing the shorthand \(k \equiv k^{DMS}(\gamma)\) and \(k_i^{DMS} \equiv k^{DMS}(\gamma_i^{DMS})\), we observe from Figure 3.5 that \(|dk| \equiv |k - k_i^{DMS}| \sim |d\gamma|^{1/2}\) for \(|d\gamma| \ll 1\). This behavior is a consequence of the (numerical) observation that the amplitude of the wavetrain determines uniquely its wavenumber \(k\). This wavenumber selection process is well understood in time-independent systems that are Hamiltonian in space (such as the Swift-Hohenberg equation [38]) and is a consequence of the presence of the fronts connecting the wavetrain to the background state \(A^-\). Thus a saddle-node bifurcation in the amplitude also corresponds to a saddle-node bifurcation in the wavenumber (Fig. 3.5). Now we write the steady LS at \(\gamma \approx \gamma_i^{DMS}\) as a connection between the background state \(A^-\) and a periodic state \(P\) and back again,

\[
A(x; k) \equiv A^- \xrightarrow{F(-n\pi-kx;k)} P(kx - 2m\pi; k) \xrightarrow{F(kx-n\pi;k)} A^-,
\]

where \(F\) (\(P\)) denotes the front (roll) profile (\(P\) is even and \(2\pi\)-periodic in the first argument), \(n\) denotes the total number of rolls and \(m\) ranges from \(-n/2\) to \(n/2\). In (4.7), the parametric dependence on \(k\) has been kept, but terms at \(O(d\gamma)\) or higher orders have been discarded. Differentiating with respect to \(k\) for each \(x\), we get an eigenfunction homoclinic to zero,

\[
\frac{dA(x; k)}{dk} = 0 \quad \Rightarrow \quad xP' + P_k \xrightarrow{xP' + F_k} 0,
\]

(4.8)
Figure 4.8: The Eckhaus, defect and mixed modes ($A_E(x)$, $A_d(x)$ and $A_m(x)$) at (a,c,e) $N = 67$; (b,d,f) $N = 72$. The Eckhaus and defect modes are computed at $|d\gamma| = 2 \times 10^{-5}$. 
Figure 4.9: Growth rates of the Eckhaus and defect modes ($\sigma_E$ and $\sigma_d$) as functions of $|d\gamma|$ at (a,c) $N = 67$; (b,d) $N = 72$. 
where $'$ and the subscript $k$ denote, respectively, partial derivatives with respect to the first argument and with respect to $k$. Modulo normalization, Eq. (4.8) evaluated at $\gamma_i^{DMS}$ now yields the Eckhaus mode $A_E(x)$. We see that along the wavetrain $A_E(x) = xP' + P_k \neq 0$, with the $xP'$ term responsible for the linear change in the amplitude of the mode with $x$ shown in Figs. 4.8(a) & 4.8(b). These observations confirm the Eckhaus nature of this mode. Incidentally, the same scaling law, $|dk| \sim |d\gamma|^{1/2}$, also applies to homoclinic snaking in the 2:1 FCGLE (Figure 3 of [129]), where the selected wavenumbers $k(\gamma)$ form a closed curve in the $(\gamma, k)$-plane (Figure 5 of [129]). Using the same construction as above, it can be established that the Eckhaus mode must again take the form of (4.8). In contrast, for homoclinic snaking in the Swift-Hohenberg equation [42], the wavenumber $k$ does not changes between the upper and lower branches. Thus $|dk| \sim |d\gamma|$ and hence $P' = 0$, i.e., the associated amplitude mode vanishes on the wavetrain and remains localized at the pair of fronts as reviewed in \S2.4.2.

As we have seen the Eckhaus mode on the uniform amplitude segment changes rapidly and dramatically on the defect portion of the DMS branch. This fact greatly complicates the calculation of the depinning speed. Since the use of the marginal eigenfunction $(U_m, V_m)$ fails to reproduce the numerical results further from the saddle-node we proceed as in [38] but allow for a general superposition of near-marginal modes:

$$
\begin{bmatrix}
    U_1(x, t) \\
    V_1(x, t)
\end{bmatrix} = \sum_j a_j(t) \begin{bmatrix}
    U_j(x) \\
    V_j(x)
\end{bmatrix},
$$

where $a_j(t)$ are slowly evolving real amplitudes and the summation index $j$ ranges over a subset of $\{E, d, m\}$. At the next order we have

$$
\mathcal{L} \begin{bmatrix}
    U_2 \\
    V_2
\end{bmatrix} = \partial_t \begin{bmatrix}
    U_1 \\
    V_1
\end{bmatrix} + |d\gamma|^{1/2} \left( \begin{bmatrix}
    1 & -\beta \\
    \beta & 1
\end{bmatrix} \begin{bmatrix}
    3U_1^2 + V_1^2 & 2U_1V_1 \\
    2U_1V_1 & U_1^2 + 3V_1^2
\end{bmatrix} \begin{bmatrix}
    U_0 \\
    V_0
\end{bmatrix} - \begin{bmatrix}
    \text{sgn}(d\gamma) \\
    0
\end{bmatrix} \right).
$$

The solvability condition requires that the RHS is orthogonal to all null eigenvectors of $\mathcal{L}^\dagger = \mathcal{L}^T$, resulting in the following set of ODE:

$$
\sum_j \alpha_j \frac{da_j}{dt} = |d\gamma|^{1/2} \left( \beta_t \text{sgn}(d\gamma) + \sum_j \beta_j a_j^2 + \sum_h \sum_{j<h} \beta_{h,j}^k a_h a_j \right).
$$

Here the $\alpha$ and $\beta$ coefficients are (the integration limits are always $\pm \infty$)

$$
\alpha_j^i \equiv \int (U_i^\dagger U_j + V_i^\dagger V_j) \, dx, \quad \beta_i^j \equiv \int U_i^\dagger \, dx,
$$

$$
\beta_i^j \equiv -\int \left[ \begin{bmatrix}
    U_i^\dagger \\
    V_i^\dagger
\end{bmatrix} \begin{bmatrix}
    1 & -\beta \\
    \beta & 1
\end{bmatrix} \begin{bmatrix}
    3U_j^2 + V_j^2 & 2U_jV_j \\
    2U_jV_j & U_j^2 + 3V_j^2
\end{bmatrix} \begin{bmatrix}
    U_0 \\
    V_0
\end{bmatrix} \, dx,
$$

$$
\beta_{h,j}^i \equiv -2\int \left[ \begin{bmatrix}
    U_i^\dagger \\
    V_i^\dagger
\end{bmatrix} \begin{bmatrix}
    1 & -\beta \\
    \beta & 1
\end{bmatrix} \begin{bmatrix}
    3U_hU_j + V_hV_j & U_hV_j + U_jV_h \\
    U_hV_j + U_jV_h & U_h^2 + 3V_h^2
\end{bmatrix} \begin{bmatrix}
    U_0 \\
    V_0
\end{bmatrix} \, dx.
$$

We find that the linear combination of all three modes fits the computed quantity $A(x, t) - A_0(x)$ well with $A_0(x)$ evaluated right at $\gamma_i^{DMS}$. However, since the depinning
of the front is linked to phase slips triggered by the Eckhaus instability we neglect the contributions from $A_m$ and $A_d$ and evaluate the above integrals numerically. The signs of $A_E(x)$ and $A_E^\dagger(x)$ are chosen such that $a_E > 0$. When (4.11) describes depinning, the signs of the other coefficients are $d\gamma < 0$ and $\beta_E\beta_E^E < 0$ for $\gamma_1^{DMS}$, and $d\gamma > 0$ and $\beta_E\beta_E^E > 0$ for $\gamma_2^{DMS}$. By construction the amplitude function $a_E(t) = -\infty$ right after the first phase slip, and we postulate that $a_E(t) = 0$ right before the second phase slip since at this moment $A(x,t)$ is expected to be marginally Eckhaus unstable and thus should coincide with $A_0(x)$. In either case, (4.11) has no fixed point and $a_E(t)$ passes from $-\infty$ to 0 in the finite time

$$T = \frac{\pi a_E^E}{2(\beta_E\beta_E^E d\gamma)^{1/2}}.$$  \hspace{1cm} (4.15)

However, in practice we find that the integral $\beta_E^E$ tends to 0 as $N$ increases. It is hard to attain the level of precision in the computation of the eigenfunctions $A_E(x)$ and $A_E^\dagger(x)$ required to evaluate this integral, so instead we express $T$ in terms of the eigenvalue $\sigma_E$, which is numerically more reliable. When Eq. (4.11) describes steady states, the sign of $d\gamma$ is opposite to that for depinning while the other coefficients remain the same. In this case Eq. (4.11) has the following pair of fixed points with eigenvalues

$$a_0^E = \pm \text{sgn}(\beta_E^E) \left(\frac{\beta_E\text{sgn}(d\gamma)}{\beta_E^E}\right)^{1/2} \quad \text{with} \quad \sigma_E = \pm \frac{2(\beta_E\beta_E^E d\gamma)^{1/2}}{a_E^E}.$$  \hspace{1cm} (4.16)

The lower sign corresponds to the steady states on the uniform segment with $\sigma_E < 0$ the growth rate of the Eckhaus mode $A_E(x)$. It then follows from Eqs. (4.15) and (4.16) that the predicted relation between $T$ and $\sigma_E$ is

$$T = \frac{\pi}{|\sigma_E|}.$$  \hspace{1cm} (4.17)

To test the result (4.17), we postulate a relation between $T$ and $|d\gamma|$ of the form

$$\frac{1}{T^2} = m(|d\gamma| - \tilde{\gamma})$$  \hspace{1cm} (4.18)

and determine the slope $m$ and the intercept $\tilde{\gamma}$ (both as functions of $N$) from DNS of Eq. (1.11). The results are compared with the predicted values $m = f_E^2/\pi^2$ and $\tilde{\gamma} = 0$ that follow from Eq. (4.17). As shown in Figure 4.10, for $\gamma_1^{DMS}$ (panel (a)) the actual and predicted slopes $m$ agree almost perfectly, while for $\gamma_2^{DMS}$ (panel (b)) the predicted $m$ is larger than the actual value, or equivalently the predicted depinning period $T$ is shorter than the actual period. Aside from this discrepancy, we find in DNS of Eq. (1.11) that the prediction (4.18) fails for very small $|d\gamma|$, and more generally that the intercept $\tilde{\gamma}$ does not vanish. It is precisely in this regime that the marginal mode $A_m$ should be employed. As shown in Figure 4.10, for $\gamma_2^{DMS}$ (panel (d)) $\tilde{\gamma}$ depends on $N$ periodically, while for $\gamma_1^{DMS}$ (panel (c)) this dependence looks random, but its Fourier transform again suggests periodicity with a relatively small period.
Figure 4.10: The slope $m$ and the intercept $\tilde{\gamma}$ in (4.18) as functions of $N$ for (a) & (c) $\gamma^DMS_1$ and (b) & (d) $\gamma^DMS_2$. ‘o’: DNS; ‘+’: weakly nonlinear theory.
4.4 Discussion

The above study of the depinning of Type-I and Type-II LS leads to two open questions. First, the unstable pulsating fronts that connect the background equilibrium $A^-$ to a strongly Eckhaus unstable periodic state are not directly amenable to DNS but may be computed using special numerical continuation techniques \[49\]. The result could shed light on the process of dynamical wavelength selection for both the 1:1 FCGLE (1.11) and SH23 (2.64). On the other hand, the analytic techniques for computing the speeds of pulsating fronts proposed in Ref. \[56\] in periodically modulated neural field models may also be adaptable to the above pattern-forming PDEs.

Second, the weakly nonlinear theory for slow depinning is currently incomplete largely due to the coexistence of multiple marginal eigenfunctions near the limits $\gamma_{1,2}^{DMS}$ of the DMS region. In this regard it may be useful to compute the time it takes for a phase slip to occur as a function of the distance from the Eckhaus boundary.

4.5 Appendix: Movement of eigenvalues for the RB bifurcation

Let $A_0 \equiv U_0 + iV_0$ be an equilibrium solution to (4.2) and denote $R_2 \equiv U_0^2 + V_0^2$. Linearizing around $A_0$, we get the following equation for the spatial eigenvalues $\lambda$,

\[(1 + \alpha^2)\lambda^4 + 2c\lambda^3 + (2M + c^2)\lambda^2 + 2cN\lambda + Q = 0,\]  
\[(4.19)\]

where

\[M = \mu + \alpha\nu - 2(1 + \alpha\beta)R_2, \quad N = \mu - 2R_2,\]  
\[(4.20)\]

\[Q = \mu^2 + \nu^2 - 4(\mu + \beta\nu)R_2 + 3(1 + \beta^2)R_2^2.\]  
\[(4.21)\]

When $c = 0$ Eq. (4.19) becomes

\[(1 + \alpha^2)\lambda^4 + 2M\lambda^2 + Q = 0,\]  
\[(4.22)\]

with the solutions

\[\lambda^2 = \frac{-M \pm \sqrt{M^2 - (1 + \alpha^2)Q}}{1 + \alpha^2}.\]  
\[(4.23)\]

In order that the unperturbed eigenvalues are all imaginary, the coefficients must satisfy

\[Q > 0, \quad M > 0, \quad S^2 \equiv M^2 - (1 + \alpha^2)Q > 0 \quad (S > 0),\]  
\[(4.24)\]

in which case Eq. (4.23) has two negative roots

\[\lambda^2 = -k^2 \equiv \frac{-M \pm S}{1 + \alpha^2}.\]  
\[(4.25)\]

For $|c| \ll 1$, the solutions to Eq. (4.19) can be written as

\[\lambda = ik_\pm + c(\delta_\pm^r + i\delta_\pm^i) + O(c^2) \quad \text{and} \quad c.c.\]  
\[(4.26)\]
At $O(c^1)$, we get the following condition

$$2(ik_{\pm})^2(1 + \alpha^2)(\delta_r^\pm + i\delta_i^\pm) + (ik_{\pm})^2 + 2M(\delta_r^\pm + i\delta_i^\pm) + N = 0, \quad (4.27)$$

which leads to

$$\delta_r^\pm = \frac{k^2_{\pm} - N}{2(M - k^2_{\pm}(1 + \alpha^2))}, \quad \delta_i^\pm = 0. \quad (4.28)$$

Thus $\delta_r^\pm \neq 0$ whenever $k^2_{\pm} \neq N$ implying that the eigenvalues move off the imaginary axis for $c \neq 0$. Thus $c = 0$ is a (multiple) bifurcation point.

These eigenvalues move in opposite directions whenever

$$\delta_r^+ \delta_r^- = -\frac{(M - (1 + \alpha^2)N)^2 - S^2}{4S^2(1 + \alpha^2)^2} = -\frac{1}{4S^2(1 + \alpha^2)} \left( (1 + \alpha^2)N^2 + Q - 2MN \right) < 0. \quad (4.29)$$

In the last factor, $Q > 0$ and $M > 0$ as required by (4.24), while the sign of $N$ determines the Hopf stability of $A_0$ in (1.11) [130]. In the case under consideration $\mu < 0$ and the $A_0$ are Hopf-stable. Thus $N < 0$ and the eigenvalues do indeed move in opposite directions.

From an alternative viewpoint, the line of (spatial) Hopf bifurcations in the $(R_2, c)$ plane intersects the line $c = 0$ at $(R_{2TW}^T, 0)$, with $R_{2TW}^T$ computed in Ref. [130]. At this location $\delta_r^+ \delta_r^-$ changes sign implying that the right side of Eq. (4.29) vanishes at $R_2 = R_{2TW}^T$. This is indeed the case. In contrast, when $\mu > 0$ the state $A_0$ may be Hopf unstable in time. In this case a localized initial condition consisting of an inner plateau of $A_0$ embedded in a homogeneous background state may evolve into a localized spatiotemporally chaotic state resulting in a stochastically driven front connecting the $A_0$ state to the background state. This topic will be part of Chapter 9.
Chapter 5

Homoclinic snakes bounded by a saddle-center periodic orbit

5.1 Introduction

In order to explain the new phenomenon to be studied in this chapter, we mention two recent and seemingly unrelated examples that point to the existence of different forms of homoclinic snaking in one spatial dimension. Yulin & Champneys [186] studied homoclinic snaking in a spatially discrete system that also contains a constant offset. In this system, snaking curves of localized solutions occur within an S-shaped bistability region bounded by fold (saddle-center) bifurcations of the background and patterned states. As a second parameter is varied, the width of the snake increases, until one side of the snake “crashes” into the parameter value corresponding to the saddle-center bifurcation of the patterned state. This codimension-two transition results in a change of topology and transforms the homoclinic snake into a stack of C-shaped isolas. These isolas involve what one might describe as hybrid localized states that contain an additional small-amplitude modulation in the center of the excited state.

In Chapter 3 [129], we found a form of snaking in the 1:1 resonant CGL equation that involves a single connected branch along which primary and hybrid localized states alternate. In this example, the additional central modulation within the hybrid states takes the form of small defect and hence this behavior has been dubbed defect-mediated snaking.

In §5.3 below, we provide a theoretical underpinning for defect-mediated snaking in the context of planar reversible maps. In particular, we show that the bifurcation structure arises because the behavior near both the left and right boundaries of the snake resembles the behavior near the boundary of the C-shaped isolas in the first example that results from the saddle-center transition. Since the 1:1 resonant CGL equation is non-variational, the theory of planar maps does not directly apply, and must be extended (see §5.6) into the realm of appropriate three-dimensional reversible maps such as arise in the analysis of spatial dynamics in this example. We also present, in §5.5, another example based on the Swift-Hohenberg (S-H) equation, that also undergoes a transition like that in the first example.

The chapter is organized as follows. We begin, in §5.2–5.4, with a description of
the above behavior within the realm of reversible planar maps. Specifically, after setting
the mathematical scene in §5.2, we show in §5.3 how the new kind of snaking leads to a
scaling in which the location of the nth fold scales like $|\lambda|^{-1/4}$ to leading order. §5.4 then
goes on to study the codimension-two situation in which regular snaking transitions into
saddle-center mediated snaking. §5.5 considers the implications of the above analysis for
PDEs with a spatially conserved Hamiltonian, taking a variational extension of the S-H
equation as an example. §5.6 describes an extension of the theory to deal with spatially
reversible problems with no conserved Hamiltonian and compares the resulting theory with
numerical results for the 1:1 resonant CGL equation. §5.7 draws conclusions and points to
directions for future work.

### 5.2 Mathematical setting

Consider a planar diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$, which is reversible, that is, there
eexists an involution $R$ such that

$$R \circ F = F^{-1} \circ R,$$

and assume that the fixed point set $S := \text{Fix} R$ is one-dimensional. Suppose that the map
$F$ depends on a single parameter $\lambda$ such that at $\lambda = 0$ there is a symmetric fixed point
$A_0 \in S$, the linearization around which can be represented by a non-semisimple pair of
eigenvalues at 1. Furthermore, assume that $\lambda$ unfolds this singularity in a generic way such
that for $\lambda > 0$ there is a pair of nearby symmetric fixed points, one of which is a saddle and
the other is a center, while for $\lambda < 0$ there are no nearby fixed points. In what follows we
shall let $A(\lambda)$ refer to the continuous family of saddle points of the map for small $\lambda > 0$,
and extend the definition so that $A_0 = A(0)$.

Suppose that global co-ordinates $(x, y)$ are chosen for all $\lambda$ such that the reversing
transformation can be written

$$R : (x, y) \to (x, -y), \quad \text{so that} \quad S = \{(x, y) : y = 0\}.$$ 

By standard properties of reversible maps, any trajectory of the dynamical system $(x, y) \to
F(x, y; \lambda)$ that intersects $S$ is necessarily symmetric under $R$. That is, any point $(x_0, 0) \in \mathbb{R}^2$
is necessarily part of a trajectory $\{(x_n, y_n)\}_{n=-N}^{N}$ for which $x_{-n} = x_n$ and $y_{-n} = -y_n$; see
[118] for further properties of reversible maps.

Next, consider a small neighborhood $U$ of $A_0$ and $\Lambda$ of $\lambda = 0$ in phase and para-
meter space. Then, by standard results (see e.g. [117, Thm. 1.1]), co-ordinate transformations
can be chosen such that the map can be approximated, up to exponentially small terms,
by the time-one map of the vector field normal form for the equivalent singularity. In the
present case the vector field can be shown to be topologically equivalent to its 2-jet, viz.,

$$\begin{align*}
\dot{x} &= y, \\
\dot{y} &= \lambda - x^2,
\end{align*}$$

(5.1)

(5.2)

corresponding to the Hamiltonian

$$H = \frac{y^2}{2} - \lambda x + \frac{x^3}{3}.$$ 

(5.3)
The dynamics of Eqs. (5.1)-(5.2) are depicted in Fig. 5.1. When \( \lambda > 0 \), there is a saddle point at \( A(\lambda) = (-\sqrt{\lambda}, 0) \) and a center at \( (+\sqrt{\lambda}, 0) \). When \( \lambda = 0 \) the equilibrium is degenerate and there exist trajectories of the flow (and hence invariant sets of the time-one map) that asymptote (algebraically) to \( A_0 = (0, 0) \) as \( t \to \pm \infty \). These trajectories are given explicitly by \( y = \pm \sqrt{-2/3} x^3 \) for \( x < 0 \) and correspond to stable and unstable sets to the origin of the corresponding time-one map. As \( \lambda \) is increased from zero these sets have continuous continuations into the negative-\( x \) components of the stable and unstable manifolds of \( A \). For this reason we shall refer to the stable (unstable) set asymptotic to the origin of the map at \( \lambda = 0 \) as the singular stable (unstable) manifold of \( A_0, W_{\text{sing}}^{s(u)}(A_0) \).

For simplicity, after a smooth change of coordinates if necessary, we shall suppose that the dynamics outside of \( U \) is unaffected by variation of \( \lambda \in \Lambda \).

Furthermore, assume there is an additional saddle point \( B \in S \) of the map \( F \) that is outside \( U \), such that the two one-dimensional sets \( W^u(B) \) and \( W^s(A_0) \) intersect transversally (see Fig. 5.2(a)). Using the motivating example in §5.1, we identify \( B \) with the background state, and \( A \) with the excited state.

As is clear from Fig. 5.2(b), transversal intersection of \( W^u(B) \) and \( W_{\text{sing}}^s(A_0) \) implies transversal intersection of \( W^u(B) \) and \( W^s(A(\lambda)) \) for sufficiently small \( \lambda > 0 \). Figure 5.2(b) also indicates that at \( \lambda \geq 0 \) there must necessarily be an infinite discrete family of homoclinic orbits to \( B \), which occur whenever \( W^u(B) \) intersects the \( x \)-axis \( S \). Note that there are two kinds of homoclinic connections, those with their point of symmetry at \((x_0, 0)\) with \( x_0 < 0 \) and those for which \( x_0 > 0 \). We shall refer to those with \( x_0 < 0 \) as primary homoclinic orbits and to those with \( x_0 > 0 \) as hybrid homoclinic orbits. This nomenclature is inspired by the situation for \( \lambda > 0 \), where the primary homoclinic orbits approach the core state \( A \) monotonically up to their point of symmetry, whereas the hybrid ones contain extra iterates close to the homoclinic loop of the normal form.

We shall be interested in what happens to this set of \( B \)-homoclinic orbits for \( \lambda < 0 \). Specifically, in the next section we analyze the bifurcation that occurs for \( \lambda < 0 \) when \( W^u(B) \) becomes tangent to \( S \) so that a pair of intersection points are created. These intersections persist for \( \lambda > 0 \) at which point one corresponds to a single-pulse homoclinic orbit and the other to a hybrid. As \( \lambda \) tends to zero from below there is an infinite number of such bifurcations that create the situation shown in Fig. 5.2. §5.4 then considers the codimension-two case where we break the transversality hypothesis about the intersection of \( W^u(B) \) and \( W_{\text{sing}}^s(A_0) \), and introduce an additional parameter that unfolds this tangency.
Figure 5.2: (a) The assumed intersection between $W^u(B)$ and $W^s(A_0)$. (b) The corresponding case for $\lambda > 0$ showing the infinite number of intersections between $W^u(B)$ and $S$.

Figure 5.3: (a) A typical set $\Gamma$ (blue curve) and its computed forward iterates $\Gamma^n, n = 1, 2, \ldots$ (red curves) for $\lambda = 0.01$ with the appropriate level set of $H$ shown as a thin solid line. (b) A similar computation for $\lambda = -0.002$.

5.3 Codimension-one bifurcation

Locally, in the normal form co-ordinates, the singular stable manifold is given by

$$W^s_{\text{sing}}(A_0) = \{(x, y) = (-s, \sqrt{(2/3)s^{3/2}}); s > 0\}. $$

Let $(-\delta, \sqrt{(2/3)\delta^{3/2}})$, for $\delta > 0$, be a point of intersection of this manifold with $W^u(B)$ in $U$. Let the local segment $\Gamma$ of $W^u(B)$ be given by

$$\Gamma = \{(x, y) = (-\delta + k_1\theta, \sqrt{(2/3)\delta^{3/2}} + k_2\theta + O(\theta^2)); \theta \in [-\varepsilon, \varepsilon]\}$$

for some $0 < \varepsilon \ll \delta$, where transversality implies $(k_1, k_2) \cdot e_v \neq 0$, where $e_v$ is the normal vector to $W^s(A_0)$ defined by Eq. (5.9) below.
Figure 5.3(b) depicts what happens to forward iterates $\Gamma^n$, $n > 0$, of $\Gamma$ under the normal form map for $\lambda < 0$. Let $(x_n(\theta), y_n(\theta))$ be the co-ordinates of $\Gamma^n$. A point of tangency of $\Gamma^n$ with $S$ occurs if

$$y_n(\theta) = 0, \quad \text{and} \quad y'_n(\theta) = 0.$$  \hspace{1cm} (5.4)

Now, from Eq. (5.3) we calculate explicitly that

$$y_n(\theta) = \sqrt{Y(x_n; \lambda, c(\theta, \lambda))},$$

where

$$Y = 2\lambda x_n - \frac{2}{3}x_n^3 + 2c(\theta, \lambda),$$

and $c(\theta, \lambda)$ is the value of the first integral $H$. We are interested in the behavior for small $\lambda$, with $\delta$ fixed, implying that the value $c$ of the first integral $H$ selected by the initial condition $(x_0, y_0) \in \Gamma$ is

$$c(\theta, \lambda) = \delta\lambda + k_1(\delta^2 - \lambda)\theta + k_2\sqrt{(2/3)\delta^2\theta} + O(\theta^2).$$ \hspace{1cm} (5.5)

Since $c$ is approximately affine in $\theta$ in the following we vary $c$ instead of the parameter $\theta$.

We can now solve Eqs. (5.1)-(5.2) for the time $t$ to reach $x = x_n$, obtaining the implicit expression

$$t(x_n; \lambda, c) = \int_{-\delta + k_1 \theta}^{x_n} \frac{1}{\sqrt{Y(x; \lambda, c)}} \, dx = n.$$ \hspace{1cm} (5.6)

The first of the conditions in Eq. (5.4) is satisfied if $x_n$ in the upper limit of the integral (5.6) satisfies $Y(x_n; \lambda, c) = 0$. Consider now the second condition in Eq. (5.4). Since

$$\frac{dy}{dc} = \frac{dt}{dx} \frac{dx}{dc} \frac{dy}{dt},$$

where, by Eq. (5.1), $\frac{dy}{dt}$ is strictly negative when $\lambda < 0$, it follows that the second condition is satisfied if and only if

$$\frac{dt}{dc}(x_n; \lambda, c) = 0.$$

Moreover, it is easy to see that such a point must be a maximum of $t$ with respect to $c$. Hence to find a homoclinic tangency we need to solve for $\lambda$, $c$ such that

$$\max_c t(x_n; \lambda, c) = n, \quad \text{where} \quad x_n \text{ solves } Y(x_n; \lambda, c) = 0.$$

To compute an asymptotic expression for the integral (5.6) in the limit of small $\lambda$ (and hence large $n$) we use the rescaling

$$x = (-\lambda)^{1/2} \xi, \quad c = (-\lambda)^{3/2} \chi,$$

and rewrite Eq. (5.6) as

$$t(x_n; \lambda, c) = (-\lambda)^{-1/4} \int_{(-\delta + k_1 \theta)/(-\lambda)^{1/2}}^{\Phi(\xi; \chi) = 0} \frac{1}{\sqrt{\Phi(\xi; \chi)}} \, d\xi \equiv (-\lambda)^{-1/4} \tau(\chi),$$ \hspace{1cm} (5.7)
where
\[ \Phi(\xi; \chi) \equiv -2\xi - \frac{2}{3}\xi^3 + 2\chi. \]

We now seek to find the maximum of \( \tau(\chi) = (-\lambda)^{1/4}n \).

Note that as \( \lambda \to 0^- \), \( (-\delta + k_1\theta)/(-\lambda)^{1/2} \to -\infty \), so \( \tau(\chi) \) is independent of \( \delta \) and \( \theta \) to leading order. Numerically, we find that \( \tau(\chi) \) has a unique maximum at \( \tau_0 = 3.6384 \), for which \( \chi = \chi_0 \approx 0.61015 \). Hence, as \( \lambda \to 0 \) the \( n^{th} \) fold and its corresponding \( \lambda \)-value are asymptotically related by
\[ n = \tau_0(-\lambda)^{-1/4} + \text{h.o.t.} \]

Equivalently,
\[ \lambda_n = -\tau_0^4 n^{-4} + o(n^{-4}) = -175.24n^{-4} + \text{h.o.t.} \quad (5.8) \]

In summary, we have shown that there is a sequence of \( \lambda \) values, \( \lambda_n \) say, which converges to 0 from below according to the scaling (5.8) as \( n \to \infty \). For \( \lambda = \lambda_n \), the \( n^{th} \) iterate of a local piece of \( W^u(B) \) within \( U \) intersects \( S \) tangentially. For \( \lambda > \lambda_n \) there is a pair of transverse intersections, each corresponding to a homoclinic orbit to \( B \). Each of these orbits has \( 2n \) iterates within the neighborhood \( U \) of the excited state \( A_0 \). The two orbits differ in that for \( \lambda > 0 \) their point of symmetry \((x_0, 0)\) lies on opposite sides of \( x = 0 \). Thus for one \( x_0 > 0 \) and its point of symmetry lies close to the point of symmetry of the separatrix of the normal form while for the other \( x_0 < 0 \). The former correspond to hybrid localized states while the latter correspond to primary localized states. Both types of orbits are present even when \( \lambda < 0 \) but are connected by folds that accumulate algebraically in \( \lambda = 0 \) at the rate predicted by Eq. (5.8).

### 5.4 Codimension-two bifurcation

Next we shall consider an unfolding of the homoclinic orbits to \( B \) that arise from the codimension-two point for which \( W^u(B) \) is quadratically tangent to the singular stable manifold \( W^s_{\text{sing}}(A_0) \). To that end, we introduce an additional parameter \( \beta \) that unfolds the quadratic tangency in a generic way. To describe the tangency, we introduce a small arc \( \Gamma(\beta) \) of \( W^u(B) \) within \( U \). As in the previous section, at the critical parameter values \( \lambda = \beta = 0 \) \( \Gamma(0) \) intersects \( W^s_{\text{sing}}(A_0) \) at the point \( \gamma_0 : (x, y) = (-\delta, \sqrt{(2/3)}\delta^{3/2}) \), for some small \( \delta > 0 \). To describe the tangency, it is convenient to introduce local right-handed co-ordinates \((u, v)\) such that the unit vector \( e_u \) is tangent to the flow at \( \gamma_0 \), and \( e_v \) is normal to the flow. Specifically,
\[ e_u = \frac{(\sqrt{(2/3)}\delta^3, -\delta^2)}{\sqrt{(2/3)}\delta^3 + \delta^4}}, \quad e_v = \frac{(\delta^2, \sqrt{(2/3)}\delta^3)}{\sqrt{(2/3)}\delta^3 + \delta^4}} \quad (5.9) \]

Then the assumption of quadratic tangency implies that we can write
\[ \Gamma(0) = \{(u, v) = (\theta, \kappa\theta^2 + O(\theta^3)) : \theta \in [-\varepsilon, \varepsilon]\} \]
for some $0 < \varepsilon \ll \delta$, where $\kappa \neq 0$ is an $O(1)$ constant. In what follows, we shall assume that $\varepsilon$ is sufficiently small that the cubic term in $\theta$ can be dropped. Furthermore, we suppose that a parameter $\beta$ unfolds this tangency in a generic way:

$$\Gamma(\beta) = \{(u, v) = (\theta, \beta + \kappa \theta^2 + O(\theta^3, \beta \theta)) : \theta \in [-\varepsilon, \varepsilon]\}. \tag{5.10}$$

We look for homoclinic orbits to $B$. Such an orbit occurs when there is a point of intersection between a forward image $\Gamma^n(\beta)$, $n > 0$, of $\Gamma(\beta)$ under the map $F(\cdot; \lambda)$ that intersects the symmetric section $S = \{(x, y) : y = 0\}$. In particular we shall look at loci in the $(\beta, \lambda)$-plane of points of tangency between $\Gamma^n$ and $S$.

In what follows we compute such curves numerically by solving appropriate boundary value problems using AUTO, and draw whatever analytical insights we can in various distinguished limits. There are two topologically distinct cases to consider, depending on the sign of $\kappa$ in Eq. (5.10). The case $\kappa > 0$ is referred to as an outer tangency while $\kappa < 0$ is an inner tangency. The distinction arises from whether $\Gamma(0)$ lies inside or outside the cusp formed by $W^s_{\text{sing}}(A_0)$ and $W^u_{\text{sing}}(A_0)$.

### 5.4.1 Outer tangency case

Figure 5.4(a) depicts $\Gamma(0)$ and its numerically computed first 10 iterates $\Gamma^n$, $n = 1, \ldots, 10$, for $\lambda = \beta = 0$ and $\kappa > 0$. We see that, for sufficiently large $n$, $\Gamma^n$ intersects the set $S$ (which is just the $x$-axis in the figure) at locations with $x > 0$. These intersections persist even when $\lambda$ is slightly negative, as can be seen from the computation of $\Gamma^{10}$ in Fig. 5.4(b). Intersections can also occur for $\lambda > 0$ provided $\beta$ is sufficiently negative, as in Fig. 5.4(d). In addition, in both panels (b) and (d) there are also intersections between $\Gamma^{10}$ and $S$ with $x < 0$.

Figure 5.4(b) reveals that there are four points on $\Gamma^{10}$ that cross the $x$-axis. As $\lambda$ is slightly reduced from the value depicted, these homoclinic orbits will disappear in pairs as the two branches of the red curve $\Gamma^{10}$ become tangent to the $x$-axis. We shall call such tangencies side tangencies. From the geometry, we see that the side tangencies corresponding to each distinct branch of $\Gamma^{10}$ occur for nearby values of $\lambda$, but for distinct values of $\theta$. Thus $\theta$ parametrizes points in $\Gamma^{10}$. Looking now at panels (c) and (d) of Fig. 5.4, we note that between these two values of $\beta$, four points of intersection between $\Gamma^{10}$ and the $x$-axis are created. These are created at $\beta$-values where the ‘tip’ of the highly folded curve $\Gamma^{10}$ becomes tangent to the $x$-axis, either for $x > 0$ or for $x < 0$. We call such events creating homoclinic orbits tip tangencies. A tip tangency for $x < 0$ creates a pair of primary homoclinic orbits and one for $x > 0$ creates a pair of hybrid homoclinic orbits. See [47] for a related problem involving folding of manifolds close to a heteroclinic cycle in which the interplay between side and tip tangencies is important.

### 5.4.2 Inner tangency case

The case of an inner tangency between $W^u(B)$ and $W^s_{\text{sing}}(A_0)$, i.e., $\kappa < 0$, can be analyzed in exactly the same way as the outer case. Figure 5.5 shows the numerically determined loci for both kinds of tangencies between $\Gamma^n$ and $S$ in the $(\lambda, \beta)$ plane, where $n = 7, \ldots, 10$. 
Figure 5.4: (a) The set $\Gamma$ (blue curve) and its first 10 iterates $\Gamma^n$, $n = 1, \ldots, 10$ (red curves) for $\lambda = \beta = 0$. Also depicted as thin black lines are the stable and unstable manifolds of $A$. (b)–(d) Similarly depicted $\Gamma$ and its tenth iterate $\Gamma^{10}$ for: (b) $\lambda = -0.002$, $\beta = 0$; (c) $\lambda = 0.02$, $\beta = 0.02$; (d) $\lambda = 0.02$, $\beta = -0.0022$. Other parameters are $\kappa = 50$, $\delta = 0.2$ and $\varepsilon = 0.02$. 
5.4.3 Scaling of the tangency curves

We shall now present an approximate analysis to explain what is observed in this figure, namely the scaling of the two types of tangency curves and their codimension-2 meeting points as \( n \to \infty \).

As argued in §5.3, side tangencies correspond to extrema of the time taken for points on \( \Gamma(\beta) \) to reach the symmetric section \( S \). The flight time from \( y = +\infty \) to any point on \( \Gamma(\beta) \) is given by

\[
\Delta t = \int_{-\infty}^{-\delta} \frac{1}{\sqrt{Y(x; \lambda, c)}} dx \approx \int_{-\infty}^{-\delta} \frac{1}{\sqrt{-2x^3/3}} dx = \sqrt{6}/\delta.
\]

This is a constant offset independent of \( \lambda \) or \( c \). The flight time \( t(\lambda, c) \) from \( y = +\infty \) to the symmetric section \( y = 0 \) is then related to \( n \) by

\[
t(\lambda, c_{ex}) = n + \Delta t,
\]

where \( c_{ex} \) denotes the value of \( c \) such that \( t \) attains an extremum.

Consider a point \( P : (u, v) = (\theta, \beta + \kappa \theta^2) \in \Gamma(\beta) \). In terms of \((x, y)\), the displacement of \( P \) from \( \gamma_0 \) is

\[
(\Delta x, \Delta y) = \theta e_u + (\beta + \kappa \theta^2)e_v,
\]

where \( e_u \approx (1, -\sqrt{3\delta/2}) \) and \( e_v \approx (\sqrt{3\delta/2}, 1) \). The Hamiltonian \( c \) at \( P \) can be calculated similarly to Eq. (5.5). In this calculation we assume that \( \delta \ll 1 \) and take \( \lambda, \beta \) to be smaller still, viz. \( \lambda = O(\delta^2), \beta = o(\delta^{3/2}) \). The condition \( \theta \in [-\varepsilon, \varepsilon] \), where \( \varepsilon \ll \delta \), provides a restriction on \( \theta \). However, in order to describe the tangency between \( W^u(B) \) and \( W^s_{\text{sing}}(A_0) \)
as computed in Fig. 5.4(a) the curvature of $W^u(B)$ must exceed that of $W^s_{\text{sing}}(A_0)$ implying that $\kappa \delta^{1/2} \gg 1$. In the following we therefore assume that $\kappa \epsilon^2 \sim \beta = o(\delta^{3/2})$ and obtain

$$c(P) \approx c(\gamma_0) + \sqrt{\frac{2}{3} \delta^3 (\beta + \kappa \theta^2)} - \frac{1}{4} \delta \theta^2 - \lambda \theta + \text{h.o.t.},$$

where $c(\gamma_0) = \lambda \delta$. As $\theta$ varies in $[-\epsilon, \epsilon]$ the extremum of $c$, $c = \bar{c}$ say, is attained at $\theta/\delta = O((\lambda/\delta^2)(1/\kappa \delta^{1/2})) \ll 1$ and

$$\bar{c} \approx \lambda \delta + \sqrt{\frac{2}{3} \delta^3 \left( \beta - \frac{3 \lambda^2}{8 \kappa \delta^3} \right)} + \text{h.o.t.} \approx \lambda \delta + \sqrt{\frac{2}{3} \delta^3 \beta} + \text{h.o.t.} \quad (5.12)$$

Thus $\bar{c} = \bar{c}(\beta)$ and the extremum depends only on $\beta$. A tip tangency occurs for $c = \bar{c}$ with

$$t(\lambda, \bar{c}) = n + \Delta t, \quad (5.13)$$

where we now think of $\bar{c}$ as the unfolding parameter, instead of $\beta$. In particular we consider in the following the loci of the tangency curves in the $(\lambda, \bar{c})$ plane instead of the $(\lambda, \beta)$ plane. If we identify each point $P$ with its value, then the parabola $\Gamma(\beta)$ maps, in the case of outer (inner) tangency, into a double covering of the semi-infinite interval $I_c : [\bar{c}, \infty)$ $(I_c : (-\infty, \bar{c})$ on the $c$-axis.

Consider first the asymptotic scaling as $n \to \infty$ of the (approximate) codimension-2 meeting point $(\lambda_n^*, \bar{c}_n^*)$ of the $n$-th side and tip tangency curves. The above discussion shows that such points are defined by $c_{ex} = \bar{c}$, where $\bar{c}$ is a local extremum of $t(\lambda_n^*, c)$. This extremum must be the local maximum for $\lambda < 0$ as calculated in §5.3. Equation (5.11) is equivalent to Eq. (5.6) and leads to the asymptotic scaling

$$\lambda_n^* = -175.24(n + \Delta t)^{-4}, \quad \bar{c}_n^* = 1415.4(n + \Delta t)^{-6}. \quad (5.14)$$

This prediction is consistent with the numerical results in Fig. 5.5 in which it can be seen the codimension-two points converge algebraically on $\lambda = \beta = 0$.

Consider now the asymptotics of the side tangencies. At the $n$-th side tangency $(\lambda_n^*, \bar{c}_n^*)$, the extremum of $t$ at $c_{ex}$ is the same as above, such that $(\lambda_n^*, c_{ex}) = (\lambda_n^*, \bar{c}_n^*)$ except that $c_{ex}$ is now no longer equal to $\bar{c}$, but $c_{ex} \in \text{int}(I_c)$. This implies that, to leading order, the $n$-th side tangency occurs on the half-line

$$\lambda_n^* = \lambda_n^*, \quad \bar{c}_n^* \leq \bar{c}_n^* \quad (\text{outer}), \quad \bar{c}_n^* \geq \bar{c}_n^* \quad (\text{inner})$$

that extends from the codimension-2 point. Again this is consistent with the numerical observations in Fig. 5.5 in which the parabolic curves of side tangencies become closer to double covers of the horizontal lines $\lambda = \text{const.}$ as $n$ increases.

Finally, consider the asymptotics of the tip tangencies. Let $(\lambda_n^t, \bar{c}_n^t)$ be parameter values at which the $n$-th tip tangency occurs. The condition for a tip tangency, $c = \bar{c}$ where $\bar{c}$ is the extremum value of $c$ within $\Gamma$, combined with Eq. (5.13), requires that $\max_{c \in \mathbb{R}} t(\lambda_n^t, c) \geq n$. This implies that a tip tangency only occurs for $\lambda_n^t \geq \lambda_n^*$, which reduces to $\lambda_n^t \geq 0$ as $n \to \infty$. In the $n \to \infty$ limit Eq. (5.13) becomes $t(\lambda, c_{ex}^t(\lambda)) = \infty$, so
that the tip of $P$ must lie on the stable manifold $W^s(A)$ of the saddle equilibrium $A$. Thus the limiting curve for tip tangency is defined by $\tau$ being equal to the $H$-value of $A$, i.e.,

$$c^l_n(\lambda) \rightarrow \tau^l_\infty(\lambda) = \frac{4}{3} \lambda^{3/2} \text{ as } n \rightarrow \infty. \quad (5.15)$$

For large but finite $n$, Eq. (5.13) leads to two solutions for $\tau^l_n$ at $\tau^l_n < \tau^l_\infty$ and $\tau^l_n > \tau^l_\infty$. The former, denoted by $\tau_n^{lt}$, describes a trajectory to the left of $W^s(A)$ and corresponds to a tip tangency for hybrid homoclinic orbits. The latter, denoted by $\tau_n^{lt}$, describes a trajectory to the right of $W^s(A)$ and corresponds to the tip tangency for primary homoclinic orbits. The expressions for $\tau_n^{lt}$ and $\tau_n^{lt}$ can be calculated from the linearized flow around $A$,

$$z_+(t) = z_+(0) \exp(\Lambda t), \quad z_-(t) = z_-(0) \exp(-\Lambda t), \quad \Lambda = \sqrt{2\lambda^{1/4}}, \quad (5.16)$$

where $z_+$ and $z_-$ are the coordinates along $W^u(A)$ and $W^s(A)$, and $\Lambda$ is the unstable eigenvalue of $A$.

For primary homoclinic orbits the initial condition is $(z_+(0), z_-(0)) \sim (\Delta \tau_n, 1)$, where $\Delta \tau_n \equiv \tau^l_\infty - \tau^l_n > 0$ and we assume $\delta$ is an $O(1)$ constant as $\beta \rightarrow 0$. We can now solve Eq. (5.16) up to the time taken to intersect the symmetric section $S$, i.e., such that $z_+ = z_-$. Setting this time equal to $n$, we find that the $n$-th tip tangency for primary homoclinic orbits occurs at

$$\Delta \tau_n = \tau^l_\infty - \tau^l_n \sim \exp(-2\Lambda n).$$

For hybrid homoclinic orbits the initial condition is again $(z_+(0), z_-(0)) \sim (\Delta \tau_n, 1)$ where $\Delta \tau_n \equiv \tau_n - \tau^l_\infty > 0$, but this time the trajectory passes through the full neighborhood of $A$, i.e., the trajectory follows the positive $x$ component of the unstable manifold of $A$ before intersecting the symmetric section $S$ to the right of this separatrix at $(z_+(n), z_-(n)) \sim (1, \Delta \tau_n)$. Setting the time taken equal to $n$, we find that the $n$-th tip tangency for hybrid homoclinic orbits occurs at

$$\Delta \tau_n = \tau_n^{lt} - \tau^l_\infty \sim \exp(-\Lambda n).$$

To summarize, we have found the following scalings:

1. The (approximate) codimension-two points where the tip tangencies and side tangencies meet converge as $n \rightarrow \infty$ to $(\beta, \lambda) = (0, 0)$. The convergence is algebraic in $n$ with asymptotic scaling

$$\left(\beta_n^*, \lambda_n^*\right) \sim (n^{-6}, n^{-4}). \quad (5.17)$$

2. The curves of side tangencies converge as $n \rightarrow \infty$ to (a double covering of) the half-line

$$\{(\lambda, \beta) : \lambda = 0, s\beta > 0\}, \quad (5.18)$$

where $s = -\text{sign}(\kappa)$. The convergence is again algebraic in $n$ with asymptotic scaling $\lambda_n^* = \lambda_n^* \sim n^{-4}$ as $n \rightarrow \infty$. 

3. The curves of tip tangencies converge as \( n \to \infty \) to (a double covering of) the half-line

\[
\{(\lambda, \beta) : \beta = -\alpha \lambda, \lambda > 0\},
\]

where \( \alpha = \sqrt{3/(2\delta)} \) is a positive constant. The convergence to the line of tip tangencies corresponding to primary homoclinic orbits is exponential with asymptotic scaling

\[
\Delta \beta^{pt}_n \sim \exp(-2\sqrt{2}\lambda^{1/4}n),
\]

whereas the convergence of those corresponding to hybrid homoclinic orbits is also exponential but with asymptotic scaling

\[
\Delta \beta^{ht}_n \sim \exp(-\sqrt{2}\lambda^{1/4}n).
\]

In the context of homoclinic snaking, the above results suggest that while the folds on the primary and hybrid snakes due to tip tangencies both converge exponentially to the same limit, they converge from different directions and the exponent for the hybrid snake is half that of the primary snake. Note also that the loci of such tip tangencies are the same for outer and inner tangencies, although in one case each tangency creates a pair of homoclinic orbits as \( \beta \) increases, and in the other it destroys them. The dependence of the constant \( \alpha \) in Eq. (5.19) on the (arbitrary) parameter \( \delta \) is not a problem since \( \beta \) also depends on the choice of \( \delta \). If \( \tilde{\mathcal{F}} \), the extremal value of the normal form Hamiltonian along the segment \( \Gamma \), is used as a more natural choice of the unfolding parameter, the line (5.19) is replaced by the curve (5.15) that is independent of \( \delta \).

As pointed out above, homoclinic orbits to \( B \) in the 2D reversible map \( F \) correspond to localized states in a fourth-order reversible and Hamiltonian ODE for a scalar function \( u(x) \). Without loss of generality, we can regard the discrete time \( n \) in the former as being proportional to locations of maxima in \( u(x) \) in the latter. In this connection, the homoclinic orbits to \( B \) that take integer iterations to reach the symmetric section \( S \), as studied in \$5.3 \) and \$5.4 \), correspond to localized states with maxima in \( u(x) \) at \( x = 0 \). On the other hand, those localized states with minima in \( u(x) \) at \( x = 0 \) correspond to homoclinic orbits to \( B \) that take half-integer iterations to reach \( S \). As a result, the expressions for the \( n \)-th folds in the minima case are simply those in the maxima case with \( n \) replaced by \( n + 1/2 \). In other words, the folds on the minima branch(es) should interpolate between the folds on the maxima branch(es).

### 5.5 Applications: Example 1

The bifurcation results, although derived in the context of reversible maps, will now be used to describe the formation of localized patterns in PDE models. The analysis of solutions to variational PDEs often leads to spatial dynamics problems that are both reversible and conserve a first integral. If the spatial dynamics system is four-dimensional, then taking a Poincaré section within a level set of the first integral shows that the spatial dynamics can be described by a planar map. Moreover, the map is area-preserving and the Poincaré section can be chosen such that it is also reversible, thus putting us exactly in the framework of the previous analysis. See [182] for additional details.
The example presented in this Section is based on the Swift-Hohenberg (S-H) equation for a real field \( u \). In the best-known version, the nonlinear terms consist of simple powers in \( u \). However, attempts to derive the 2D S-H equation from weakly nonlinear Rayleigh-Bénard convection [65, 128] show that the nonlinear terms take the form of convolutions in Fourier space. Hence in this Section, we attempt to include additional nonlinear terms in the S-H equation such that the resulting system is still variational as well as spatially Hamiltonian (and reversible). These terms may be more relevant in the 2D case, but we will only focus on 1D with spatial dynamical variable \( x \). The primary goal is to study a variant of the standard snakes-and-ladders picture of homoclinic snaking [42], that results from interaction with a saddle-center periodic orbit as studied in the previous Sections.

### 5.5.1 The extended S-H equation

In order that the PDE remains quasi-linear, we shall consider terms with two \( x \) derivatives or none. The latter simply consist of \( u^i \) with \( i = 0, 2, 3, 5 \), if no higher powers of \( u \) are deemed necessary. The former, however, must take specific forms to preserve the Hamiltonian structure. To see this, we observe that an extra term \( T_j \) at \( j \)-th order in \( u \) descends from a corresponding extra term \( E_j \) in the spatial Hamiltonian \( H \) via the relation \( T_j u_x = \partial E_j / \partial x \). Thus \( E_j \) is \( (j + 1) \)-th order in \( u \) with two \( x \) derivatives, but the factor \( u_{xxx} \) (which would lead to a factor \( u_{xxxx} \) in \( T_j \)) is ruled out. Hence we must have \( E_j \equiv u^{j-1}u_x^2 \), which implies \( T_j = (j - 1)u^{j-2}u_x^2 + 2u^{j-1}u_{xx} \).

In the absence of the \( u \rightarrow -u \) symmetry, the conventional S-H equation possesses both quadratic \((u^2)\) and cubic \((u^3)\) nonlinearities. The inclusion of the additional terms \( T_2 = u_x^2 + 2uu_{xx} \) and \( T_3 = 2uu_x^2 + 2u^2u_{xx} \) leads to

\[
u_t = [r - (1 + \partial_{xx})^2]u + b_2u^2 - u^3 + \alpha(u_x^2 + 2uu_{xx}) + \gamma(uu_x^2 + u^2u_{xx}). \tag{5.22}\]

This system has spatial Hamiltonian

\[
H = \frac{1}{2}(r - 1)u_x^2 - u_x^2 - uu_xu_{xxx} + \frac{1}{2}u_{xx}^2 + \frac{1}{3}u_x^3 - \frac{1}{4}u^4 + \alpha uu_x^2 + \frac{\gamma}{2}u^2u_x^2, \tag{5.23}\]

and a candidate free energy density is

\[
\mathcal{L} = \frac{1}{2}r u_x^2 + \frac{1}{2}(u_{xx} + u)^2 - \frac{b_2}{3}u_x^3 + \frac{1}{4}u^4 + \alpha uu_x^2 + \frac{\gamma}{2}u^2u_x^2. \tag{5.24}\]

We remark that the variational structure implies the conservation law [124]

\[
\partial_x \mathcal{L} - \partial_x (u_x \mathcal{L}_{ux} + \mathcal{L}_{ux} \partial_x u_x - u_x \partial_x \mathcal{L}_{ux}) = 0. \tag{5.25}\]

The reader may indeed verify that \( H \) may be computed from \( \mathcal{L} \) by

\[
H = u_x \mathcal{L}_{ux} + \mathcal{L}_{ux} \partial_x u_x - u_x \partial_x \mathcal{L}_{ux} - \mathcal{L}. \tag{5.26}\]

The total free energy \( F \equiv \int \mathcal{L} dx \) is always non-increasing by definition, but we need to rearrange \( \mathcal{L} \) to manifest that \( F \) is bounded from below. In the \( \gamma > 0 \) case, completing squares and integrating by parts, we may rewrite \( \mathcal{L} \) as

\[
\mathcal{L} = \frac{1}{2}r u_x^2 + \frac{1}{2}(u_{xx} + u)^2 - \frac{b_2}{3}u_x^3 + \frac{1}{4}u^4 + \frac{\gamma}{2}(u + \frac{\alpha}{\gamma})^2u_x^2 + \frac{\alpha^2}{2\gamma}uu_{xx}. \tag{5.27}\]
Then we combine terms and complete squares again to obtain
\[
\mathcal{L} = \frac{1}{2}(r + \frac{\alpha^2}{\gamma} + \frac{\alpha^4}{4\gamma^2})u^2 + \frac{1}{2}(u_{xx} + (1 + \frac{\alpha^2}{2\gamma})u)^2 - \frac{b_2}{3}u^3 + \frac{1}{4}u^4 + \frac{\gamma}{2}(u + \frac{\alpha}{\gamma})^2u_x^2. \tag{5.28}
\]
Thus \( \mathcal{L} \) is the conventional expression (2.67) with adjusted coefficients plus a nonnegative term. In the \( \gamma < 0 \) case, \( \mathcal{F} \) is unbounded from below if we take, say, a localized state with inner oscillations \( u(x) \sim e^{-2\cos(\epsilon^{-1}x)} \) and let \( \epsilon \to 0 \). In the borderline case \( \gamma = 0 \), we integrate by parts in (5.24) thereby replacing \( uu_x^2 \) by \(-u^2u_{xx}/2\), and then complete squares to get
\[
\mathcal{L} = -\frac{1}{2}(r + \frac{|\alpha|}{\sqrt{2}} - |\alpha|)u^2 + \left(\frac{1}{2} - \frac{\sqrt{2}}{4} |\alpha| \right) (u_{xx} + \frac{1}{1 - |\alpha|/2}u)^2 - \frac{b_2}{3}u^3
\]
\[
+ \left(\frac{1}{4} - \frac{\sqrt{2}}{8} |\alpha| \right)u^4 + \frac{1}{4} |\alpha|(2^{1/4}u_{xx} - 2^{-1/4}\text{sgn}(\alpha)u_x^2)^2. \tag{5.29}
\]
In this expression, the coefficients of \( u_{xx}^2 \) and \( u^4 \) have been tuned to vanish simultaneously at \( |\alpha| = \sqrt{2} \). For \( |\alpha| < \sqrt{2}, \mathcal{F} \) is bounded from below as in the \( \gamma > 0 \) case. For \( |\alpha| > \sqrt{2} \), \( \mathcal{F} \) is unbounded from below if the localized state has inner oscillations taking the form of a suitable Weierstrass elliptic function, such that both \( u_{xx} \) and \( u^2 \) are large while the last term in (5.29) vanishes identically.

### 5.5.2 Snakes mating

In this section, we study the effect of \( T_3 \) on the snaking branches. Numerical continuation is performed with \( r \) as the active bifurcation parameter, and presented at slices of constant \( \gamma \). The other parameters are fixed at \( b_2 = 1.8, \alpha = 0 \).

The primary and hybrid snakes

As first discovered in Ref. [42], the bifurcation diagram for localized states in the quadratic-cubic S-H equation exhibits the “snakes-and-ladders” structure for a suitable range of \( b_2 \), including \( b_2 = 1.8 \). This structure persists under perturbation by terms like \( T_3 \) which respect the symmetry \( x \to -x \). As shown in Fig. 5.6(a) for \( \gamma = -0.04 \), there are two branches of localized states bifurcating subcritically from \( r = 0 \), referred to as \( L_0 \) and \( L_\pi \). The localized states on \( L_0 \) (\( L_\pi \)) have local maxima (minima) in the middle. As \( r \) is decreased, either branch enters into a snaking region where it undergoes an infinite number of saddle-node bifurcations. A localized state in this region resembles a periodic wave train embedded in a zero background, as shown in Fig. 5.6(b). Each time the branch goes through a C-shaped segment, an extra roll is created on either end of the localized state by continuous deformation of the front (compare panel (i) with (ii) or (iii) with (iv)).

The localized states on \( L_0 \) differ from those on \( L_\pi \) by exactly one wavelength in the middle (compare panel (i) with (iii) or (ii) with (iv)).

In addition to the above pair of snaking branches (henceforth referred to as primary), there is a pair of hybrid snaking branches that bifurcate into the same snaking region from \( r = -\infty \), as shown in Fig. 5.7(a) and Fig. 5.8(a). They will be referred to as \( L_0^h \) and

\[\text{(Continued...)}\]
Figure 5.6: (a) Bifurcation diagrams showing the two primary snaking branches ($L_0$ and $L_\pi$) at $\gamma = -0.04$. The “ladders” are not shown here. (b) Sample profiles on the 6-th C-shaped segment of the $L_0/L_\pi$ branch are shown in (i-ii)/(iii-iv).

$L_h^b$ since similar to $L_0$ and $L_\pi$, the localized states on them have respectively maxima and minima in the middle, as shown in Fig. 5.7(b) and Fig. 5.8(b). But in contrast to the primary snaking branches, a localized state on either hybrid snaking branch possesses a large amplitude and narrow core. As $\gamma$ approaches 0, the core becomes a singularity at $x = 0$ so the hybrid snaking branches no longer persist in this limit, but for $\gamma$ close to 0 one may be able to capture the shape of the core by asymptotic methods. Indeed, this shape hardly changes as $r$ varies for either $L_h^b$ or $L_\pi^h$, so the only essential difference between these two branches is that the localized states on $L_h^b$ have a one-peak core while those on $L_\pi^h$ have a two-peak core. For either hybrid branch, before it enters into the snaking region, the localized state consists of the core embedded in a zero background. After it does, extra rolls are created around the core by the same growth mechanism as for primary branches.

Mating and breakup into isolas

As $\gamma$ is further decreased, the primary snake $L_0$ ($L_\pi$) and the hybrid snake $L_h^b$ ($L_\pi^h$) mate with each other and create a stack of isolas, as shown in Fig. 5.9(a) (Fig. 5.10(a)) for $\gamma = -0.64$. In either case, the $i$-th isola appears to be formed by reconnecting the $(i+2)$-th C-shaped segment of the primary snake to the $(i+1)$-th C-shaped segment of the hybrid snake ($i \in \mathbb{N}$), while the first two C-shaped segments of the primary snake appear to reconnect to the first C-shaped segment of the hybrid snake, thus creating a non-snaking branch extending from $r = 0$ to $r = -\infty$. But in fact two more C-shaped segments are created on the lower part of the hybrid snake as $\gamma$ is decreased from $-0.04$ to $-0.64$, so that the $i$-th isola corresponds to the $(i-1)$-th C-shaped segment on the hybrid snake at $\gamma = -0.04$ instead of the $(i+1)$-th.

As shown in Fig. 5.9(b) and Fig. 5.10(b), a localized state on an isola resembles its “ancestor” on the mating snakes, apart from the fact that the core on the hybrid snake is much less spiky on the isola and may be more properly called a defect. Thus henceforth
Figure 5.7: (a) A bifurcation diagram showing the hybrid snaking branch $L^h_0$ at $\gamma = -0.04$. (b) Sample profiles on the 6-th C-shaped segment of the $L^h_0$ branch are shown in (i-ii), with their cores and fronts featured in (iii-iv).

Figure 5.8: (a) A bifurcation diagram showing the hybrid snaking branch $L^h_\pi$ at $\gamma = -0.04$. (b) Sample profiles on the 6-th C-shaped segment of the $L^h_\pi$ branch are shown in (i-ii), with their cores and fronts featured in (iii-iv).
we shall refer to the C-shaped segment reminiscent of the primary (hybrid) snake as the uniform (defect) segment. During a single traversal of a higher isola in one of the two possible directions, an extra roll is added on either end through the uniform segment, but subsequently this addition is undone through the defect segment to close the loop.

5.5.3 Explanation using spatial dynamics

The above results can be explained by treating the steady state of the extended S-H equation (5.22) as a 4D reversible ODE in the dynamical variable $x$. To facilitate the discussion, we will denote the left and right limits of the snakes or isolas by $r_-$ and $r_+$ (both are functions of $\gamma$), and denote by $\gamma^*$ the value of $\gamma$ at which the primary and hybrid snakes mate. In this 4D reversible ODE, the unstable manifold $W^u(0)$ of the zero background is 2D and the center-stable manifold $W^{cs}(P)$ of a periodic state $P$ is 3D. The localized states inside the snaking region are created by the codimension-0 intersections between $W^u(0)$ and $W^{cs}(P)$. In particular, the limits of the snaking region are determined by the codimension-1 tangencies between $W^u(0)$ and $W^{cs}(P)$. Because the fronts between 0 and $P$ are identical for the primary and hybrid snakes, they necessarily have the same snaking limits.

As for the isolas, we observe that $r_+$ for $\gamma < \gamma^*$ corresponds to where the defect bifurcates from the uniform wave train, which implies that the periodic state undergoes a saddle-center bifurcation precisely at $r_+$. The origin of this bifurcation is illustrated in Fig. 5.11. Because (5.22) is spatially Hamiltonian, the collection of periodic states which can connect to the zero background must satisfy $H = 0$. In the case $\gamma = 0$, it is known that these periodic states form part of a branch that bifurcates subcritically from $r = 0$. This picture persists when $\gamma$ is decreased to $-0.4$, as shown in Fig. 5.11(a), where this periodic branch is labeled $P_1$. In addition to $P_1$, there exists another periodic branch $P_2$ that connects to $r = -\infty$. As $\gamma$ is further decreased to $-0.44$, $P_1$ and $P_2$ reconnect into two new periodic branches $P_3$ and $P_4$, as shown in Fig. 5.11(b). The collection of periodic
Figure 5.10: (a) A bifurcation diagram showing the stack of isolas created from mating between the $L^\pi$ and $L^h_\pi$ snakes at $\gamma = -0.64$. Only the first 9 figure-8 isolas are shown. (b) Sample profiles on the 9-th figure-8 isola are shown in (i-iv).

states on the snaking branches, which previously lies between the 1st and 2nd folds of $P_1$ (denoted by $r_1$ and $r_2$), now lies between $r_1$ and the newly created 2nd fold of $P_3$ (denoted by $r_3$). As $\gamma$ is decreased still further, $r_3$ decreases until at $\gamma = \gamma^*$, $r_3$ meets $r_+$ and causes the primary and hybrid snakes to mate and break up into the stack of isolas whose right limit $r_+$ is always identical to $r_3$. For $\gamma < \gamma^*$, $r_3 = r_+ \text{ decreases until eventually } r_- \text{ meets } r_+$ and destroys the isolas.

A 4D reversible ODE with a Hamiltonian (like (5.22)) can be reduced to a 2D reversible map (denoted by $F$). In $F$, the zero background and the periodic state become the pair of fixed points denoted respectively by $B$ and $A$. As shown in Fig. 5.2, when both equilibria are hyperbolic, the unstable manifold $W^u(B)$ of $B$ and the stable manifold $W^s(A)$ of $A$ are both 1D. Inside the snaking region $r \in (r_-, r_+)$, $W^u(B)$ intersects $W^s(A)$ transversely to the left of $A$ and creates a codimension-0 heteroclinic tangle. To the right of $A$, $W^s(A)$ and $W^u(A)$ form a homoclinic orbit to $A$ which intersects $S$ at a point denoted by $\bar{A}$. The intersections between $W^u(B)$ and $S$ fall into two categories, namely those closer to $A$ and those closer to $\bar{A}$. These correspond respectively to the localized states on the primary and hybrid snaking branches.

As $r$ varies, the heteroclinic tangle can be destroyed by either an inner tangency ($W^u(B)$ is tangent to and below $W^s(A)$) or an outer tangency ($W^u(B)$ is tangent to and above $W^s(A)$). The former (latter) is responsible for creating the snaking limit $r_+$ ($r_-$), which can be inferred as follows for primary snakes. As the outer tangency is approached, the intersections $W^u(B) \cap S$ near $A$ are pairwise destroyed starting from those closer to $B$, with no such intersections left when the outer tangency is reached. This is precisely the expected behavior as the left limit $r_-$ is approached. In contrast, the intersections $W^u(B) \cap S$ near $A$ are destroyed only after the inner tangency is reached, starting from those further from $B$. This is again consistent with the expected behavior as the right limit $r_+$ is approached. In summary, as one traces either primary snake upwards, both limits $r_\pm$ are approached in the direction of transforming the manifold configuration from inner
tangency to outer tangency. In the case of hybrid snakes, the reasoning is analogous and the conclusion is that as one traces either hybrid snake upwards, both limits $r_{\pm}$ are again approached in the same direction but this direction is opposite to that of the primary snakes. The stack of isolas is created from the primary and hybrid snakes when the hyperbolic fixed point $A$ annihilates the elliptic fixed point inside the homoclinic orbit to the right of $A$ via a saddle-center bifurcation at $r_3$, before the heteroclinic tangle deforms into the inner tangency configuration.

Thus the theory developed in §5.3 and 5.4 applies directly to the present setting and leads to predictions on the fold accumulation behavior. Specifically, every second fold resulting from side tangency on the stack of isolas should accumulate algebraically, and every fold resulting from tip tangency should accumulate exponentially with those involving hybrid localized states accumulating more slowly than those involving primary localized states. Although we will not compare these predictions against numerics in this Section, we mention in this connection a periodically forced CGL equation studied by Bortolozzo et al. in Ref. [31], where the slow exponential convergence of successive folds on the hybrid snake has been mistaken as slanted snaking in the spirit of Ref. [70]. Recently, asymmetric localized states associated with the hybrid snakes and the stack of isolas have also been found to form ladders in a variant of the Bortolozzo example [48], but an explanation for these structures using spatial dynamics is still lacking.

5.6 Theory for non-conservative reversible systems

In systems with a conserved Hamiltonian the Hamiltonian can be used to reduce a 4D reversible conservative ODE to a 2D reversible map. In this Section we turn to a 4D reversible ODE in which no underlying conserved Hamiltonian is present. By taking an appropriate Poincaré section $\Sigma$, we can reduce the system to a 3D reversible map. In this map we can again study the convergence of side tangencies near an analogue of the saddle-center bifurcation. However, for this purpose the theory in §5.3 requires modification, as
discussed next.

A periodic wavetrain in a purely reversible PDE will be a stationary state whenever it is invariant under the reflection $x \mapsto -x$. This wavetrain may undergo an Eckhaus instability as parameters are varied. In the spatial dynamics approach this bifurcation is equivalent to a reversible saddle-center bifurcation of a periodic orbit [99, 129]. Even in the absence of a global conserved quantity, the normal form near such a bifurcation is integrable and can be represented by the same normal-form as the Hamiltonian case studied in §5.2 with $H$ and $\lambda$ playing the role of the two conserved quantities. The dynamics of this normal form can be represented in a three-dimensional Poincaré section which contains a two-dimensional symmetric section $S$, see Fig. 5.12, and by the same arguments as in §5.2 can be represented as the time-one map of an equivalent flow.

As before we assume the existence of a distant equilibrium $B$ with 2D stable and unstable manifolds, and consider iterations of a segment $\Gamma$ of the unstable manifold of $B$ under the 3D map. Within the normal form flow near the saddle-center, we consider the family $\Omega(\lambda)$ of 2D invariant submanifolds corresponding to $\lambda =$const. on which the dynamics are equivalent to the 2D saddle-node normal form map with parameter $\lambda$. The preceding analysis reveals that for given $\lambda \leq 0$, $\Omega(\lambda)$ is foliated by trajectories $M(\lambda, c)$ parametrized by the “energy” $c$ of the normal form Hamiltonian. In particular, the $\lambda < 0$ dynamics exhibit self-similarity in the sense that the time for a distant point to reach the symmetric section along $M$ scales as

$$t(\lambda, c) = \tau(\chi)(-\lambda)^{-1/4}, \quad \text{where } \chi = c(-\lambda)^{-3/2}.$$  

The function $\tau(\chi)$ peaks at $(\chi_0, \tau_0) = (0.61015, 3.6384)$ and decays to zero as (see §5.3)

$$\tau(\chi) \sim |\chi|^{-1/6} \quad \text{as } \chi \to \pm \infty.$$  

On the other hand, the absence of a global first integral implies that $\Gamma$ generically does not lie in any single $\Omega(\lambda)$, see Fig. 5.12. The control parameter that determines the relative position between $\Omega(\lambda)$ and $\Gamma$ will be denoted by $\gamma$.

To analyze the resulting situation we define the collection of slowest trajectories (corresponding to $\chi = \chi_0$) as the 2-manifold $\Upsilon := \{M(\lambda, c_0(\lambda)) : \lambda \leq 0\}$. By construction, we are interested in the critical parameter value $\gamma_\infty$ at which $\Gamma$ intersects $\mathcal{Y}$ at $\lambda = 0$, with this intersection corresponding to $t = \infty$. Let us define $\Delta \gamma \equiv \gamma_\infty - \gamma$. Then for $\gamma < \gamma_\infty$, we suppose that $\Gamma \cap \Upsilon$ lies at the point $\lambda_\Gamma$. In the generic case, we expect that $\lambda_\Gamma \propto \Delta \gamma$ to leading order, specifically $\lambda_\Gamma = -l\Delta \gamma$, for some $l > 0$. The correction to this linear relation for $c \neq c_0$ can be calculated by factoring out the direction along the trajectory and considering the dynamics in the $(\lambda, c)$ plane. In such a plane, we can write leading-order expressions for the curves $\Upsilon$ and $\Gamma$:

$$\Upsilon := \{(\lambda, c) : c = \chi_0(-\lambda)^{3/2}\}$$

$$\Gamma := \{(\lambda, c) : \lambda - c \tan \phi = -l\Delta \gamma, l > 0\},$$  

where $\phi$ describes the deviation angle from the vertical axis $\lambda = 0$. 
Figure 5.12: The family of invariant submanifolds \( \Omega(\lambda) \) near an Eckhaus bifurcation. On each \( \Omega(\lambda) \) the dynamics are equivalent to a saddle-center normal form. Also shown is the unstable manifold segment \( \Gamma \) which generically does not lie within a single \( \Omega(\lambda) \).

In the conservative case the existence of a global first integral implies that \( \phi = 0 \) and hence that \( \lambda \propto \Delta \gamma \). In the general reversible case we expect that \( \phi \neq 0 \). In this case the intersection occurs at

\[
-\lambda = l\Delta \gamma - \chi_0 \tan \phi l^{3/2} (\Delta \gamma)^{3/2}.
\]

Finally, after approximating the iterate number \( n \) of the underlying Poincaré map by the continuous time \( t \) in the normal form and recalling that \( t = \tau_0 (-\lambda)^{-1/4} \), we arrive at the prediction

\[
n = \tau_0 \left( l^{-1/4} (\Delta \gamma)^{-1/4} + \frac{1}{4} \chi_0 \tan \phi l^{1/4} (\Delta \gamma)^{1/4} \right),
\]

or, equivalently,

\[
n = k_1 \Gamma + k_2 / \Gamma, \quad k_1 > 0,
\]

where \( \Gamma \equiv (\Delta \gamma)^{-1/4} \). Thus the leading order correction, as measured by \( k_2 \), provides a measure of the angle \( \phi \).

5.6.1 Example 2: The 1 : 1 forced CGL equation

In this section we discuss primary and hybrid localized states in the context of the 1 : 1 forced CGL equation (1.11). The steady state of this PDE, when written in terms of \( A \equiv U + iV \), is a 4D reversible ODE as in Example 1, but here there is no conserved Hamiltonian. In Chapter 3 we observed a new mechanism for the growth of localized structures as one follows the solutions through parameter space. In this mechanism, referred
to as defect-mediated snaking (DMS), the localized structure grows along a single snaking branch (denoted by $L_0$) by repeated splitting of the central cell instead of adding cells at either end as in standard snaking. This snaking mechanism is shown in Fig. 3.4. In this figure, the segment containing primary (hybrid) localized states are shown as solid (dashed) lines. As shown in Fig. 3.5, the limits of the observed pinning region are defined by Eckhaus bifurcations of the periodic wavetrain whose wavenumber matches the local wavenumber within the localized structure at this parameter value. As a result all the folds on the $L_0$ branch correspond to side tangencies and the folds corresponding to tip tangencies characteristic of standard snaking are absent. This fact is reflected in the profiles along the branch shown in Fig. 3.4(b). The figure shows the succession of solution profiles as one follows $L_0$ to larger $L_2$ norm, and reveals that $L_0$ consists in succession of all four solution types, primary and hybrid localized states with maxima and minima at $x = 0$, in contrast to Example 1 in which we found a stack of isolas consisting of branch segments with primary and hybrid localized states.

In Fig. 5.13 we show a detail of the $L_0$ branch, focusing on the rate of accumulation of successive $L_0$ folds at the left edge of the pinning region (Fig. 5.13(a)). To leading order we expect the successive folds to follow the predicted scaling relation (5.30)

$$n = k_1 \sigma + k_2 / \sigma, \quad \sigma \equiv |\gamma_n - \gamma_\infty|^{-1/4}, \quad (5.31)$$

where $n$ is the index of the fold (odd for left folds), $\gamma_n$ is the location of the $n$-th fold and $\gamma_\infty$ is the location of the saddle-center corresponding to the Eckhaus boundary. Since measurements are made for finite $n$ we fit our numerical results to

$$n = k_1 \sigma + k_2 / \sigma + n_0, \quad (5.32)$$

where $n_0$ is a constant offset corresponding to $\Delta t$ in Eq. (5.11). In order that $n_0 \sim O(1)$ we have added an extra term $w n_0^2$ to the sum of squared errors in $n$ to be minimized, where the weight $w$ has been chosen as $w = 1/64$. The result is shown in Fig. 5.13(b) with the best fit parameters $k_1 = 1.96$, $k_2 = -44.5$, and $n_0 = -1.06$. As is consistent with the remark ending §5.4, the accumulation of every fold (i.e. $n$ in steps of 2) follows the scaling relation equally well, even though the localized state at the $n$-th fold with $n = 4m - 1$ ($n = 4m - 3$), $m \in \mathbb{N}$, has a local minimum (maximum) in $V(x)$ at $x = 0$.

Figure 5.13(c) compares the numerically determined slope $n'(\Gamma)$ with the prediction from Eq. (5.31). Although the limiting value $\gamma_\infty$ cannot easily be measured directly, we have been able to adjust its value to make the slope $n'(\Gamma)$ almost constant for larger $\Gamma$ outside the range of this figure. The decrease in the slope $n'(\Gamma)$ as $\Gamma$ increases is visible for smaller $\Gamma$ used in this figure, and the dependence appears to be captured reasonably well by Eq. (5.31).

To gain further insight into the spatial dynamics of DMS, we have also computed the 2D unstable manifold $W^u(A^-)$ intersected with the 3D Poincaré section $\Sigma \equiv \{(U, V, U_x, V_x) : U_x = 0\}$. Figure 5.14 shows a collection of segments comprising $W^u(A^-) \cap \Sigma$, which are computed near the $n$-th fold for increasing $n$, at a fixed value of $\gamma$ slightly to the left of the DMS region. The two intersections between a segment and the symmetric section $V_x = 0$, if present, create the pair of primary and hybrid localized states on the DMS branch at the corresponding $n$ and $\gamma$. As shown in Fig. 5.14(a), the collection of segments consists
of two clusters with the left (right) cluster consisting of those segments with $n = 4m - 1$ ($n = 4m - 3$), $m \in \mathbb{N}$, and these segments create localized states with a local minimum (maximum) in $V(x)$ at $x = 0$. Away from the symmetric section, the segments from the same cluster turn out to converge to a single trajectory, but more relevant for our purpose is the behavior near the symmetric section as shown in Fig. 5.14(b,c). For very small $n$, we expect the shape of the segment to undergo little change as $\gamma$ varies near $\gamma_\infty$. At fixed $\gamma$, the evolution of points on this segment as $n$ increases is then gradually taken over by the slow dynamics near the saddle-center. Finally, the value of $n$ at which the last point crosses the symmetric section is the index of the fold that occurs at this chosen value of $\gamma$. This picture of fold accumulation is qualitatively the same as in Example 1, despite the fact that one cannot reduce the 4D non-Hamiltonian ODE, namely the steady state of Eq. (1.11), to a 2D map. The existence of a slowest trajectory is also evident from the formation of sharp turns in segments with larger $n$, as shown in Fig. 5.14(d/e).

5.7 Discussion

In this chapter we have studied the properties of localized states associated with a heteroclinic cycle between a trivial state and a periodic state of saddle-center type. Our analysis, using appropriate return maps, demonstrates the presence of two types of localized states near this cycle, primary homoclinics and hybrid homoclinics. Each of these states is reflection-symmetric, and has either a maximum or a minimum at the point of symmetry. The primary homoclinics represent the localized states familiar from standard snaking scenarios, i.e., a periodic wavetrain connected to the trivial state by a pair of fronts at either end. The hybrid states differ in having an extra defect or peak in the center and so resemble
the states present in defect-mediated snaking. We have seen that when these states extend beyond the saddle-center bifurcation, corresponding to an Eckhaus instability, the folds on the corresponding snaking branch accumulate algebraically as $n^{-4}$. In addition we have explored, by analyzing an appropriate codimension-two problem, how the branches of primary and hybrid states reconnect. We have checked the predicted algebraic convergence of folds in two numerical examples and in both cases found reasonable agreement with the exponent predicted by our theory. We have not, however, attempted to confirm quantitatively the predictions of the codimension-two analysis. On the other hand, our numerical exploration of Example 1 does show that with increasing parameter $\eta$ the primary and hybrid snaking branches do indeed reconnect, forming a stack of figure-eight isolas. As one follows an isola one passes in succession through a primary state to a hybrid state and back again. In fact, there are two stacks of isolas one corresponding to states with maxima at $x = 0$ and the other to states with minima at $x = 0$. In contrast, in Example 2, the 1:1 forced complex Ginzburg-Landau equation, we found instead a single snaking branch of localized states which visits each of the above four states in succession. The reason for this difference can be traced to the fact that in Example 1 only one boundary of the snaking region for isolas corresponds to a side tangency while in Example 2 both boundaries correspond to Eckhaus points. In the spatial dynamics point of view these Eckhaus points correspond to saddle-center points that yield side tangencies [129]. It may also be the case that the cut taken in parameter space influences which kind of global bifurcation diagram is seen, much as in figs. 2 and 6 of Ref. [37].

As already mentioned, near the folds the solution typically changes from one resembling a primary localized state to one resembling a hybrid state. Asymptotically near such folds one expects to find bifurcations to non-symmetric localized states. Such states
should therefore be present outside the bistability region, in the regime of algebraic accumulation, and it would be of interest to study the properties of these states in this regime. In this connection we emphasize the crucial role played by the local wavenumber within the localized structure. This wavenumber, \( k(r) \), say, depends on the value of the parameter \( r \) within the pinning region \( (r_- < r < r_+) \), sometimes strongly. For our scenario to apply the edge of the pinning region must correspond to values \( r = r_\pm \) such that the periodic wavetrains with wavenumber \( k_\pm = k(r_\pm) \) are at threshold for Eckhaus instability. Thus it is the parameter dependence of the local wavenumber that ultimately determines the width of the pinning region. This dependence in turn follows from a conserved Hamiltonian [42] if such a Hamiltonian exists, but in systems with no conserved Hamiltonian the wavenumber selection process remains to be understood.

We wish to mention that qualitatively similar hybrid states exist in other systems but their origin may be quite different. For example, in the quadratic-cubic Swift-Hohenberg equation there are regimes (see fig. 20 of Ref. [42]) containing a large amplitude central oscillation embedded in a background of small oscillations that are in turn connected at either end to the trivial state. These states differ from the ones considered here in that the central oscillation is of large amplitude. Indeed, the states identified in [42] form as a result of an overlap of two distinct pinning regions, with the large amplitude localized states in the first pinning region falling under the influence of a second Maxwell point and ultimately forming a heteroclinic cycle involving the second, small amplitude oscillation. These states cannot be readily related to the unfolding of a saddle-center although the large amplitude central oscillation may be viewed as being due to a large amplitude homoclinic to the small oscillations. In that system such homoclinics are associated with heteroclinic cycles between two different periodic orbits, and the present chapter suggests possible approaches to studying the associated defect or hybrid states.

Finally, and perhaps most importantly, we have not studied the temporal stability properties of the localized states, primary, hybrid and non-symmetric, in the parameter regime where the spatially periodic state undergoes an Eckhaus instability. In an infinite wavetrain this instability leads to phase slips and a change in the wavenumber of the wavetrain. This process typically continues until the wavenumber falls again in the Eckhaus stable range and the wavetrain stabilizes. This picture is modified in the presence of fronts on either side of the structure in ways that are not fully understood. In addition, near the folds the fronts undergo a depinning transition [42] and this instability can interact with the Eckhaus instability to modify the speed of the resulting fronts. This interesting topic cannot be addressed within the spatial dynamics approach adopted here but will be studied elsewhere.
Chapter 6

Localized phase-winding states

6.1 Introduction

As discussed in the previous chapters, many pattern forming systems contain a multiplicity of stationary spatially localized states. The spatial dynamics of such systems is characterized by homoclinic orbits organized around heteroclinic cycles between a fixed point and a periodic orbit. These states occupy an extended region in parameter space called the snaking region. A similar behavior is present in systems which contain heteroclinic orbits between two fixed points, though in this case the solutions collapse to a single point in parameter space. In this chapter we analyze a novel hybrid example involving phase winding states (PWS) in the 1:1 forced complex Ginzburg-Landau equations (FCGLE), which combines aspects from both types of behavior: homoclinic snaking associated with a heteroclinic connection between a fixed point and a periodic orbit which collapses to a point in parameter space.

Phase-winding states (PWS) are spatially periodic states in the FCGLE with $U = \Re(A)$ and $V = \Im(A)$ out of phase as a function of $x$. They generically travel but those with vanishing speed can form the inner part of stationary localized states. Localized PWS of even or odd parity were first discovered in the 2:1 FCGLE [35]. In this chapter we again focus on the 1:1 FCGLE, in which there are two flavors of even localized PWS. In either case, localized PWS are of codimension-1 and thus undergo collapsed snaking. As the collapsing point is approached, the growth of localized PWS is mediated by a central pacemaker defect.

This chapter is organized as follows. §6.2 studies localized phase-winding states from the perspective of spatial dynamics. §6.3 addresses aspects of temporal dynamics related to phase-winding states. §6.4 lists a number of open questions.

Before presenting the numerical results, we illustrate in Fig. 6.1 the typical profiles of solutions studied in §6.2. These solutions are all steady states and can be interpreted using spatial dynamics as follows. A phase kink (PK) is an asymmetric homoclinic orbit to the upper equilibrium $A^+\ (\text{Fig. 6.1(a)})$. A phase-winding state (PWS) is an asymmetric periodic orbit (Fig. 6.1(b)). A localized phase-winding state (LPWS) is a heteroclinic cycle between the lower equilibrium $A^-$ and a phase-winding state. The LPWS fall into two flavors, referred to as primary and secondary LPWS. The former exist at relatively small
forcing $\gamma$ and thus possess PWS with an almost circular phase portrait in the $(U, V)$ plane (Fig. 6.1(c)). The latter exist at larger $\gamma$ and thus the phase portrait of its associated PWS resembles a distorted circle (Fig. 6.1(d)).

### 6.2 The 1:1 case

Arguably the simplest localized state (LS) in a bistable system with two stable equilibria $A^{\pm}$ consists of a pair of fronts between $A^{\pm}$ assembled back-to-back, or equivalently an inner equilibrium $A^{+}$ embedded in an outer equilibrium $A^{-}$. More complicated LS can be created if a bifurcation occurs on one of the two equilibria (e.g. $A^{+}$) as this codimension-1 front between $A^{\pm}$ is followed in two parameters. As shown in Chapter 3, in the case of a supercritical Turing bifurcation, a collection of periodic states is created near the bifurcation point and the resulting bifurcation diagram for the branch of LS is known as defect-mediated snaking [129]. In this chapter we focus on the case of a saddle-node bifurcation, which leads to one branch of symmetric pulses that bifurcates into a pair of phase kinks (PK) (§6.2.1). A new type of LS that we refer to as localized PWS (LPWS) is born when multiple pairs of PK replace the inner plateau of $A^{+}$ (§6.2.2). After introducing bifurcation of PK, bifurcation of PWS as building blocks of LPWS is then studied (§6.2.3). Bifurcation of LPWS themselves follows, highlighting their growth mediated by a central pacemaker defect (§6.2.4). In this section we fix the parameters $\alpha = 4$, $\beta = 1$, and $\mu = 1$ and vary $\nu$ and $\gamma$.

#### 6.2.1 Phase kinks bifurcating from saddle-nodes

The branch following from the upper saddle-node bifurcation at $\nu = 1.2$ is shown in Fig. 6.2. The weakly nonlinear solution (panel (i)) becomes more localized and larger in amplitude towards the first fold (panel (ii)), as is typically seen in such continuations. However, before reaching a well-defined second fold, the branch appears to linger at a codimension-1 point denoted by $\gamma_{1}^{PK}$. At $\gamma_{1}^{PK}$, the solution resembles a pair of phase kinks.
Figure 6.2: The branch following from the saddle-node bifurcation ($\alpha = 4$, $\beta = 1$, $\mu = 1$, and $\nu = 1.2$). (a) Bifurcation diagram in terms of the $L^2$-norm $N$. The black and blue curves are respectively the branches of equilibria and localized states. (b) Sample solution profiles plotted for $U$ and $V$ as functions of $x$ (left) and on the $(U,V)$ phase plane (right). On the left panels $U$ and $V$ are respectively shown in red and green. On the right panels the red/green/blue dot marks the location of $A^+/A^0/A^-$. placed back-to-back (panel (iii)), which migrate apart as the branch is followed. A closer inspection of the bifurcation diagram (inset of Fig. 6.2(a)) reveals that the branch slightly overshoots before reaching $\gamma_{1PK}^+$.  

6.2.2 Interaction between front and saddle-node

As shown in Fig. 6.3, the codimension-1 fronts between $A^\pm$ bifurcate from a codimension-2 cusp point and their existence curve $\gamma^*(\nu)$ can be traced out as long as both $A^\pm$ exist. However, $A^+$ undergoes a saddle-node bifurcation when the continuation reaches $\gamma_{SN}^+$ (panel (i)) at $\nu = \nu_{SN}^*$. At this point the branch folds back such that the $A^+$ plateau is replaced by the symmetric pulse (panel (ii)) in Fig. 6.2. Even though both $\nu$ and $\gamma$ are varied here, this pulse subsequently bifurcates into a pair of phase kinks (panel (iii)) at $\nu = \nu_{PK}^*$, precisely as in Fig. 6.2 since the bifurcation diagrams in $\gamma$ are qualitatively the same for the range of $\nu$ covered. As long as a phase kink interacts with neighboring structures weakly through its exponentially decaying tails, its profile resembles one on an infinite domain so that the continuation appears to be “stuck” at $\nu_{PK}$, which is just what happens as either phase kink migrates towards its closest bounding front.  

The subsequent continuation can be viewed at http://www.youtube.com/watch?v=1koPP0Nd_KU or Fig. 6.4. The plot of $\nu$ as a function of the continuation step consists of a series of bump-plateau pairs. At the $n$-th bump, as $\nu$ increases from $\nu_{PK}^*$ to $\nu_{SN}^*$ and then decreases back to $\nu_{PK}^*$, $n$ phase kinks compress into the bounding front and then dilate into $n$ phase kinks again, while in the meantime the central piece of $A^+$ is replaced by a symmetric pulse embedded in $A^+$ via branch switching at $\nu_{SN}^*$. At the $n$-th plateau, as $\nu$ gets “stuck” at $\nu_{PK}^*$, the $n$ phase kinks dilate and collide with the central symmetric pulse
converting it to a new pair of phase kinks, and the resulting \((n + 1)\) phase kinks compress towards the bounding front to start a new cycle. As \(n\) increases, the wavelength of the phase kinks decreases when \(\nu_{PK}^*\) is reached, and the spatial extent of \(A^+\) also decreases when \(\nu_{SN}^*\) is reached. As a result, the continuation spends less and less “time” bouncing between \(\nu_{PK}^*\) and \(\nu_{SN}^*\), and eventually increases \(\nu\) past \(\nu_{SN}^*\) monotonically. The corresponding solution profiles are referred to as localized phase-winding states (LPWS), where phase-winding states (PWS) refer to finite-wavelength periodic states reminiscent of phase kinks. As \(\nu\) increases, the growth of LPWS is mediated by a central pacemaker defect.

In summary, LPWS is created by a spatial analogy of the familiar sniper bifurcation in the time domain [163]. This state has a number of unusual properties. It is composed of a pair of PWS, i.e. spatially periodic states with a phase shift between the real and imaginary parts that are “glued” together at the central defect in such a way that the complete solution retains reflection symmetry. Although time-independent constant amplitude periodic states with this type of phase shift may exist in systems like the FCGLE they are present at isolated parameter values only; at all other parameter values such states are expected to drift in time, i.e. to take the form of a rotating wave.

6.2.3 Phase winding states (PWS)

In this section, we study the bifurcation structures of PK and PWS.

Bifurcation of phase kinks

The codimension-1 family of PK can be followed in \(\nu\) and \(\gamma\). As shown in Fig. 6.5, there are two flavors of PK, namely the previous one that exists on \(\gamma_{PK}^+(\nu)\), and a new one that is connected to the first via a saddle-node bifurcation at \((\nu_{PK}^+, \gamma_{PK}^+)\) and exists on what
Figure 6.4: (a) Details of the transition from $\gamma^*$ to $\gamma_L^{LP}$ for the two-parameter continuation in $(\nu, \gamma)$ as shown in Fig. 6.3(a). Since $\gamma$ appears to be a single-valued function of $\nu$, in this figure we have plotted $\nu$ as a function of the continuation step. (b) Sample profiles during the transition from localized phase kinks to localized phase-winding states (the sequel to Fig. 6.3(b)).
Figure 6.5: The existence curve $\gamma_{1,2}^{PK}(\nu)$ for the family of PK embedded in $A^+$, and sample profiles on this curve.

we refer to as $\gamma_{2}^{PK}(\nu)$. The primary PK line $\gamma_{1}^{PK}$ bifurcates from $(\nu, \gamma) = (\beta\mu, 0)$, while the secondary PK line $\gamma_{2}^{PK}$ terminates at a codimension-2 point $(\nu_{PK}^-, \gamma_{PK}^-)$ where the solution profile consists of a primary PK and an asymmetric pulse. Overall the bifurcation diagram takes a noose shape that resembles a bifurcation sequence of periodic orbits in other ODEs known as the noose bifurcation [104]. The crucial difference is that in those systems the noose ties to itself in a period-doubling bifurcation, while in our case the codimension-2 point $(\nu_{PK}^-, \gamma_{PK}^-)$ represents the intersection between the existence curves of the primary PK and the asymmetric pulse, both of which are codimension-1 solutions. Since the origin of this asymmetric pulse is unclear, we cannot directly relate this codimension-2 point to period-doubling.

Bifurcation of PWS

Both PK and PWS achieve $2\pi$ phase winding, though the former lives on an infinite domain by definition while the latter has no such restriction. Thus a PK is a special kind of PWS and can be continued in $\gamma$ and the wavenumber $k$ to obtain an entire family of PWS. As shown in Fig. 6.6 for $\nu = 1.2 < \nu_{PK}^+$, the primary PK connects to a stationary plane wave solution to the CGLE at $\gamma = 0$. In contrast, the secondary PK connects to a reflection symmetric periodic state $SP^*$ through one fold.

When the primary and secondary PK annihilate in a saddle-node bifurcation at $\nu_{PK}^+$, the two branches of PWS in Fig. 6.6 should reconnect into a single Z-shaped branch. As $\nu$ further increases, the branch of PWS becomes single-valued via a hysteresis bifurcation. As shown in Fig. 6.7 for $\nu = 1.5 > \nu_{PK}^+$, this branch connects a stationary plane wave solution to the CGLE to a reflection symmetric periodic state $SP^*$.

Every point on the bifurcation diagram of PWS represents two PWS differing in chirality, or equivalently related by $x \to -x$. The PWS where $V$ leads $U$ ($U$ leads $V$) and thus winds in the CW (CCW) direction in the $(U,V)$ phase plane will be referred to as $PWS^+$ ($PWS^-$). The (spatial) stability of a PWS is determined by the configuration of
Figure 6.6: PWS branches that follow from (a) the primary and (b) the secondary PK at $\nu = 1.2$. 
its Floquet multipliers (denoted collectively by $\Lambda$). For $PWS^-$ we have $\Lambda = (1, e^{-2s}, e^{s \pm it})$ with $s > 0$ and $0 < t < \pi/2$, while for $PWS^+$ the Floquet multipliers are the inverses of $PWS^-$ and we have $\Lambda = (1, e^{2s}, e^{-s \pm it})$. The Floquet multipliers of state $SP^*$ are all equal to 1 indicating that the bifurcation from $SP^*$ that creates the PWS is a pitchfork bifurcation of a symmetric periodic orbit in a spatially reversible system. This pitchfork bifurcation happens at a codimension-1 point on the Eckhaus boundary.

**6.2.4 Simple localized PWS**

As explained in §6.2.2, localized PWS (LPWS) can be created when a heteroclinic orbit between $A^\pm$ interacts with a saddle-node on $A^+$, and the inner PWS may be considered as copies of the primary PK with finite wavelength. On the other hand, as discussed in §6.2.3, two branches of PWS are found to follow from the two flavors of PK that exist for $\nu \in (\nu_{PK}^-, \nu_{PK}^+)$. This leads to the question whether there are LPWS formed by PWS from the secondary PWS branch. The answer is yes but these states exist at different parameter values, such that they cannot be found by continuing the front between $A^\pm$ in $\nu$ and $\gamma$. Instead, we need to follow symmetric LS created near local bifurcation points as exemplified by Fig. 6.2. As might be guessed from the fact that LPWS are always embedded in $A^-$, the bifurcations leading to LPWS are subcritical Turing bifurcations on $A^-$. This bifurcation creates two branches of LS, which will be referred to in terms of their spatial phases $\phi = 0$ and $\phi = \pi$. One-parameter continuation of these weakly nonlinear LS in $\gamma$ then provides a shortcut to finding LPWS (which also happens to be how LPWS was first discovered).

**Primary LPWS**

Fig. 6.8 shows the $\phi = \pi$ branch followed from the lower Turing bifurcation when $\nu = 1.2$. Initially, a modulated pulse (panel (i)) develops from $A^-$. As $\gamma$ further decreases the branch undergoes collapsed snaking towards $\gamma = \gamma_{LP}^1 = 0.0734$, referred to as the collapsing point. At $\gamma_{LP}^1$, the solution (panel (ii)) can be written as $A^- \leftrightarrow PWS_{1^-} \leftrightarrow PWS_{1^+}$.
Figure 6.8: Bifurcation diagram for the $\phi = \pi$ branch of localized states when $\nu = 1.2$. The branch originates from the (subcritical) Turing bifurcation at $\gamma_T$.

$PWS_+^\pm \leftrightarrow A^-$, where $PWS_-^/PWS_+^\pm (F^-/F^\pm)$ represents respectively the PWS (the front) on the left/right half-domain and $PD$ represents the central defect. As we follow the primary LPWS branch additional wavelengths of $PWS_+^\pm$ are repeatedly inserted by $PD$, pushing $F^\pm$ further and further towards the boundaries until the available domain is filled with PWS, and the connection to $A^-$ is lost (panel (iii)). Alternatively, we may think of the defect as the spatial analog of a pacemaker producing a 1D target pattern [116]. In the $(U,V)$ phase plane $F^\pm$ corresponds to the trajectory that spirals out of the equilibrium $A^-$ and approaches the periodic orbit, while $PD$ corresponds to the outside tail that winds around the periodic orbit to create more copies of this orbit as the LPWS branch is followed upwards.

We can think of the defect as a heteroclinic connection between two periodic states related by reflection in the 4D phase space [6]. Such a defect forms a continuous family parametrized by its own phase at fixed parameter values. The reason lies in the dimensions of the stable and unstable manifolds of $PWS_+^\pm$, which are in turn determined by their Floquet multipliers (as discussed in §6.2.3). The stable manifold of $PWS_+^\pm$ is three-dimensional and the symmetric section is two-dimensional, so they form a continuous, structurally stable, family of intersections ($PD$). On the other hand, the unstable manifold of $PWS_+^\pm$ and the stable manifold of $A^-$ are both two-dimensional, so they only form codimension-1 intersections ($F^\pm$). As a corollary, we assert that the location of the collapsing point $\gamma_1^{LP}$ is determined by $F^\pm$ rather than $PD$.

Secondary LPWS

The behavior of the $\phi = \pi$ branch when $\nu = 1.5$ is shown in Fig. 6.9. The branch doubles back at a sharp fold (panel (i)), which results from branch reconnection at a smaller $\nu$. Subsequently the branch continues towards larger $\gamma$ and undergoes a series of folds (panel (ii)) before collapsing at $\gamma = \gamma_2^{LP} = 0.1980 > \gamma_1^{LP}$ (panel (iii)). The solutions at $\gamma_2^{LP}$ have
the same structure as the primary LPWS and will be referred to as the secondary LPWS, and the PWS that form secondary LPWS will be denoted by \( PWS^\pm \). When \( \nu = 1.5 \), the Floquet multipliers of \( PWS^\pm \) are \( \Lambda = (1, e^{2s}, e^{-s+is'}) \) with \( 0 < s' < s \), while those of \( PWS^+ \) are \( \Lambda = (1, e^{2s}, e^{-s+it}) \) with \( s > 0 \) and \( \pi/2 < t < \pi \). For other values of \( \nu \), e.g. \( \nu = 2 \), the stable Floquet multipliers \( \Lambda^\pm \) are complex for both \( PWS^+ \) and \( PWS^- \).

If we compute \( \gamma_{LP} (\nu) \) by continuing the secondary LPWS in \( \nu \) and \( \gamma \) (Fig. 6.10), the branch enters the coexistence region of multiple equilibria and approaches \( \gamma^* (\nu) \) for smaller \( \nu \), and extends to smaller \( \gamma \) and approaches \( \gamma_{LP} (\nu) \) for larger \( \nu \). The primary and secondary LPWS both involve heteroclinic connections between \( A^- \) and a PWS, \( PWS^\pm \) for the former and \( PWS^\pm \) for the latter. As shown in Figs. 6.6 & 6.7, these two PWS belong to the same family of orbits for larger \( \nu \) and different families for smaller \( \nu \).

**Modulated LPWS**

Now we study the bifurcation behaviors of LPWS for larger \( \nu \). Since we find it numerically difficult to continue \( \gamma_{LP} (\nu) \) to \( \nu > 2 \), we will only study bifurcation diagrams in \( \gamma \) for a fixed \( \nu \). The \( \phi = \pi \) branch following from the Turing bifurcation for \( \nu = 3 \) and \( L = 1200 \) is shown in Fig. 6.11. The LPWS on this branch consist of \( F^\pm \) and \( PD \) that span multiple wavelengths of PWS, and we refer to these solutions as modulated LPWS. This feature results from the Floquet multipliers of the underlying PWS being near unity. In contrast to the primary/secondary LPWS branches, there is no discernible collapsing point even for the large \( L \) chosen. As suggested by Fig. 6.11, \( \gamma \) decreases towards 0 exponentially as a function of the width of the LPWS. In this process, the bounding fronts generate far more rolls of PWS than the pacemaker defect via a steady-state analog of the familiar front propagation of a periodic pattern into a Turing-unstable equilibrium. This growth mechanism of modulated LPWS can be viewed at [http://www.youtube.com/watch?v=34ku5S4E0Ak](http://www.youtube.com/watch?v=34ku5S4E0Ak).

The modulated LPWS on the \( \phi = 0 \) branch differ from those on the \( \phi = \pi \) branch.
Figure 6.10: The locus $\gamma_{2}^{LP}$ of secondary LPWS on the $(\nu, \gamma)$ plane, together with $\gamma^*$ and $\gamma_{1}^{LP}$. (i) $\nu = 1.5$. (ii) $\nu = 1.2$. (iii) Vicinity of $\gamma^*$.

Figure 6.11: Bifurcation diagram for the $\phi = \pi$ branch of localized states created in the Turing bifurcation when $\nu = 3$. (i) $\gamma = 1 \times 10^{-3}$. (ii) $\gamma = 1 \times 10^{-4}$. (iii) $\gamma = 1 \times 10^{-5}$. 
by half a wavelength, while their bifurcation diagrams almost coincide with each other. From another perspective, these two branches are related by the approximate symmetry $A \rightarrow -A$ for $\gamma \approx 0$. Based on the bifurcation diagrams at successively larger $\nu$, it is likely that these two branches of modulated LPWS come from the primary/secondary LPWS at smaller $\nu$. Further analysis is needed in the regime $\gamma \ll 1$ to determine whether these branches are truly approaching 0 or collapsing towards extremely small $\gamma$’s, and in the latter case, the multiplicity of collapsing points. Due to the presence of a slowly varying envelope, we expect modulated LPWS to be well described by a multi-scale expansion as carried out in Ref. [52] for regular homoclinic snaking. Numerical continuation at several intermediate values of $2 < \nu < 3$ could also be useful for interpreting the transition from primary and secondary LPWS to modulated LPWS.

### 6.3 Temporal dynamics

In this section, we study time-dependent structures in the original PDE in relation to the steady states found in §6.2.

#### 6.3.1 Traveling waves

Traveling wave solutions take the form $A(x,t) = A(\xi = x - ct)$ and hence correspond to solutions of the following ODE

\begin{align}
-cU_\xi &= \mu U - \nu V - (U^2 + V^2) (U - \beta V) + (U_{\xi\xi} - \alpha V_{\xi\xi}) + \gamma, \\
-cV_\xi &= \mu V + \nu U - (U^2 + V^2) (V + \beta U) + (V_{\xi\xi} + \alpha U_{\xi\xi}).
\end{align}

The time-independent solutions studied in §6.2 correspond to solutions to (6.1-6.2) with $c = 0$. In particular, we can follow PWS to $c \neq 0$ at fixed $\gamma$ by varying both $c$ and the wavenumber $k = 2\pi/L$, $L$ being the spatial period. As shown in Fig. 6.12, the branch of traveling waves (TW) bifurcates from an equilibrium (panel (i)). As the amplitude increases, the phase speed $c$ changes monotonically and passes through $c = 0$ at PWS± (panel (ii)) before the amplitude saturates as $|c| \rightarrow \infty$ (panel (iii)). Thus a PWS can be interpreted as a traveling wave that happens to have zero wavespeed. Evidently, if a TW (e.g. PWS+) is a solution to (6.1-6.2), then its symmetric partner (e.g. PWS−) is also a solution to (6.1-6.2) with $c$ replaced by $-c$. Therefore, each branch in Fig. 6.12 has a mirror image with respect to the vertical axis $c = 0$, which together form a pair of TW branches related by spatial reflection. The collection of TW forms a surface $c(\gamma,k)$ parametrized by $\gamma$ and the wavenumber $k = 2\pi/L$, which could be visualized similarly to Figure 3.5. On the analytical side, a weakly nonlinear theory for TW bifurcating from the equilibrium $A^-$ can be formulated. An interesting prediction is that a branch of standing waves (SW) bifurcates together with the pair of TW branches. These standing waves could be related to time-periodic defects in §6.3.2. However since standing waves are periodic in both space and time, numerical continuation of the SW branch is much harder than TW and will not be pursued in this chapter.

To study the temporal stability of TW, we time evolve the original PDE taking multiple wavelengths (typically 60) of the solution profiles on the TW branch in Fig. 6.12
as initial conditions. There exists \( c^a \approx -0.65(\pm 0.01) \) such that the wave train is stable for \( c < c^a \), while for \( c > c^a \) an amplitude instability leads eventually to a uniform oscillation. This amplitude instability implies the existence of an unstable mode in the spectrum of a single roll of TW.

### 6.3.2 Dynamics of defects

If the initial condition for time evolution consists of more than one species of TW, the subsequent dynamics will involve defects at the domain walls. Among the collection of TW, the most relevant for our purposes are TW with \( c = 0 \), namely \( PWS^\pm \). These two PWS can form two types of defects, respectively referred to as the \((+-)-\)defects and the \((-+)-\)defects. The steady-state versions of both have been encountered in §6.2. The steady \((+-)-\)defects are equivalent to the pacemaker defects responsible for the growth of localized structures along the LPWS branches. They form a continuous family parametrized by their own phases at fixed \( \gamma \). The steady \((+-)-\)defects are created during the elimination of \( A^- \) at the uppermost portions of the LPWS branches (e.g. the boundary defect in panel (iii) of Fig. 6.8(b)). They only form a discrete family at fixed \( \gamma \). To explore the temporal dynamics, we assemble the initial condition as 24\(+\)12\(-\), where \( n \sigma \) \((n \in \mathbb{N}, \sigma \in \{+, -\})\) denotes \( n \) wavelengths of \( PWS^\sigma \), and rotate the phases \( \Phi^\pm \) of both \( PWS^+ \) and \( PWS^- \). Two sample initial conditions are shown in Fig. 6.13.

A series of time evolutions at \( \nu = 1.5 \) based on the PWS branch in Fig. 6.7 can be viewed at [http://www.youtube.com/watch?v=f88RuUDZ0wk](http://www.youtube.com/watch?v=f88RuUDZ0wk), where the defects are initialized to be symmetric, i.e. \( \Phi^+ = \Phi^- \). In these time evolutions \( \gamma \) is varied from 0 to 0.35 in steps of 0.05, \( \Phi^\pm \) is varied from \(-\pi\) to \(3\pi/4\) in steps of \(\pi/4\), and the results are plotted on the middle half of the spatial domain to highlight the defects. For \( \gamma = 0 \) to 0.15, \( PWS^\pm \) are temporally stable, whereas for \( \gamma = 0.2 \) to 0.35, \( PWS^\pm \) are subject to an amplitude instability that produces uniform oscillations (denoted by \( UO \)). For any \( \gamma \) or \( \Phi^\pm \), the \((+-)-\)defect always pairwise annihilates \( PWS^\pm \) to produce \( UO \) (Figure 6.14). The left (right) half of this process may be regarded as \( PWS^+ \) (\( PWS^- \)) connected to \( UO \) via a...
bending layer formed by the collection of TW profiles with \( c > 0 \) (\( c < 0 \)) existing between \( PWS^+ \) (\( PWS^- \)) and \( UO \) in Fig. 6.12.

As shown in Figure 6.14, at \( \gamma = 0 \) the (\( -+ \))-defect is temporally stable regardless of \( \Phi^{\pm} \), as expected from the phase rotation symmetry \( A \rightarrow Ae^{i\phi_0} \) of the FCGLE at \( \gamma = 0 \). Later, when \( UO \) meets the (\( -+ \))-defect, a new bending layer is created that connects \( UO \) to \( PWS^+ \) to the right of this defect again through TW with \( c > 0 \). Ideally the propagation speed \( v_b \) of any bending layer should be related to the wavelength \( \lambda_{PWS} \) of \( PWS^{\pm} \) and the period \( T_{UO} \) of \( UO \) as \( v_b = \lambda_{PWS}/T_{UO} \). As \( t \to \infty \), this bending layer becomes straight leaving behind a TW with \( c > 0 \).

As shown in Figure 6.15, at \( \gamma = 0.15 \) the (\( -+ \))-defect can be either stable or Hopf unstable depending on \( \Phi^{\pm} \). In addition, after \( UO \) meets \( PWS^+ \), the phase differences between these two types of oscillations are eliminated in successive phase slips that tend to happen near the original location of the (\( -+ \))-defect.

As shown in Figure 6.16(a), at \( \gamma = 0.2 \) the (\( -+ \))-defect is more unstable than
Figure 6.15: Space-time plots of $V(x,t)$ for $\nu = 1.5$ showing the symmetric defects at (a) $\gamma = 0.15, \Phi^\pm = -\pi$; (b) $\gamma = 0.15, \Phi^\pm = 0$.

$PW S^\pm$ and can evolve into a patch of $UO$ that invades the PWS on both sides via successive phase slips. As shown in Figure 6.16(b), at $\gamma = 0.35$ instabilities develop too fast for any coherent structure to emerge from the defects, but the inhomogeneity in the initial condition later produces 1D spirals in the domain-filling $UO$, which could persist for tens of oscillation periods before vanishing.

Finally we remark that $UO$ becomes unstable for larger $\gamma$ and coexists with propagating structures. This happens quite independently of the initial condition and will not be pursued in this chapter. However in Chapter 9 we will study in more detail the instability of $UO$ and the spatiotemporal chaos that results from this instability.

## 6.4 Discussion

As seen above, localized phase-winding states (LPWS) typically exist for small forcing $\gamma$. In the limit $0 < \gamma \ll 1$, the FCGLE can be treated as perturbations of the CGLE and a phase equation can be derived [57]. The LPWS found in the 1:1 and 2:1 FCGLE can be related to each other by homotopic continuation. It may also be possible to demonstrate the existence of LPWS for the 3:1 and 4:1 cases.

The pacemaker defect (PD) needs to be better understood. In analogy to Eq. (3.14), a phase for PD can be defined in terms of either $U$ or $V$. By imposing proper boundary conditions, the family of PD homoclinic to any given PWS can be followed. In the sense of Ref. [154], the PD resemble a trivial version of contact defects with the left and right group velocities $c_\theta^\pm$ and the defect velocity $c_d$ all identically 0. These contact defects are parametrized by the asymptotic wavenumber $k_- = k_+$.

In addition to the above simple LPWS with a central $(-+)$-defect, we have also found composite LPWS with a central $(+-)$-defect and two $(-+)$-defects on the sides. In fact there may exist an infinite hierarchy of multi-pulse states formed by different numbers...
of defects and numbers of rolls separated by these defects, which we leave as a topic for future work.

The behavior of LPWS in the 2:1 FCGLE largely resembles the 1:1 case, although there are at least the following differences. In the 2:1 case there is only one locus of LPWS in the \((\nu, \gamma)\) plane, but the additional up-down symmetry leads to the existence of both even and odd parity LPWS. Also absent from the 1:1 case is the bifurcation to TW from the so-called Ising-Bloch transition points which are present in the 2:1 case.

The dynamics of antisymmetric defects \(\Phi^+ = \Phi^- + \pi\) need to be further explored. New dynamics are also expected in the \(\nu = 1.2\) case, where \(PWS\) become \(PK\) as their wavelengths diverge. Another possibly interesting place is near the pitchfork bifurcation, where symmetric periodic states undergo a reversibility breaking (RB) bifurcation to yield \(PWS\). However for the parameters used in this chapter, these periodic states are all subject to an amplitude instability and thus fail to reveal dynamical features on a longer time scale that may result from the proximity to the Eckhaus boundary.
Chapter 7

Classification of 1D localized states

7.1 Introduction

Many experiments over the past two centuries have been performed to study parametrically driven oscillatory systems [81]. In a single periodically driven nonlinear oscillator interesting behavior, such as hysteresis, frequency locking and chaos, is generally associated with the presence of a strong resonance between the natural frequency $\omega$ of the oscillator and the driving frequency $\Omega$, viz., $\Omega : \omega = n : 1$, $n = 1, 2, 3, 4$ [84]. Analogous behavior is found in spatially extended systems when these undergo spatially homogeneous oscillations, but new types of dynamical behavior characterized by finite spatial coherence are also possible.

The best studied resonances are the 2:1 resonance ($n = 2$), usually referred to as the parametric or subharmonic resonance, and the 1:1 resonance ($n = 1$), referred to as the direct or harmonic resonance. In the Faraday system, i.e., surface gravity-capillary waves excited by the vertical oscillation of a container, this instability generates many different types of standing waves, including spatially localized oscillations called oscillons [122]. As reviewed in §1.1, related oscillons have been observed in vertically vibrating granular media [174]. In contrast, in self-excited chemical systems the subharmonic resonance can produce hole-like structures in an oscillating background that have been called reciprocal oscillons. In addition, different types of fronts separating states that are phase-locked to the drive and either in phase or out of phase with the drive have also been observed [119].

Similar behavior is associated with the 1:1 resonance [57, 53, 105, 151] and this resonance has been the focus of many studies of dissipative solitons or cavity solitons in the nonlinear optics literature [150, 162, 132]. The systems of interest are typically optical resonators or nonlinear slabs of material such as a saturable absorber with a feedback mirror. In these systems the envelope $E(x,t)$ of the optical frequency electric field obeys the equation

$$\frac{\partial E}{\partial t} = E_I + (\mu + i\nu + F(|E|^2))E + i\alpha \nabla^2 E,$$

(7.1)

where $E_I$ is the amplitude of the coherent driving field, $\nu$ is the detuning with respect to the closest cavity resonance, and $\mu < 0$ and $\alpha$ represents damping and the effects of diffraction, respectively. As reviewed in §1.1 the nonlinearity $F$ is typically taken to be
\( F = i \beta |E|^2 \) [127, 90] although the nonlinearity \( F = -2C(1 - i\Delta)/(1 + \Delta^2 + |E|^2) \) is also used [26, 186]; when \( |E| \ll 1 \) the resulting equation reduces to (7.1) but with a dissipative nonlinearity; the coefficient \( \alpha \) is generally taken to be real thereby neglecting diffusion.

As also pointed out in §1.1, much of the behavior observed in these systems can be described using a slight generalization of Eq. (7.1), the forced complex Ginzburg-Landau equation (FCGLE) [57], and this equation may be thought of as the normal form for the dynamics close to a strong resonance. As such all examples of strong resonance in spatially extended systems can be reduced to the corresponding FCGLE provided only that the system is near threshold for oscillations, and the forcing is weak and its frequency sufficiently close to resonance. Consequently, it is invaluable to have a detailed and systematic understanding of the properties of the resulting normal forms as a function of the coefficients. In a recent paper [41] the authors have provided an almost complete classification of the different types of phase-locked homogeneous states and the associated spatially structured states described by the FCGLE for the 2:1 resonance. The present chapter attempts to carry out a similar analysis for the 1:1 resonance.

The classification naturally divides into two parts corresponding, respectively, to damped and self-excited systems in the absence of forcing. In §7.2 we review the forced complex Ginzburg-Landau equation describing the 1:1 resonance in one spatial dimension (1D) and the concept of spatially homogeneous phase-locked states, i.e. the states that oscillate with the frequency of the forcing. In §7.3 we perform detailed analysis of the properties of the spatially homogeneous states in both time and space, emphasizing their relevance to the existence and stability of different types of spatially localized states. In §7.4 we place the spatially localized states discovered in the previous chapters in the context of §7.3, and in addition show how various branches of localized states terminate. Finally in §7.5 we outline a classification program based on the conditions for the existence of localized states in the parameter space, and point out the usefulness of considering the large forcing limit. The localized states found in this limit persist into both self-excited (\( \mu > 0 \)) and damped (\( \mu < 0 \)) oscillatory regimes, but we will not present the detailed numerical results. The analytical computations that serve as the starting point for numerical continuation are relegated to two appendices.

### 7.2 The forced complex Ginzburg-Landau equation

We consider a 1D continuous system near a bifurcation to spatially homogeneous oscillations with natural frequency \( \omega \) in the presence of spatially homogeneous forcing with frequency \( \Omega \). It is well-known that interesting dynamics occur with strong resonances of the form \( \Omega : \omega = n : 1 \) where \( 1 \leq n \leq 4 \). Among these possibilities the subharmonic resonance \( \Omega \approx 2\omega \) and the harmonic resonance \( \Omega \approx \omega \) are most suitable for experimental studies. In this chapter we study the resonance tongue associated with the harmonic resonance and refer the interested reader to [41] for the relevant results in the subharmonic case.

Inside the resonance tongue the system responds to the forcing with oscillations at frequency \( \Omega \), corresponding to phase-locked states. Outside the resonance tongue the frequency difference \( |\omega - \Omega| \) is too large and the response frequency is no longer locked to the forcing frequency [93]. As detailed in §1.1, in a spatially extended system we may
include large-scale spatial modulation in the theory and suppose that a dynamical observable \( w \) takes the form

\[
  w = w_0 + A e^{i \Omega t} + \text{c.c.} + \cdots,
\]

(7.2)

where \( w_0 \) represents the equilibrium state, \( A(x,t) \) is a complex amplitude and the ellipses denote higher order terms. The oscillation amplitude \( A(x,t) \) then obeys the FCGLE [57]

\[
  A_t = (\mu + i \nu)A - (1 + i \beta)|A|^2A + (1 + i \alpha)A_{xx} + \gamma.
\]

(7.3)

Here \( \mu \) represents the distance from onset of the oscillatory instability, \( \nu \) is the detuning from the unforced frequency, and \( \alpha, \beta \) and \( \gamma \) are real coefficients representing dispersion, nonlinear frequency correction and the forcing amplitude, respectively. In the absence of forcing, oscillations grow when \( \mu > 0 \) but decay when \( \mu < 0 \).

The spatially uniform states satisfy an ODE in \( t \) for the complex amplitude \( A \). This equation has been analyzed in detail in [84, 87]. In general, steady-state solutions of Eq. (7.3) are all phase-locked in the sense that they correspond to observables \( w \) in the original system that oscillate at exactly the driving frequency \( \Omega \). In the following we will refer to a phase-locked state independent of \( x \) as a uniform state. In the amplitude-phase representation, we write a uniform state as \( A = R \exp(i\phi) \) and define \( R_2 \equiv R^2 \). It follows from Eq. (7.3) that \( R_2 \) satisfies the cubic equation

\[
  (1 + \beta^2)R_3^2 - 2(\mu + \beta \nu)R_2^2 + (\mu^2 + \nu^2)R_2 - \gamma^2 = 0,
\]

(7.4)

and \( \phi \) satisfies

\[
  \cos \phi = \frac{1}{\gamma}(R^3 - \mu R), \quad \sin \phi = \frac{1}{\gamma}(\nu R - \beta R^3).
\]

(7.5)

Eq. (7.4) has in general either one or three positive solutions when \( \gamma \neq 0 \). A complete description of the uniform states and their stability properties in both space and time is given in the following section; these provide a key to the origin and presence of localized states in this system.

### 7.3 Properties of the uniform states

Localized states typically involve orbits homoclinic to a particular uniform state or a heteroclinic cycle between two uniform states. The properties of the uniform states are therefore important indicators of the properties of the corresponding localized states, namely their structural and temporal stability. The analysis here is most easily carried out in terms of the real and imaginary parts of the complex amplitude, \( A \equiv U + iV \). The FCGLE respects the parity transformations

\[
  (U, V, \mu, \nu, \alpha, \beta, \gamma) \rightarrow (U, -V, \mu, -\nu, -\alpha, -\beta, \gamma), \quad (A, \gamma) \rightarrow (-A, -\gamma).
\]

Therefore we only need to consider \( \beta \geq 0, \gamma > 0 \). Moreover, through application of a proper scaling transform we can show that only the distinct signs of \( \mu \) give rise to distinct behaviors. Thus when we consider the self-excited oscillatory regime (\( \mu > 0 \)) we fix \( \mu = 1 \), while in the damped oscillatory regime (\( \mu < 0 \)) we fix \( \mu = -1 \).
7.3.1 Bifurcation analysis

We first consider the spatially uniform states obtained from setting $A_{xx} = 0$ in Eq. (7.3). The real and imaginary parts of the equation form the following two-dimensional (2D) dynamical system

$$U_t = \mu U - \nu V - (U - \beta V)(U^2 + V^2) + \gamma, \quad V_t = \nu U + \mu V - (V + \beta U)(U^2 + V^2).$$

(7.6)

The trace and determinant of the Jacobian matrix describing the linearization of the vector field about the steady state $A = R \exp(i\phi)$ are given by

$$\text{tr} J = 2(\mu - 2R^2), \quad \text{det} J = \mu^2 + \nu^2 - 4(\mu + \beta \nu)R^2 + 3(1 + \beta^2)R^2.$$ 

Saddle-node bifurcation

Saddle-node bifurcation occurs when $\text{det} J = 0$. Introducing $A \equiv 1 + \beta^2, B \equiv \mu + \beta \nu, C \equiv \mu^2 + \nu^2$, we can express the condition in terms of $R^2$. The larger (smaller) root of the equation is denoted by $R_{2+}^{SN}$ ($R_{2-}^{SN}$) and is given by

$$R_{2\pm}^{SN} = \frac{2B \pm \sqrt{4B^2 - 3AC}}{3A}$$

(7.7)

provided that $R_{2\pm}^{SN}$ is real and positive. As $\gamma$ varies the saddle-node bifurcations lie on the curve

$$4(4B^2 - 3AC)^3 = (18ABC - 16B^3 - 27A^2\gamma^2)^2.$$ 

In general we have two saddle-node bifurcations occurring at two different values of $\gamma$, corresponding to $R_{2+}^{SN}$ ($R_{2-}^{SN}$) and denoted by $\gamma_{SN}^+$ ($\gamma_{SN}^-$). Simultaneous occurrence of the two saddle-node bifurcations corresponds to the codimension-two hysteresis bifurcation, i.e., the onset of a hysteresis loop in the $(\gamma, |A|)$ bifurcation diagram. Algebraically the hysteresis bifurcation occurs when Eq. (7.4) has three identical roots, yielding

$$B = \frac{3}{2} \gamma^{2/3} A^{2/3}, \quad C = 3\gamma^{4/3} A^{1/3}.$$ 

The hysteresis bifurcation defines two critical values of $\nu$ denoted by $\nu_{SN}^{+}$:

$$\nu_{SN}^{+} = \frac{\nu \sqrt{3} \beta + 1}{\beta - \sqrt{3}}, \quad \nu_{SN}^{-} = \frac{\nu \sqrt{3} \beta - 1}{\beta + \sqrt{3}}.$$ 

These locations in the $(\nu, \gamma)$ plane are referred to as cusps (Fig. 7.1). Near these locations, referred to as nascent bistability in the optics literature [171], the FCGLE can be reduced to the real Swift-Hohenberg equation, a prototypical equation for studies of time-independent spatially localized structures [112, 42].
Hopf bifurcation

Hopf bifurcation occurs when $\text{tr} J = 0$ and $\det J > 0$. In terms of $R_2$ the condition becomes

$$R_2 = \frac{\mu}{2}, \quad 3AR_2^2 - 4BR_2 + C > 0.$$  \hfill (7.8)

Since we require $R_2 \geq 0$, Hopf bifurcation is absent for $\mu < 0$. After eliminating $R_2$ in favor of $\gamma$, the locus of Hopf bifurcations becomes

$$(1 + \beta^2)\mu^3 + 4\mu\nu(\nu - \beta\mu) - 8\gamma^2 = 0, \quad \nu^2 - 2\beta\mu\nu - \frac{1}{4}\mu^2(1 - 3\beta^2) > 0.$$  

It follows from the second condition that $\nu$ must lie in the range $\nu > \nu^+_H$ or $\nu < \nu^-_H$, where

$$\nu^+_H \equiv \mu(\beta + \frac{1}{2}\sqrt{1 + \beta^2}), \quad \nu^-_H \equiv \mu(\beta - \frac{1}{2}\sqrt{1 + \beta^2}).$$

Simultaneous occurrence of the saddle-node and the Hopf bifurcation gives rise to the well-known codimension-two Takens-Bogdanov bifurcation. The locus of Takens-Bogdanov bifurcations in terms of $\gamma$ is

$$\nu^2 = (\mu^3 - 6\gamma^2)^2, \quad \beta^2 = \frac{(\mu^3 - 6\gamma^2)^2}{\mu^3(8\gamma^2 - \mu^3)} \quad \text{with} \quad \beta\nu = \frac{(\mu^3 - 6\gamma^2)^2}{\mu^2(8\gamma^2 - \mu^3)}.$$  

We can readily identify $\nu^+_H$ as the critical values of $\nu$ for the Takens-Bogdanov bifurcation.

The problem has enough free parameters to create a codimension-three hysteresis-Hopf bifurcation, corresponding to the point of tangency between $\nu^-_{SN}$ and $\nu^-_H$. The hysteresis-Hopf bifurcation (also known as the cusp-Hopf bifurcation; for detailed dynamics see [94]) forms the organizing center for the temporal dynamics. The locus in terms of $\gamma$ is

$$\beta = \frac{1}{\sqrt{3}}, \quad \nu = 0, \quad \mu^3 = 6\gamma^2.$$  

Geometric interpretation

For fixed $\beta$, $\mu$ and $\nu$ (note that $\alpha$ does not affect the behavior of the 2D dynamical system in $t$ only), we can plot $|A|$ vs $\gamma$ in a bifurcation diagram. Fig. 7.1 shows the critical curves in the $(\nu, \gamma)$ plane, $(\nu, R_2)$ plane and the corresponding bifurcation diagrams. Each representation of critical curves in terms of $R_2$ or $\gamma$ has its own advantage and disadvantage. First, note that each $R_2 > 0$ corresponds to a unique point in the bifurcation diagram, while a particular $\gamma$ can correspond to either one or three points in this diagram. Therefore, the critical curves in the $(\nu, R_2)$ plane uniquely characterize the properties of every portion of the bifurcation diagram, while the critical curves in the $(\nu, \gamma)$ plane need to be carefully labelled to indicate, in the regime of multiple equilibria, whether the bifurcation occurs on the upper or the lower branch. Second, the algebraic relations characterizing the critical curves in the $(\nu, R_2)$ plane are much simpler than their counterparts in the $(\nu, \gamma)$ plane. The former consist of solely conic sections and straight lines, while the latter in general involve higher-order curves. This implies that the characterization of the relative positions of these critical curves in terms of $R_2$ is relatively straightforward, while in terms of $\gamma$ the
characterization is rather difficult. Lastly, we remark that although the \((\nu, R_2)\) plane offers these advantages over the \((\nu, \gamma)\) plane, the \((\nu, R_2)\) plane in its primitive form does not tell us which values of \(R_2\) correspond to the same \(\gamma\).

First we study the \((\nu, R_2)\) plane with several concepts borrowed from analytic geometry. The saddle-node bifurcation curve given by Eq. (7.7) represents a conic section whose shape depends on \(\beta\). In the whole \((\nu, R_2)\) plane, for \(0 \leq \beta < \sqrt{3}\) the conic section is an ellipse (Figs. 7.1(b) & 7.1(e)) while for \(\beta > \sqrt{3}\) the conic section is a hyperbola (Figs. 7.1(h) & 7.1(k)). However, only the upper half-plane \(R_2 \geq 0\) is physically relevant. In the \(0 \leq \beta < \sqrt{3}\) case, the ellipse always lies entirely in \(R_2 \geq 0\) for \(\mu > 0\) (Figs. 7.1(b) & 7.1(e)) but entirely outside \(R_2 \geq 0\) for \(\mu < 0\) (not plotted). Therefore, saddle-node bifurcation is absent for \(\mu < 0, 0 \leq \beta < \sqrt{3}\). In the \(\beta > \sqrt{3}\) case, the upper branch of the hyperbola always lies entirely in \(R_2 \geq 0\) while the lower branch of the hyperbola always lies entirely outside \(R_2 \geq 0\), regardless of the sign of \(\mu\) (Figs. 7.1(h) & 7.1(k)). Moreover, the center of the hyperbola lies outside \(R_2 \geq 0\) for \(\mu > 0\) (Fig. 7.1(h)) but in \(R_2 \geq 0\) for \(\mu < 0\) (Fig. 7.1(k)). The hyperbola has two asymptotes that intersect the horizontal axis at

\[
\nu_{HY}^{\pm} \equiv \pm \frac{2}{\sqrt{\beta^2 - 3}}\mu.
\]

The region bounded by either the ellipse or the upper branch of the hyperbola in the upper half-plane represents the middle branch of the bifurcation diagram in the region of multiple equilibria. The ellipse is bounded by \(\nu_{SN}^{-} < \nu < \nu_{SN}^{+}\), while the upper branch of the hyperbola lies in the range \(\nu > \nu_{SN}^{-}\) for \(\mu > 0\) and \(\nu > \nu_{SN}^{+}\) for \(\mu < 0\), where \(\nu_{SN}^{\pm}\) are the critical \(\nu\)'s for the hysteresis bifurcation.

The Hopf bifurcation curve given by Eq. (7.8) represents a horizontal straight line \(R_2 = \mu/2\), which is physically meaningful only when \(\mu > 0\) (Figs. 7.1(b), 7.1(e) & 7.1(h)). The Hopf line intersects the saddle-node conic section at two points that correspond to the Takens-Bogdanov bifurcation. Within the saddle-node conic section the equilibrium is hyperbolic and Hopf bifurcation cannot occur. Outside the saddle-node conic section the equilibrium is stable above the Hopf line, unstable below the Hopf line. In other words, an non-hyperbolic equilibrium is Hopf stable if and only if its amplitude is large enough to satisfy \(R_2 > \mu/2\).

For \(\mu > 0\), the saddle-node conic section opens up from an ellipse into a hyperbola as \(\beta\) is increased through \(\sqrt{3}\). On the other hand, as \(\beta\) is decreased from \(\sqrt{3}\), the ellipse becomes less tilted. In consequence, although the left cusp lies below the Hopf line and the right cusp lies above the Hopf line (Fig. 7.1(e)) for \(1/\sqrt{3} < \beta < \sqrt{3}\), the former rises above the Hopf line (Fig. 7.1(b)) as \(\beta\) is decreased through \(1/\sqrt{3}\), the codimension-three hysteresis-Hopf bifurcation point.

In the \((\nu, R_2)\) plane, the qualitatively different scenarios concerning the saddle-node and Hopf bifurcations are now exhausted. Next we should determine the temporal stability of the uniform states in the \((\nu, \gamma)\) plane. To establish the correspondence, first note that the values of \(\nu\) at the intersections between the Hopf line and the saddle-node conic section correspond to the values of \(\nu\) at the tangency points between the Hopf and saddle-node critical curves on the \((\nu, \gamma)\) plane. At these tangency points, the saddle-node and Hopf bifurcations occur on the same branch and coalesce to form the Takens-Bogdanov
bifurcation. We will refer to this type of collision between two bifurcations as a local collision. However, there also exist transverse intersections between the Hopf and saddle-node critical curves in the \((\nu, \gamma)\) plane that remain undetectable in the primitive \((\nu, R_2)\) plane. At these transverse intersections, the saddle-node bifurcation occurs on one branch while the Hopf bifurcation occurs on another branch. We will refer to this type of collision between two bifurcations as a remote collision. To detect a remote collision between two bifurcations on the \((\nu, R_2)\) plane, we have to project the first bifurcation onto the competing branch and compare the projection with the second bifurcation. This is most easily done for the saddle-node bifurcation; the resulting critical curve is

\[
R_{2+}^{SP} = \frac{2}{3A} \left( B \mp (4B^2 - 3AC)^{1/2} \right),
\]

where \(R_{2+}^{SP}\) (\(R_{2-}^{SP}\)) denote the projection of \(R_{2+}^{SN}\) (\(R_{2-}^{SN}\)). These quantities solve

\[
3A(R_2^{SP})^2 - 4BR_2^{SP} + 4 \left( C - \frac{B^2}{A} \right) = 0. \tag{7.9}
\]

The projection of a saddle-node conic section is therefore another conic section which envelops the original conic section while tangent to the original conic section at the cusp point(s) (Figs. 7.1(b), 7.1(e), 7.1(h) & 7.1(k)).

The geometry of the projection implies that there are two transverse intersections and that the two tangency points lie between them. It follows that the three-equilibria region in the \((\nu, \gamma)\) plane (the region bounded by the upper and lower saddle-node critical curves) is split into three parts: the region containing the left cusp (region \(L_t\)), the region below the right Hopf curve containing no cusp (region \(N_t\)) and the region above the right Hopf curve containing the right cusp for \(0 \leq \beta < \sqrt{3}\) or unbounded for \(\beta > \sqrt{3}\) (region \(R_t\)). In each region the temporal stabilities of the upper and lower branches are uniquely determined. In region \(L_t\), the upper and lower branches are both stable for \(0 \leq \beta < 1/\sqrt{3}\) and both unstable for \(\beta > 1/\sqrt{3}\). In region \(N_t\), the upper branch is stable while the lower branch is unstable. In region \(R_t\), the upper and lower branches are both stable. Whenever both the upper and lower branches are stable, we have bistability between two uniform states, a necessary prerequisite for the existence of temporally stable fronts connecting these states.

### 7.3.2 Spatial eigenvalues

To study the spatially nonuniform solutions of Eq. (7.3), we rewrite the FCGLE in the operator form

\[
\begin{bmatrix}
U_t \\
V_t
\end{bmatrix} = (\mathcal{L} + \mathcal{N}) \begin{bmatrix}
U \\
V
\end{bmatrix} + \begin{bmatrix}
\gamma \\
0
\end{bmatrix}, \tag{7.10}
\]

where \(\mathcal{L}\) and \(\mathcal{N}\) denote the linear and nonlinear terms, respectively:

\[
\mathcal{L} = \begin{bmatrix}
\mu & -\nu \\
\nu & \mu
\end{bmatrix} + \begin{bmatrix}
1 & -\alpha \\
\alpha & 1
\end{bmatrix} \partial_{xx}, \quad \mathcal{N} = -(U^2 + V^2) \begin{bmatrix}
1 & -\beta \\
\beta & 1
\end{bmatrix}. \tag{7.11}
\]

As seen in the previous chapters, the steady states of the system satisfy a fourth order ODE in \(x\) that can be studied using a combination of bifurcation theory and numerical
Figure 7.1: The uniform states. The first and second columns show, respectively, the critical curves in the \((\nu, \gamma)\) plane and \((\nu, R_2)\) plane. Blue curves denote saddle-node bifurcations while green curves denote Hopf bifurcations. The gray curves in the second column denote projections of the saddle-node bifurcations onto the competing branch. The third column shows the bifurcation diagrams corresponding to constant \(\nu\) slices (dotted vertical lines in the first two columns) in matching colors. Hopf bifurcations are denoted, respectively, by a dashed magenta line for Hopf bifurcations on the left (i.e. smaller \(\nu\)) and dashed black line for Hopf bifurcations on the right (i.e. larger \(\nu\)). First row: \(\mu = 1, \beta = 0\) (region: \(\mu > 0, 0 \leq \beta < 1/\sqrt{3}\)); slices: \(\nu = 0.3\) (red), \(\nu = 0.54\) (green), \(\nu = 0.6\) (blue). The symmetry-related Hopf bifurcation on the left is not shown. Second row: \(\mu = 1, \beta = 1\) (region: \(\mu > 0, 1/\sqrt{3} < \beta < \sqrt{3}\)); slices: \(\nu = 0.28\) (cyan), \(\nu = 1.5\) (red), \(\nu = 3\) (green), \(\nu = 4\) (blue). Third row: \(\mu = 1, \beta = 2\) (region: \(\mu > 0, \beta > \sqrt{3}\)); slices: \(\nu = 0.75\) (cyan), \(\nu = 3\) (red), \(\nu = 4\) (green). Fourth row: \(\mu = -1, \beta = 3\) (region: \(\mu < 0, \beta > \sqrt{3}\)); slices: \(\nu = 15\) (red). The black dashed lines in the \((\nu, R_2)\) plane in the last two rows are the asymptotes for the blue hyperbola.
branch following techniques. Eq. (7.10) is invariant under \( x \to -x \) and hence is reversible in space. In the following we think of \( x \) as a time-like variable and classify the states of interest as either homoclinic or heteroclinic orbits depending on the behavior of \( A(x) \) as \( x \to \pm \infty \). Homoclinic orbits to a uniform state correspond to oscillons while heteroclinic orbits between two uniform states correspond to fronts. In this section we consider the stability of uniform steady states in space.

**Local bifurcations**

The concept of spatial eigenvalues of equilibria and their relevance to the existence of homoclinic and heteroclinic orbits have been reviewed in §2.3.1. To compute spatial eigenvalues we perturb the uniform steady state by \( \epsilon e^{\lambda x} \ (\epsilon \ll 1) \) and linearize. Eq. (7.10) yields the characteristic equation

\[
(1+\alpha^2) \lambda^4 - 2(2(1+\alpha \beta) R_2 - (\mu + \alpha \nu)) \lambda^2 + (\mu^2 + \nu^2 - 4(\mu + \beta \nu) R_2 + 3(1+\beta^2) R_2^2) = 0. \tag{7.12}
\]

This is a quadratic equation in \( \lambda^2 \) and the equilibrium type depends on the sign of the constant term and on the discriminant (denoted by \( \Delta \)). The constant term vanishes at the saddle-node bifurcation; in the current setting this bifurcation corresponds to a transition between type 3 and type 2 equilibria, or between type 3 and type 4 equilibria. As shown in Appendix 7.6 the former gives rise to a branch of spatially nonuniform hole-like states. The discriminant distinguishes between transitions from type 1 to type 2 equilibria (Belyakov-Devaney bifurcation) or from type 1 to type 4 equilibria (Turing bifurcation or modulational instability in the optics literature). The latter generates two branches of localized states (Appendix 7.7). The solution to \( \Delta = 0 \) determines two new critical values of \( R_2 \) [53]

\[
R_{2\pm}^T \equiv \frac{\nu - \alpha \mu}{2(\beta - \alpha) \pm \sqrt{(1+\alpha^2)(1+\beta^2)}}. \tag{7.13}
\]

We denote the critical values of \( \gamma \) corresponding to \( R_{2\pm}^T \) by \( \gamma_{\pm}^T \). If one defines

\[
z(\alpha, \beta) \equiv (1+\alpha^2)(1+\beta^2) - 4(\beta - \alpha)^2, \tag{7.14}
\]

then the discriminant of Eq. (7.12) becomes

\[
\Delta \equiv 4z(\alpha, \beta)(R_2 - R_{2\pm}^T)(R_2 - R_{2\mp}^T). \tag{7.15}
\]

In the Turing bifurcation case, the critical wavenumber is

\[
k_{i}^2 = \frac{(\mu + \alpha \nu) - 2(1+\alpha \beta) R_2}{1+\alpha^2}. \tag{7.16}
\]

**Geometric interpretation**

In this section, we discuss the significance of the critical curves in the \((\beta, \alpha)\) plane in determining the configuration of the critical curves \( \gamma_{SN}^\pm \) and \( \gamma_T^\pm \) in the \((\nu, \gamma)\) plane. Six groups of critical curves in the \((\beta, \alpha)\) plane are identified and plotted together in Fig. 7.3. Here, the \((\nu, R_2)\) plane again facilitates much of the argument by providing a simpler
geometric picture. The \((\nu, R_2)\) upper half-plane is partitioned by the boundaries \(R_{2SN}^+\) and \(R_{2T}^+\) into distinct regions, each of which has a distinct spatial type (type 1 to 4 in Fig. 2.12). From Eq. (7.13) we readily identify \(R_{2T}^\pm\) as two lines originating from \((\nu, R_2) = (\alpha \mu, 0)\). These will be referred to collectively as Turing lines, although the lines can represent either Turing or Belyakov-Devaney bifurcation. In the whole \((\nu, R_2)\) plane, the Turing lines are always tangent to the saddle-node conic section. Let us denote the value of \(\nu\) at the tangency point between \(R_{2T}^+\) (\(R_{2T}^-\)) and the saddle-node conic section by \(\nu_T^\pm (\nu_T^-)\) for future reference. The tangency point represents the analog of the Takens-Bogdanov bifurcation in our reversible 4D dynamical system in \(x\).

Our analysis is much simplified in the \((\nu, R_2)\) plane largely because the intersection between the Turing lines and the horizontal axis \((\alpha \mu, 0)\) depends only on \(\alpha\) and \(\mu\) (not \(\beta\)), while the shape of the saddle-node conic section depends only on \(\beta\) and \(\mu\) (not \(\alpha\)). Recall that the cusps of the saddle-node conic section occur at \(\nu = \nu_{SN}^\pm\). When \(\alpha \mu = \nu_{SN}^\pm\), the Turing lines are vertical and we have the analog of the cusp-Hopf bifurcation in our reversible 4D dynamical system. We denote the corresponding critical values of \(\alpha\) by \(\alpha_{SN}^\pm\),

\[
\alpha_{SN}^+ = \nu_{SN}^+ / \mu = -\sqrt{3} \beta + 1 / \beta - \sqrt{3}, \quad \alpha_{SN}^- = \nu_{SN}^- / \mu = \sqrt{3} \beta - 1 / \beta + \sqrt{3}.
\]

Note that \(\alpha_{SN}\) are the solutions to \(z(\alpha, \beta) = 0\) with \(z(\alpha, \beta)\) defined in Eq. (7.14). When \(\alpha\) crosses over \(\alpha_{SN}^\pm\), the type of the uniform states surrounding the cusp in the \((\nu, R_2)\) plane makes a transition from type 2 to type 4 or vice versa. However, when the cusp lies in the lower half-plane, the corresponding \(\alpha_{SN}\) has no physical meaning. In consequence, only \(\alpha_{SN}^+\) and the upper branch of \(\alpha_{SN}^\pm\) are significant for \(\mu > 0\), and only the lower branch of \(\alpha_{SN}^\pm\) is significant for \(\mu < 0\).

When the saddle-node conic section is an ellipse, the Turing lines are both tangent to the ellipse for arbitrary \(\alpha\) (Figs. 7.2(g) & 7.2(h)). The spatial types can then be unambiguously labelled for each region delimited by \(R_{2SN}^\pm\) and \(R_{2T}^\pm\). However, when the saddle-node conic section is a hyperbola, the Turing lines can be tangent to either the upper or the lower branch of the hyperbola. The issue is settled by considering the relative position between the origin of the Turing lines \((\alpha \mu, 0)\) and the asymptotes of the hyperbola, which intersect the horizontal axis at \((\nu_{HY}^\pm, 0)\). We define

\[
\alpha_{HY}^\pm = \nu_{HY}^\pm / \mu = \pm 2 / \sqrt{\beta^2 - 3}.
\]

For \(\alpha_{HY}^- < \alpha < \alpha_{HY}^+\), the Turing lines are tangent to the upper (lower) branch of the hyperbola when \(\mu > 0\) (\(\mu < 0\)) (Figs. 7.2(a) & 7.2(d)). For \(\alpha > \alpha_{HY}^+\), \(R_{2T}^+ (R_{2T}^-)\) is tangent to the upper branch when \(\mu > 0\) (\(\mu < 0\)) (Figs. 7.2(b) & 7.2(e)). For \(\alpha < \alpha_{HY}^-\), \(R_{2T}^+ (R_{2T}^-)\) is tangent to the upper branch when \(\mu > 0\) (\(\mu < 0\)) (Figs. 7.2(c) & 7.2(f)).

The two groups of critical curves \(\alpha_{SN}\) and \(\alpha_{HY}\) completely characterize the local collisions between the saddle-node bifurcations (represented by \(R_{2SN}^\pm\)) and the Turing/Belyakov-Devaney bifurcations (represented by \(R_{2T}^\pm\)). The situation of remote collisions between these bifurcations yields additional groups of critical curves on the \((\beta, \alpha)\) plane. First we study the remote collisions between \(R_{2SN}^\pm\) and \(R_{2T}^\pm\) in the \((\nu, R_2)\) plane. This amounts to comparing the conic section represented by \(R_{2SP}^\pm\), the projection of \(R_{2SN}^\pm\) onto the other branch,
Figure 7.2: Equilibria in the \((\nu, R_2)\) plane. The bold face digits label the spatial type in the region delimited by \(R^{SN}_{2\pm}\) and \(R^{T}_{2\pm}\) using the scheme in Fig. 2.12. The red (orange) portion of the saddle-node conic section represents \(R^{SN}_{2\pm}\) (\(R^{SN}_{2-}\)). In the first two rows, the black dashed lines are the asymptotes for the hyperbola represented by \(R^{SN}_{2\pm}\). The green (cyan) Turing line represents \(R^{T}_{2\pm}\) (\(R^{T}_{2-}\)). The gray curves represent \(R^{SP}_{2\pm}\). The red (green) dashed line represents \(\nu^{\pm}_T\) (\(\nu^{\pm}_T\)). Parameters: (a) \(\alpha = -0.3, \beta = 3, \mu = 1\). (b) \(\alpha = 3, \beta = 3, \mu = 1\). (c) \(\alpha = -2, \beta = 3, \mu = 1\). (d) \(\alpha = -0.3, \beta = 3, \mu = -1\). (e) \(\alpha = 3, \beta = 3, \mu = -1\). (f) \(\alpha = -3, \beta = 3, \mu = -1\). (g) \(\alpha = 2, \beta = 1, \mu = 1\). (h) \(\alpha = 2, \beta = 1, \mu = -1\).
with the Turing lines $R_{2\pm}^T$. The portion of the conic section $R_{2\pm}^{SP}$ in the upper half-plane is tangent to the horizontal axis at $(\nu, R_2) = (\beta \mu, 0)$ for $\mu > 0$. When the tangency point coincides with the origin of the Turing lines $(\alpha \mu, 0)$, we have a new critical curve $\alpha = \beta$, which is significant only for $\mu > 0$. Below (above) the line $\alpha = \beta$, the origin of the Turing lines lies to the left (right) of the origin of the saddle-node projected conic section.

To study the intersections between $R_{2\pm}^{SN}$ and $R_{2\pm}^T$ we substitute Eq. (7.13) into Eq. (7.9) and solve for $\nu$. For $R_{2\pm}^T$, the coefficient of the quadratic term in the resulting quadratic equation is

$$z_{ST}^\pm(\alpha, \beta) \equiv 4 - \frac{4\beta^2}{1 + \beta^2} + \frac{3(1 + \beta^2)}{2\alpha - 2\beta \pm \sqrt{(1 + \alpha^2)(1 + \beta^2)}}^2 + \frac{4\beta}{2\alpha - 2\beta \pm \sqrt{(1 + \alpha^2)(1 + \beta^2)}}.$$ 

Since the discriminants are in both cases nonnegative there are always two such intersections in the whole $(\nu, R_2)$ plane. The conic sections $R_{2\pm}^{SN}$ and $R_{2\pm}^{SP}$ can always be considered as being composed of a lower and an upper segment. Specifically, when a slice of constant $\nu$ intersects the conic section at two distinct points, the point with smaller (larger) absolute value of $R_2$ belongs to the lower (upper) segment. The fact that $R_{2\pm}^{SN}$ is enveloped by $R_{2\pm}^{SP}$ implies that if a Turing line is tangent to the lower (upper) segment of $R_{2\pm}^{SN}$, the same line must intersect the lower (upper) segment of $R_{2\pm}^{SP}$. In the ellipse case, this observation implies that there is no additional critical curve arising from intersections between $R_{2\pm}^{SP}$ and $R_{2\pm}^T$. However, in the hyperbola case, either segment has an upper branch and a lower branch. If the tangency point between the Turing line and $R_{2\pm}^{SN}$ occurs on the upper (lower) branch, the Turing line can either intersect the upper (lower) branch twice (in other words, fail to intersect the lower (upper) branch), or intersect the upper and lower branches once each. The dividing line between these two situations satisfies $z_{ST}^\pm(\alpha, \beta) = 0$ and defines two pairs of new critical curves on the $(\beta, \alpha)$ plane, denoted by $\alpha_{ST}^\pm$. Between the pair of curves comprising $\alpha_{ST}^+(\alpha_{ST}^-)$, $R_{2\pm}^T$ intersects both branches of $R_{2\pm}^{SP}$. In particular, when $\alpha$ is equal to $\alpha_{HY}^+(\alpha_{HY}^-)$, $R_{2\pm}^T$ reduces to one of the asymptotes for $R_{2\pm}^{SN}$, which always intersects both branches of $R_{2\pm}^{SP}$. In consequence, $\alpha_{HY}^+(\alpha_{HY}^-)$ lies between the pair of curves comprising $\alpha_{ST}^+(\alpha_{ST}^-)$.

Based on these critical curves, we can outline a procedure to determine the relative position between $\gamma_{ST}^\pm$ and $\gamma_{HY}^\pm$ in the $(\nu, \gamma)$ plane, given $\alpha$ and $\beta$. We can first draw the pair of curves $\gamma_{ST}^\pm$, whose shape depends only on $\beta$ and $\mu$. For $\mu > 0$, $\gamma_{ST}^\pm$ form a closed wedge for $0 < \beta < \sqrt{3}$ (Figs. 7.1(a) & 7.1(d)) and an open wedge for $\beta > \sqrt{3}$ (Fig. 7.1(g)), while intersecting the horizontal axis at $(\nu, \gamma) = (\beta \mu, 0)$ in both cases. For $\mu < 0$, $\gamma_{ST}^\pm$ do not exist for $0 < \beta < \sqrt{3}$ but form an open wedge above the horizontal axis for $\beta > \sqrt{3}$ (Fig. 7.1(j)). The curves $\gamma_{HY}^\pm$ both originate from $(\nu, \gamma) = (\alpha \mu, 0)$. For $\mu > 0$, the relative position between $(\alpha \mu, 0)$ and $(\beta \mu, 0)$ is determined by comparing $\alpha$ with $\beta$. The initial segments of the Turing curves $\gamma_{HY}^\pm$ are such that $\gamma_{HY}^\pm$ is counterclockwise with respect to $\gamma_{HY}^\pm$. For each Turing curve $\gamma_{HY}^\pm (\sigma = +$ or $-)$. we first determine whether $R_{2\pm}^T$ forms a tangency point with the upper or the lower branch of $R_{2\pm}^{SN}$ by comparing $\alpha$ with $\alpha_{HY}$. Next we determine whether the tangency point occurs on $R_{2\pm}^{SN}$ or $R_{2\pm}^{SP}$ by comparing $\alpha$ with $\alpha_{SN}$. If the tangency point lies on the upper branch of $R_{2\pm}^{SN}$ $(\theta = +$ or $-)$. then on the upper $(\nu, \gamma)$ plane, $\gamma_{HY}^\pm$ is tangent to $\gamma_{SN}^\pm$. Looking at the neighborhood of the tangency point between $\gamma_{HY}^\pm$ and $\gamma_{SN}^\pm$, we can readily see that the region between $\gamma_{SN}^\pm$ is split into three parts: the
region bounded by $\gamma_T^\pm$ and $\gamma_{SN}^\pm$ to the left of the tangency point (region $L_x^\sigma$), the region bounded by $\gamma_T^\pm$ and $\gamma_{SN}^\pm$ to the right of the tangency point (region $R_x^\sigma$) and the region not containing $\gamma_{SN}^\pm$ (region $N_x^\sigma$). There always exists an intersection between $\gamma_T^\pm$ and $\gamma_{SN}^\pm$ to the left of the tangency point. Therefore, if we follow $\gamma_{SN}^\pm$ towards the left, we always reach the left cusp; in other words, region $L_x^\sigma$ always contains the left cusp. However, to the right of the tangency point, we need to determine whether $\gamma_T^\pm$ forms an intersection with $\gamma_{SN}^\pm$ or fails to intersect $\gamma_{SN}^\pm$ altogether by comparing $\alpha$ with $\alpha_{ST}$. In the former case, region $N_x^\sigma$ is a closed wedge, while in the latter case, region $N_x^\sigma$ is an open wedge. In both cases, however, region $R_x^\sigma$ remains an open wedge. If the tangency point lies on the lower branch of $R_{3\theta}^\pm (\theta = + \text{ or } -)$, then on the upper $(\nu, \gamma)$ plane, there exists no tangency between $\gamma_T^\pm$ and $\gamma_{SN}^\pm$. However, we still need to determine whether $\gamma_T^\pm$ forms an intersection with $\gamma_{SN}^\pm$ or fails to intersect $\gamma_{SN}^\pm$ altogether by comparing $\alpha$ with $\alpha_{ST}$. In the former case, the region between $\gamma_{SN}^\pm$ is split into two open wedges, the wedge containing the left cusp (region $L_x^\sigma$) and the wedge not containing the left cusp (region $N_x^\sigma$).

The above procedure completely specifies the relative position between the saddle-node critical curves $\gamma_{SN}^\pm$ and either of the two Turing critical curves $\gamma_T^\pm$, but does not account for the possible intersections between $\gamma_T^\pm$ and $\gamma_T^-$. Algebraically, these intersections can be determined from the condition that $R_{3\theta}^\pm$ in both equations. Geometrically, we can project $R_{2\nu}^\pm$ onto the competing branch and compare the resulting conic section with $R_{2\nu}^\pm$ in the $(\nu, R_2)$ plane, although this picture only enhances our understanding to a very limited extent. The condition for a remote collision between $R_{2\nu}^\pm$ and $R_{2\nu}^-$. is

$$A \left( (R_{2\nu}^\pm)^2 + R_{2\nu}^\pm R_{2\nu}^- + (R_{2\nu}^-)^2 \right) - 2B \left( (R_{2\nu}^\pm)^2 + R_{2\nu}^- \right) + C = 0.$$  (7.17)

Substituting Eq. (7.13) into Eq. (7.17), we have a quadratic equation in $\nu$. Modulo a positive factor, its discriminant is

$$\Delta_T^\pm(\alpha, \beta) = -(1 + \beta + \alpha(-1 + \beta))(1 + \alpha + \beta(-1 + \alpha)),$$

and the coefficient of the quadratic term is

$$z_T(\alpha, \beta) \equiv 2 + 16\beta^2 - 2\beta^3 + 8\alpha^3\beta^-3 + \beta^2 + \alpha^4 (-3 + \beta^2)^2 + 16\alpha\beta^-1 + \beta^2 + \alpha^2 (7 + 10\beta^2 + 3\beta^4).$$

The condition $\Delta_T^\pm = 0$ defines a new pair of critical curves on the $(\beta, \alpha)$ plane,

$$\alpha_t^\pm = \frac{\beta + 1}{\beta - 1}, \quad \alpha_t^\pm = \frac{\beta - 1}{\beta + 1}.$$ 

When $\alpha$ lies above the upper branch of $\alpha_t^\pm$ or between $\alpha_t^\pm$ and the lower branch of $\alpha_t^\pm$, we have $\Delta_T^\pm > 0$ and Eq. (7.17) yields two solutions for $\nu$, i.e. the two Turing critical curves $\gamma_T^\pm$ intersect each other. Notably, this region in the $(\beta, \alpha)$ plane is contained in the region bounded by $\alpha_{SN}$, implicitly expressible as $z(\alpha, \beta) < 0$. One can justify the observation geometrically as follows. For $\gamma_T^\pm$ to intersect each other, it is necessary for the two intervals of $\nu$ where $\gamma_T^\pm / \gamma_T^\pm$ lies between $\gamma_{SN}^\pm$ to overlap with each other. A weaker necessary condition is that the two Turing lines $R_{2\nu}^\pm$ must point towards the same horizontal direction (either left or right). In light of the fact that the region outside the angle bounded
by $R_{2\pm}^T$ has spatial type 1, we require that Eq. (7.15) $< 0$ yield two open intervals, i.e. the quadratic coefficient $z(\alpha, \beta)$ must be negative.

The condition $z_{TT}(\alpha, \beta) = 0$ defines the last critical curve on the $(\beta, \alpha)$ plane, denoted by $\alpha_{TT}$. In the region bounded by $\alpha_{TT}$, we have $z_{TT}(\alpha, \beta) < 0$ and therefore the two remote collision points between $R_{2+}^T$ and $R_{2-}^T$ lie on different sides of the horizontal axis. Outside the region defined by $z_{TT}(\alpha, \beta) < 0$ and inside the region defined by $\Delta_{TT}(\alpha, \beta) > 0$, the two remote collision points both lie above (below) the horizontal axis for $\mu > 0 (\mu < 0)$. In consequence, $\alpha_{TT}^{\Delta}$ is only significant for $\mu > 0$. Finally, we observe that the critical curve $\alpha_{TT}$ is contained in the intersection between the two regions bounded by the pair of curves comprising $\alpha_{ST}^\pm$. Indeed, in the $(\nu, \gamma)$ plane, the region created from the intersection(s) between $\gamma_{TT}^\pm$ is precisely the intersection between regions $N_\nu^+$ and $N_\nu^-$. If the intersection becomes an open wedge, the regions $N_\nu^\pm$ must both become open wedges.

The above discussion suggests that the $\mu < 0$ case in the $(\beta, \alpha)$ plane is simpler than the $\mu > 0$ case owing to the absence of many of the critical curves. In fact, the critical curves in both the $(\nu, R_2)$ plane and in the $(\nu, \gamma)$ plane for the $\mu > 0$ case can be transformed into their $\mu < 0$ counterparts by changing the signs of $\mu$, $\nu$ and $R_2$ simultaneously, while bearing in mind that only the curves in $R_2 > 0$ are physically meaningful.

### 7.3.3 Temporal stability

The temporal stability of a uniform state involves the computation of the growth rate $s$ corresponding to an infinitesimal spatial perturbation of wavenumber $k$ for all $k \geq 0$. 

![Figure 7.3: The critical curves in the $(\beta, \alpha)$ plane. Different groups of critical curves are represented by different colors: $\alpha_{SN}$ (blue), $\alpha_{HY}$ (purple), $\alpha = \beta$ (gray), $\alpha_{ST}$ (cyan), $\alpha_{TT}^\Delta$ (red), $\alpha_{TT}$ (green). The upper branch of $\alpha_{ST}^+$ lies off-scale in $\alpha$ and is not shown. (a) $\mu > 0$. (b) $\mu < 0$.](image-url)
In practice we perturb the uniform state by $\epsilon e^{ikx+st}$ and linearize in $\epsilon \ll 1$. We obtain the following dispersion relation

$$s^2 - 2s(\mu - k^2 - 2R_2) + (\mu^2 + \nu^2 - 4(\mu + \beta \nu)R_2 + 3(1 + \beta^2)R_2^2) = 0.$$  

(7.18)

Geometrically, Eq. (7.18) describes an ellipse in the $(k^2, s)$ plane. If we consider the growth rate $s$ which can be complex, Eq. (7.18) has the following two solutions for $s$ as functions of $k^2$

$$s = -k^2 + \mu - 2R_2 \pm \sqrt{-(\nu - \alpha k^2)^2 + 4\beta(\nu - \alpha k^2)R_2 + (1 - 3\beta^2)R_2^2}. \quad (7.19)$$

From this expression we can readily see that the plot of $\Re(s) \text{ vs } k^2$ consists of an ellipse and two rays, while the plot of $\Im(s) \text{ vs } k^2$ consists of a line segment $\Im(s) = 0$ and a pair of hyperbolas (Fig. 7.4). Instability corresponds to $\Re(s) > 0$ for some $k^2 \geq 0$. When this is the case and $\Im(s) \neq 0$ the resulting bifurcation is a Hopf bifurcation with O(2) symmetry. As a result two branches of time-dependent spatially periodic solutions with period $2\pi/k$ are generated: a branch of standing waves and a branch of travelling waves [107].

It follows that the temporal stability of a uniform state depends only on the properties of intersection(s) between the corresponding plot of $\Re(s) \text{ vs } k^2$ and the coordinate axes. When $s = 0$ Eq. (7.18) reduces to Eq. (7.12) on replacing $k$ by $i\lambda$, and has already been discussed. We also need the intersection(s) on the vertical axis. Setting $k^2 = 0$ in Eq. (7.18), we obtain

$$s^2 - 2s(\mu - 2R_2) + (\mu^2 + \nu^2 - 4(\mu + \beta \nu)R_2 + 3(1 + \beta^2)R_2^2) = 0.$$  

(7.20)

This is nothing but the characteristic polynomial of the Jacobian at the equilibrium of the 2D dynamical system defined by Eq. (7.6). Let us define a new critical value of $R_2$ corresponding to zero discriminant of Eq. (7.20):

$$R_{2\pm}^{TE} = \frac{2\beta \pm \sqrt{1 + \beta^2}}{-1 + 3\beta^2} \nu.$$
Figure 7.5: Temporal regions in the \((\nu, R_2)\) plane. The red (orange) portion of the saddle-node conic section represents \(R_{SN_2}^{SN}\) \((R_{SN_2}^{SN})\). The purple lines represent \(R_{TE}^{TE}\). The two black dashed rays represent the Hopf line \(R_2 = \mu/2\). The bold face letters label the temporal types. Parameters: (a) \(\beta = 1, \mu = 1\). (b) \(\beta = 3, \mu = 1\). (c) \(\beta = 1, \mu = -1\). (d) \(\beta = 3, \mu = -1\).

The properties of intersection(s) on the vertical axis are completely determined by \(R_{SN_2}^{SN}, R_{TE}^{TE}\) and \(R_2 = \mu/2\) (the Hopf line), which are plotted together in the \((\nu, R_2)\) plane in Fig. 7.5. We will refer to the regions delimited by these three groups of critical curves collectively as the temporal regions. Except for the Hopf bifurcation the geometry of the temporal regions is identical to the geometry of the spatial regions with \(\alpha = 0\). Evidently there exist four different types of geometry for the temporal regions: \(\mu > 0, 0 < \beta < \sqrt{3}\) (Fig. 7.5(a)), \(\mu > 0, \beta > \sqrt{3}\) (Fig. 7.5(b)), \(\mu < 0, 0 < \beta < \sqrt{3}\) (Fig. 7.5(c)) and \(\mu < 0, \beta > \sqrt{3}\) (Fig. 7.5(d)). Each temporal region admits a unique temporal type defined as follows. When there are two intersections, corresponding to Eq. (7.20) having two real roots for \(s\), there are three possibilities regarding the signs of the roots – both positive \((B)\), one positive, one negative \((C)\) and both negative \((D)\). When there is only one intersection, corresponding to Eq. (7.20) having two complex roots for \(s\), there are two possibilities regarding the sign of the real part of the roots – positive \((A_{+})\) and negative \((A_{-})\). It follows from the definitions that temporal region \(C\) is identical to spatial region 3, which is temporally unstable. The remaining four types of temporal regions \((A_{\pm}, B\) and \(D)\) can intersect the remaining three types of spatial regions \((1, 2\) and \(4)\) to yield 12 different possible shapes of the plot of \(\Re(s)\) vs \(k^2\). When the intersection involves temporal type \(A_{+}/B\) or spatial type \(4\), inevitably there exists intersection between the plot of \(\Re(s)\) vs \(k^2\) and the first quadrant, implying temporal instability. When the intersection involves temporal type \(A_{-}/D\) and spatial type \(1/2\), one can easily show that there exists no intersection between the plot of \(\Re(s)\) vs \(k^2\) and the first quadrant, implying temporal stability.
If we only care about the temporal stability of the uniform state without worrying about the shape of the dispersion relation, then the boundary $R_{2 \pm}$ between temporal regions $A_-$ and $D$ is not critical. The only additional critical curve determining the temporal stability is the Hopf line $R_2 = \mu/2$. Similarly the boundary between spatial regions 1 and 2 or spatial regions 3 and 4 is not critical for temporal stability. In any case, as already mentioned, the only uniform states we expect as parts of a localized state are of types 1 and 2. Therefore, the uniform background state(s) of a localized state are temporally stable if and only if they lie above the Hopf line $R_2 = \mu/2$.

This completes our discussion of the spatially uniform states and their stability properties in both space and time. We are now in the position to use the above results to locate regions in parameter space with different types of localized structures.

### 7.4 Heteroclinic cycles

In the previous chapters we used the numerical continuation and bifurcation software AUTO [78] to explore parameter space. The continuation software was initiated using analytically computed localized structures. Such solutions are found near the two saddle-nodes on the branch of uniform phase-locked states (Appendix 7.6) and near the Turing bifurcation (Appendix 7.7). In each computation, we continue the analytic solution in a single parameter, taken to be $\gamma$. To compute reflection-symmetric localized states, we imposed Neumann boundary conditions on both sides of the domain. Subsequently, the entire profile can be obtained from the computed profile by appending to the latter its mirror image with respect to either the left or the right boundary.

In addition, there is a branch of front states, referred to as primary fronts (PF), that bifurcates from each type 2 cusp point. At each of these points a pair of saddle-nodes is created via a hysteresis bifurcation, and with them a region of bistability between coexisting uniform states. The uniform state with smaller/middle/larger $L^2$-norm will be denoted by $A^-/A^0/A^+$, and the region of bistability will be referred to as multiple-equilibria region (MER). The PF states can be continued in two parameters, $\gamma$ and $\nu$. In this case we again impose Neumann boundary conditions but must also impose an integral constraint to prevent AUTO from following the family of solutions generated by translation in $x$.

The PF states belong to the class of heteroclinic connections. In this section, we review the three types of heteroclinic cycles or spatially localized states formed by a pair of such connections at fixed $\alpha$, $\beta$, $\mu$ and $\nu$. In addition, we show the termination behavior of the bifurcation diagrams associated with these localized states when the connection to the background equilibrium is lost. To motivate a classification program, the relevant existence curves (for codimension-one connections)/regions (for codimension-zero connections) are shown in the $(\nu, R_2)$ plane.

In the following we primarily focus on single-pulse branches. The domain size $L$ is normally taken to be large to simulate an infinite domain. In practice, we choose $L$ at least large enough to allow interesting structures to develop. The branch behavior appears independent of $L$ if the solution approaches a background equilibrium towards at least one end.
Figure 7.6: The critical curves for spatial and temporal stability of equilibria in the \((\nu, R_2)\) plane at \(\alpha = 4\), \(\beta = 1\) and \(\mu = 1\). The colors of these curves correspond with earlier definitions. Also shown are the existence curves \(R_2^* (\nu)\) for primary fronts (PF) in dot-dashed blue, \(R_{2,1}^{LP} (\nu)\) for primary localized phase-winding states (LPWS) in solid blue and \(R_{2,2}^{LP} (\nu)\) for secondary LPWS in solid yellow.

### 7.4.1 Localized phase-winding states (LPWS) \((\alpha = 4\), \(\beta = 1\), \(\mu = 1\))

In Chapter 6, the parameter combination \(\alpha = 4\), \(\beta = 1\) and \(\mu = 1\) was chosen to illustrate the transition from PF to two flavors of localized phase-winding states (LPWS). Fig. 7.6 shows the critical curves for this case in the \((\nu, R_2)\) plane. These include the ellipse \(R_{2 \pm}^{SN}\) bounded by the left cusp at \(\nu_{SN}^-\) and the right cusp at \(\nu_{SN}^+\), and the line \(R_{2-}^T\) of Turing bifurcations tangent to \(R_{2-}^{SN}\) at \(\nu_T = 0.4914\). The PF curve \(R_2^* (\nu)\) (in dot-dashed blue) bifurcates from the left cusp at \(\nu_{SN}^-\) and continues to intersect \(R_{2-}^{SN}\) where \(A^0\) and \(A^+\) annihilate in a saddle-node bifurcation. Two LPWS curves originate near this intersection, the primary LPWS curve \(R_{2,1}^{LP} (\nu)\) (in solid blue) and the secondary LPWS curve \(R_{2,2}^{LP} (\nu)\) (in solid yellow). As detailed in Chapter 6, the former bifurcates from and hence merges smoothly with the PF curve, while the latter appears to approach but does not intersect the PF curve. As \(\nu\) increases, both LPWS curves converge to the CGLE limit \(R_2 \to 0\).

**The PF branch**

The result of branch-following in \(\gamma\) starting from the upper saddle-node bifurcation is shown in Fig. 7.7 for \(\nu = 0.4 < \nu_T\). Near the upper saddle-node bifurcation, the solution forms a pulse (panel (i)) that approaches \(A^+\) towards \(x \to \pm \infty\). With increasing \(\gamma\) the pulse grows larger in amplitude and becomes a pair of PF (panel (ii)) between \(A^+\) and \(A^-\). These PF states exist on the nearly vertical portion of the branch in Fig. 7.7. The vertical asymptote is \(\gamma = \gamma^* \equiv \gamma (R_2^*)\), referred to as the *collapsing point*. As \(\gamma\) continues to increase the pair of PF migrate towards the boundaries, as \(A^-\) develops from the middle pushing \(A^+\) aside. Once the pair of PF reach the boundaries, they decrease in amplitude and shrink into a pulse (panel (iii)) that now resembles a localized peak in an \(A^-\) background. The branch terminates in the lower saddle-node bifurcation. We remark that on an infinite domain, the
Figure 7.7: Bifurcation diagram for $\alpha = 4$, $\beta = 1$, $\mu = 1$, $\nu = 0.4$, $L = 200$. (a) The $L^2$-norm $N \equiv \sqrt{L^{-1} \int_0^L |A|^2 \, dx}$ as a function of $\gamma$. The blue curve represents the branch followed by AUTO, while the black curve represents the uniform state(s). (b) The solution profiles at three points on the bifurcation diagram in (a). The left part is the plot of $U$ and $V$ as functions of $x$, with $U/V$ represented by the solid red/green line. The right part is the plot of the projection of the solution trajectory on the 2D $(U,V)$ phase plane, with the horizontal/vertical axis representing $U/V$. The blue curve shows the trajectory and the dot(s) show the uniform state(s), with multiple equilibria distinguished by different colors.

single branch shown in Fig. 7.7 must be interpreted as two distinct branches of localized states bifurcating, respectively, from the upper and lower saddle-nodes.

The branch approaches $\gamma^*$ monotonically from below but in an oscillatory fashion from above. This is a consequence of the fact that $A^+$ has spatial type 2 with real eigenvalues while $A^-$ has spatial type 1 with complex eigenvalues.

The heteroclinic cycle at the collapsing point $\gamma^*$ inherits any instabilities of the pair of states $A^\pm$ connected by the cycle. In the neighborhood of $\gamma^*$, $A^+$ is above the Hopf line and hence temporally stable, while $A^-$ is below the Hopf line and hence unstable. Consequently we expect all PF states at these parameter values to be unstable, despite the oscillatory approach to $\gamma^*$ from above. Explicit stability computation confirms this expectation. Thus no temporally stable localized states are present at these parameter values.

The LPWS branches

As discussed in Chapter 6, the primary and secondary LPWS can be found on the $\phi = 0$ and $\phi = \pi$ branches of localized states followed from the lower Turing bifurcation (cf. Figs. 6.8 and 6.9). Fig. 7.8 shows the lower and upper portions of both branches at $\nu = 2$. On a half-domain of finite size $L$, either branch bifurcates from a wavetrain consisting of symmetric periodic states with a small amplitude. At either collapsing point $(\gamma_{L_1}^P/\gamma_{L_2}^P)$, the solution profile takes the form $A^- \leftrightarrow PWS^- \leftrightarrow PWS^+ \leftrightarrow A^-$, where $PWS^\pm$ denote
Figure 7.8: Closeup of the lower/upper (a)/(b) portion of the branch following from the Turing bifurcation ($\phi = 0/\phi = \pi$ branch (red/green), $\alpha = 4$, $\beta = 1$, $\mu = 1$, $\nu = 2$, $L = 400$), and the branch of symmetric periodic states containing $SP^*$ (blue).

As $\gamma$ increases, $PWS^\pm$ follows the PWS branch with varying period (cf. Fig. 6.7 for the $\nu = 1.5$ branch; the $\nu = 2$ branch takes the same form). As introduced in §6.3.2, the pacemaker defect $PD$ is equivalent to a stationary ($-+$)-defect and forms a continuous family at fixed $\gamma$. Hence this defect only serves as a mediator to accommodate the PWS to the Neumann boundary condition imposed on one side of the half-domain. In other words, the boundary condition on this side enables the solution profile to become homoclinic to $PWS^\pm$ at this $\gamma$. In contrast, the pair of fronts $F^\pm$ become a stationary ($+-$)-defect, which forms a discrete family at fixed $\gamma$. Hence the continuation of the $L_1^u$ and $L_2^u$ branches is equivalent to the continuation of two branches of stationary ($+-$)-defects, one having a crest and the other having a trough in the middle. In the $(U,V)$ phase plane, a stationary ($+-$)-defect lies inside the closed curve representing the PWS (cf. Fig. 6.8, panel (iii)). As the PWS branch approaches $SP^*$ (cf. Fig. 6.7, panel (iii)), the PWS becomes a double covering of a line segment representing a symmetric periodic state $SP^*$ and correspondingly the stationary ($+-$)-defect merges into $SP^*$. As a remark, we recall from Chapter 3 that a similar pair of defect branches homoclinic to symmetric periodic states bifurcate from the Eckhaus boundary and are responsible for defect-mediated snaking, but the relations between this defect family and the stationary ($+-$)-defects are unclear. The symmetric periodic state $SP^*$ should be identical to $SP^*$ if $L = n\lambda$ ($n \in \mathbb{N}$), where $\lambda$ is the wavelength of $SP^*$. Generally, the wavelength $\lambda'$ of $SP^*$ can be either $L/n$ or $L/(n - 1/2)$. This problem of wavelength selection has been addressed in Ref. [22] in the context of homoclinic snaking in the Swift-Hohenberg equation on finite domains. In
Figure 7.9: The critical curves for spatial stability of equilibria on the \((\nu, R_2)\) plane at \(\alpha = -1.5, \beta = 6\) and \(\mu = -1\). The colors of these curves correspond with earlier definitions. Also shown are the existence curve for PF in dot-dashed blue, and the limits of defect-mediated snaking (DMS) region in dot-dashed black.

In Chapter 3, the parameter combination \(\alpha = -1.5, \beta = 6\) and \(\mu = -1\) was chosen to illustrate the transition from PF to defect-mediated snaking (DMS). Fig. 7.9 shows the critical curves for this case on the \((\nu, R_2)\) plane. These include the hyperbola \(R_{2N}^{SN}\) bounded by the cusp at \(\nu_{SN}^+\), and the line \(R_{2-}^T\) of Turing bifurcations tangent to \(R_{2+}^{SN}\) at \(\nu_{T}^-\). The PF curve \(R^*_2(\nu)\) (in dot-dashed blue) originates from the cusp at \(\nu_{SN}^+\) and continues to intersect \(R_{2-}^T\) where \(A^+\) undergoes a supercritical Turing bifurcation. A DMS region bounded by two limits \(R_{2,1}^{DMS}(\nu)\) and \(R_{2,2}^{DMS}(\nu)\) (both in dot-dashed black) is created from this intersection. As \(\nu\) increases the DMS region has an additional limit and all three limits approach straight lines in the \((\nu, R_2)\) plane as \(\nu \to \infty\).
The DMS branch

As discussed in Chapter 3, DMS happens for the branch of localized states followed from the lower saddle-node bifurcation at \( \nu = 7 \) (cf. Fig. 3.4). The properties of DMS has been extensively discussed in Chapters 3 to 5, while in this section we examine the termination of DMS. In Fig. 7.10(a), the DMS region \( \Gamma : \{ \gamma \in [\gamma_{1DMS}, \gamma_{2DMS}] \} \) appears as the blue vertical line on the right. A detailed view of this region is shown as the grey curve in the inset of Fig. 7.11(a). As seen in this figure and explained in Chapter 5, the folds on the DMS branch are due to side tangencies and hence appear quite “sharp” in terms of the \( L^2 \)-norm \( N \). To highlight the profile change due to the elimination of \( A^- \), in the following discussion we will place the \( A^- \) plateau in the center. Thus apart from the wavetrain formed by a symmetric periodic state, the solution profile always has a large amplitude homoclinic orbit to this periodic state in the center, referred to as the core. Near the boundaries there may exist a small amplitude homoclinic orbit, again referred to as the defect.

As \( A^- \) is eliminated the DMS branch folds back from a uniform segment and continues to smaller \( \gamma \). In this process the wavetrain remains uniform. The branch then enters another larger pinning region \( \Gamma' \) further from the Turing bifurcation \( \gamma^T \), where the branch snakes towards lower \( N \) and back towards larger \( N \) again. The first half of the snaking curve in \( \Gamma' \) takes the form of the blue curve in the middle of Fig. 7.10(a). There exists a series of sharp folds similar to those on the DMS branch, where a uniform wavetrain acquires a central defect or vice versa. Remarkably, as \( N \) decreases the order of folds is gradually reversed with folds (iv) and (iii) corresponding respectively to folds (i) and (ii). Thus the segment between (i) and (ii) is a defect segment, while the segment between (iii) and (iv) is a uniform segment. Each time the branch goes through a defect segment, the defect deforms to annihilate one wavelength analogous to the solution behavior going down the DMS branch. The crucial difference is that due to the absence of the \( A^- \) plateau, the length of the wavetrain is always close to the size of the entire domain. As a result, the wavelength of the rolls must keep increasing (compare panels (ii) and (iii)) as the number of rolls decreases. Further down the branch the amplitude of the rolls becomes comparable to the core (panel (iv)). In the phase space both pass close to \( A^- \) and thus resemble a symmetric single-peak pulse homoclinic to \( A^- \). This pulse, referred to as pulse B, lies between the first two folds of the DMS branch denoted by \( \gamma_i^F \) and \( \gamma_j^F \). For later discussions, we will refer to the other type of single-peak pulse existing between \( \gamma_{SN} \) and \( \gamma_i^F \) as pulse A. After fold (iv) the defect fails to annihilate another wavelength (panel (v)). Instead the core undergoes a saddle-node bifurcation at fold (v) denoted by \( \gamma^{CO} \).

The second half of the snaking curve in \( \Gamma' \) takes the form of the blue curve in the middle of Fig. 7.11(a). Near \( \gamma^{CO} \), a pair of solutions with nearby \((\gamma, N)\) values but lying on different halves of the snaking curve possess the same number of peaks. However, far from \( \gamma^{CO} \) the core shapes appear quite different (compare panel (iv) in Fig. 7.10 and panel (i) in Fig. 7.11), although closer to \( \gamma^{CO} \) the core shapes become almost identical (compare panel (iii) in Fig. 7.10 and panel (ii) in Fig. 7.11). As \( N \) increases the branch undergoes a series of sharp folds and the order of folds is gradually reversed exactly like Fig. 7.10(a). However, further up the branch the core spans several wavelengths (cf. panels (iii) and (iv) in Fig. 7.11), which shrinks the effective domain for the wavetrain. As a result the branch undergoes fewer turns compared with Fig. 7.10(a) before leaving \( \Gamma' \). Afterwards the branch continues...
Figure 7.10: (a) Bifurcation diagram for $\alpha = -1.5$, $\beta = 6$, $\mu = -1$ and $\nu = 7$. The first half of the branch is shown as the blue curve; the second half is shown in grey for comparison. The branch leaves the DMS region $\Gamma$ on the right and enters the pinning region $\Gamma'$ in the middle. (b) Five profiles on the blue curve in (a). In contrast to Fig. 3.4, all profiles are shown with the core placed at $x = 0$. 
to larger $\gamma$ and reenters the DMS region $\Gamma$. However in contrast to the departure from $\Gamma$ (grey curve in the inset of Fig. 7.11(a)), during the reentry the sharp folds accumulate slowly towards $\Gamma$ from the left (blue curve in the inset). A DMS mechanism identical to the original DMS branch is then restored, but this time the core consists of, in addition to the pair of fronts connecting the wavetrain to the $A^-$ plateau, a pair of pulse $B$’s next to these fronts and a pair of pulse $A$’s in the center (panel (v)). As the branch is followed downwards, these pulses always appear to be equally spaced. However, since they interact weakly via exponentially decaying tails, the numerical continuation encounters difficulties when their separation becomes too large.

In summary, there are essentially two types of folds identified during the termination of DMS, namely the sharp folds and the fold $\gamma^{CO}$. The former result from Eckhaus bifurcations that create defects in uniform wavetrains, while the latter represents a saddle-node bifurcation of the core. Although both types of folds involve homoclinics to periodic orbits, they respectively resemble the folds created by side tangencies and tip tangencies studied in Chapter 5. Therefore we may identify the multi-pulse branch that exists in the DMS region $\Gamma$ as the companion branch $\bar{L}_0$ to the original DMS branch $L_0$, as conjectured in §5.7. As is the case with DMS, the wavelength selection mechanism in the second pinning region $\Gamma'$ remains poorly understood.
7.5 Discussion

In §7.4, we have plotted the loci of localized states in the \((\nu, R_2)\) plane and related these curves to the classification of uniform states in §7.3. For any localized state, let us denote \(R_2^\pm \equiv |A^\pm|^2\) where \(A^\pm\) denote the lower and upper uniform states at this parameter value. At a type 2 cusp in the \((\nu, \gamma)\) plane, namely the point of nascent bistability between \(A^\pm\), codimension-1 primary fronts (PF) between \(A^\pm\) are created. Two-parameter continuation in \(\nu\) and \(\gamma\) of PF reveals two classes of localized states resulting from interaction with local bifurcations of uniform states. When \(R_2^\pm\) exits the multiple equilibria region (MER), the PF becomes a pair of codimension-1 primary and secondary localized phase-winding states (LPWS) with background \(A^-\). When \(R_2^\pm\) enters a type 4 region, the PF curve broadens into a defect-mediated snaking (DMS) region (codimension-0), where the localized states are embedded in \(A^- (A^+)\). The PF curve disappears when a type 2 cusp becomes type 4 at \(\alpha = \alpha_{SN}\), but the DMS region far from this cusp could possibly persist.

As shown in Fig. 7.9, the existence loci of localized states can extend to \(\nu \to \infty\) when the MER is infinite \((\beta > \sqrt{3})\). In this case, we introduce \(t' = \nu t\), \(A' = A/\sqrt{\nu}\), \(x' = \sqrt{\nu} x\) into Eq. (7.3) and then drop the primes to get

\[
A_t = (\mu \nu + i) A - (1 + i\beta)|A|^2 A + (1 + i\alpha) A_{xx} + \frac{\gamma}{\nu^{3/2}}. \tag{7.21}
\]

In the limit \(\nu \to \infty\), Eq. (7.21) takes the simpler form

\[
A_t = i A - (1 + i\beta)|A|^2 A + (1 + i\alpha) A_{xx} + \gamma, \tag{7.22}
\]

where \(\gamma/\nu^{3/2}\) has been redefined as \(\gamma\). Localized states in Eq. (7.22) can again be found by following the PF solutions that bifurcate from type 2 cusps at \(\beta = \sqrt{3}\). Such computations lead to DMS embedded in \(A^-\) for smaller \(\alpha\) and DMS embedded in \(A^+\) for larger \(\alpha\). In the former case, DMS can interact with the lower saddle-node bifurcation at \(\gamma_{SN}\) and produce an infinite stack of isolas. A dynamical theory for this process in the spirit of Chapter 5 is left as future work.

The classification of uniform states in §7.3 not only provides a geometrical method to track the motions of bifurcation curves as the control parameters vary, but also serves as a road map for locating PF, LPWS and DMS. For example, Fig. 7.6 shows that at \((\beta, \alpha) = (1, 4)\), the origin of the Turing lines \((\nu, R_2) = (\alpha \mu, 0)\) lies to the right of the tangency \((\nu, R_2) = (\beta \mu, 0)\) between the saddle-node conic section and the horizontal axis. As a result the PF curve bifurcating from the left cusp can exit the MER and become a pair of LPWS curves. However, for \(\alpha < \beta\) the order of these two points is reversed. This time the PF curve cannot exit the MER without intersecting the type 4 region, and must therefore broaden into a DMS region instead. Numerical results that follow from this classification program, including the interplay between LPWS and DMS, are also left as future work.
7.6 Appendix: Weakly nonlinear analysis near $\gamma = \gamma_{SN}$

To find localized states near $\gamma = \gamma_{SN}$ we write $\gamma = \gamma_{SN} + \epsilon^2 \delta$, where $\epsilon \ll 1$ and $\delta \sim O(1)$. With $A = U + iV$ the localized states can be written in the form

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} + \epsilon \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} + \epsilon^2 \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} + \cdots,$$

where the first term is the uniform phase-locked state $A_i$ and the second corresponds to the space-dependent terms that decay to 0 as $x \to \pm \infty$. The uniform phase-locked state $A_i$ can be approximated by

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} + \epsilon \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} + \epsilon^2 \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} + \cdots.$$

Expanding to $O(\epsilon^2)$, we get

$$R_{21} = \left( \frac{2\delta \gamma}{-2(\mu + \beta \nu) + 3(1 + \beta^2)R_{20}} \right)^{1/2}.$$

Therefore Eq. (7.5) yields

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\mu + R_{20} \\ \nu - \beta R_{20} \end{bmatrix} R_{20} \equiv \begin{bmatrix} \eta \\ 1 \end{bmatrix} \Gamma_0, \quad (7.23)$$

$$\begin{bmatrix} U_1 \\ V_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\mu + 2R_{20} \\ \nu - 2\beta R_{20} \end{bmatrix} R_{21} \equiv \sqrt{\delta} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \Gamma_1. \quad (7.24)$$

The $x$-dependent term can be expanded as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \epsilon \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + \epsilon^2 \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} + \cdots,$$

where all quantities depend on $x$ via the slow spatial scale $X = \epsilon^{1/2} x$. The linear operator takes the form $\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1$, where

$$\mathcal{L}_0 = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix} \partial_{XX}$$

while the nonlinear term takes the form $\mathcal{N} = \mathcal{N}_0 + \epsilon \mathcal{N}_1 + \epsilon^2 \mathcal{N}_2 + \cdots$, where

$$\mathcal{N}_0 = -\left[ \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \right] \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix},$$

$$\mathcal{N}_1 = -2 \left[ \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \begin{bmatrix} U_1 + u_1 \\ V_1 + v_1 \end{bmatrix} \right] \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix},$$

$$\mathcal{N}_2 = -\left\{ \left[ \begin{bmatrix} U_1 + u_1 \\ V_1 + v_1 \end{bmatrix} \begin{bmatrix} U_1 + u_1 \\ V_1 + v_1 \end{bmatrix} + 2 \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \begin{bmatrix} U_2 + u_2 \\ V_2 + v_2 \end{bmatrix} \right] \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix}, \right\}.$$
and the constant forcing term takes the form
\[
\begin{bmatrix}
\gamma + \epsilon^2 \delta \\
0
\end{bmatrix}.
\]

At order \(\epsilon^0\) stationary solutions satisfy
\[
\{\mathcal{L}_0 + \mathcal{N}_0\} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} + \begin{bmatrix} \gamma \\ 0 \end{bmatrix} = 0
\]
which holds by virtue of the definition of \(U_0\) and \(V_0\). At order \(\epsilon\) we obtain
\[
\{\mathcal{L}_0 + \mathcal{N}_0\} \begin{bmatrix} U_1 + u_1 \\ V_1 + v_1 \end{bmatrix} = -\{\mathcal{L}_1 + \mathcal{N}_1\} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}.
\]

The \(X\)-independent terms cancel by virtue of the definition of \(U_1\) and \(V_1\). We are left with
\[
\begin{bmatrix} \mathcal{L}_0 + \mathcal{N}_0 - 2 \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} U_0^2 & U_0 V_0 \\ U_0 V_0 & V_0^2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \xi \\ 0 \end{bmatrix}
\]

implying that
\[
\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \xi \\ 1 \end{bmatrix} B(X),
\]

where \(B(X)\) is an unknown function of \(X\). Proceeding to \(O(\epsilon^2)\) we obtain
\[
\{\mathcal{L}_0 + \mathcal{N}_0\} \begin{bmatrix} U_2 + u_2 \\ V_2 + v_2 \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \end{bmatrix} = -\{\mathcal{L}_1 + \mathcal{N}_1\} \begin{bmatrix} U_1 + u_1 \\ V_1 + v_1 \end{bmatrix} - \mathcal{N}_2 \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}.
\]

The \(X\)-independent terms satisfy
\[
\{\mathcal{L}_0 + \mathcal{N}_0\} \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \end{bmatrix} = -\mathcal{N}_1^0 \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} - \mathcal{N}_2^0 \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}.
\]

Subtracting the above two equations, we have the \(X\)-dependent terms as
\[
\begin{bmatrix} \mathcal{L}_0 + \mathcal{N}_0 - 2 \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} U_0^2 & U_0 V_0 \\ U_0 V_0 & V_0^2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = -\mathcal{L}_1 \begin{bmatrix} \xi \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix}
\]

\[
\cdot \left( (U_0 U_1 + V_0 V_1) B + (\xi U_0 + V_0) B^2 \right) \begin{bmatrix} \xi \\ 1 \end{bmatrix} + (\xi U_0 + V_0) B \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} +
\]

\[
(2(\xi U_1 + V_1) B + (1 + \xi^2) B^2) \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}.
\]

Using the definitions (7.23) and (7.24) the RHS can be rewritten as
\[
\text{RHS} = -\begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \partial_{XX} B
\]

\[
+ \Gamma_0 \left( 2\sqrt{\delta} \Gamma_1 B + B^2 \right) \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix} \left( 2(\eta \xi + 1) \begin{bmatrix} \xi \\ 1 \end{bmatrix} + (1 + \xi^2) \begin{bmatrix} \eta \\ 1 \end{bmatrix} \right).
\]
Denote the matrix multiplying \[
\begin{bmatrix}
u_2 \\
v_2
\end{bmatrix}
\] on the LHS by \(\mathcal{M}\). The solvability condition is obtained by taking the scalar product with \[
\begin{bmatrix}1 & \tau
\end{bmatrix}
\] to eliminate the \(u_2\) and \(v_2\) terms, where \(\tau\) satisfies \[
\begin{bmatrix}1 & \tau
\end{bmatrix}\mathcal{M} = 0.
\] It turns out that \(\tau = 0\) after some algebraic simplification. We are left with
\[
a_{SN}B_{XX} = b_{SN}\Gamma_0\left(2\sqrt{\delta}\Gamma_1B + B^2\right)
\]
where
\[
a_{SN} = \xi - \alpha, \quad b_{SN} = (\eta + 2\xi + 3\eta\xi^2) - \beta \left(3 + 2\eta\xi + \xi^2\right).
\]
This equation admits spatially homogeneous solutions \(B = -2\sqrt{\delta}\Gamma_1\), corresponding to the other equilibrium point created in the saddle-node bifurcation. In addition, it admits a branch of \(X\)-dependent localized states
\[
B(X) = -3\Gamma_1\sqrt{\delta} sech^2 \left[ \left(\frac{\Gamma_1\sqrt{\delta}}{2a_{SN}/(b_{SN}\Gamma_0)}\right)^{1/2}X \right].
\]
The corresponding full solution is, to \(O(\epsilon^2)\),
\[
\begin{bmatrix}U \\ V\end{bmatrix} = \begin{bmatrix}U \\ V\end{bmatrix}_{\xi} - 3\Gamma_1\sqrt{\gamma - \gamma_{SN}} \begin{bmatrix}\xi \\ 1\end{bmatrix} sech^2 \left[ \left(\frac{\Gamma_1\sqrt{\gamma - \gamma_{SN}}}{2a_{SN}/(b_{SN}\Gamma_0)}\right)^{1/2}x \right].
\]
This corresponds to the localized solution with monotonic tails as reviewed in §2.3.2.

### 7.7 Appendix: Weakly nonlinear analysis near \(\gamma = \gamma_T\)

To find the localized states we write \(\gamma = \gamma_T + \epsilon^2\delta\), where \(\epsilon \ll 1\) and \(\delta \sim O(1)\). With \(A \equiv U + iV\) the localized states can be written in the form
\[
\begin{bmatrix}U \\ V\end{bmatrix} = \begin{bmatrix}U \\ V\end{bmatrix}_{\xi} + \begin{bmatrix}u \\ v\end{bmatrix}
\]
where the first term is the uniform phase-locked state \(A_{\xi}\) and the second corresponds to the space-dependent terms that decay to 0 as \(x \to \pm \infty\). The uniform phase-locked states \(A_{\xi}\) can be approximated by the series
\[
\begin{bmatrix}U \\ V\end{bmatrix}_{\xi} = \begin{bmatrix}U_0 \\ V_0\end{bmatrix} + \epsilon^2 \begin{bmatrix}U_2 \\ V_2\end{bmatrix} + \cdots
\]
while the \(x\)-dependent terms can be expanded as
\[
\begin{bmatrix}u \\ v\end{bmatrix} = \epsilon \begin{bmatrix}u_1 \\ v_1\end{bmatrix} + \epsilon^2 \begin{bmatrix}u_2 \\ v_2\end{bmatrix} + \epsilon^3 \begin{bmatrix}u_3 \\ v_3\end{bmatrix} + \cdots.
\]
All the quantities in the equation above depend on both the short spatial scale \(x\) and the long spatial scale \(X = \epsilon x\). The linear operator takes the form \(\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2\), where
\[
\mathcal{L}_0 = \begin{bmatrix}\mu & -\nu \\ \nu & \mu\end{bmatrix} + \begin{bmatrix}1 & -\alpha \\ \alpha & 1\end{bmatrix} \partial_{xx}, \quad \mathcal{L}_1 = 2\begin{bmatrix}1 & -\alpha \\ \alpha & 1\end{bmatrix} \partial_{xX}, \quad \mathcal{L}_2 = \begin{bmatrix}1 & -\alpha \\ \alpha & 1\end{bmatrix} \partial_{XX},
\]
while the nonlinear terms take the form $\mathcal{N} = \mathcal{N}_0 + \epsilon \mathcal{N}_1 + \epsilon^2 \mathcal{N}_2 + \cdots$, where

\[
\mathcal{N}_0 = - \begin{bmatrix} U_0 & V_0 \end{bmatrix} \begin{bmatrix} U_0 & 1 \\ V_0 & \beta \end{bmatrix}, \\
\mathcal{N}_1 = -2 \begin{bmatrix} U_0 & V_0 \end{bmatrix} \begin{bmatrix} u_1 & 1 \\ v_1 & \beta \end{bmatrix}, \\
\mathcal{N}_2 = -\left\{ \begin{bmatrix} u_1 & v_1 \end{bmatrix} + 2 \begin{bmatrix} U_0 & U_2 + u_2 \\ V_0 & V_2 + v_2 \end{bmatrix} \right\} \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix}, \\
\mathcal{N}_3 = -2 \left\{ \begin{bmatrix} u_1 & v_1 \end{bmatrix} \begin{bmatrix} U_2 + u_2 \\ V_2 + v_2 \end{bmatrix} + \begin{bmatrix} U_0 & U_2 + v_2 \\ V_0 & V_2 + v_2 \end{bmatrix} \right\} \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix}.
\]

At order $\epsilon^0$ stationary solutions satisfy

\[
(\mathcal{L}_0 + \mathcal{N}_0) \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} + \begin{bmatrix} \gamma \\ 0 \end{bmatrix} = 0.
\]

As before, we have

\[
\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix} -\mu + R_{20} \\ \nu - \beta R_{20} \end{bmatrix} R_{20} \equiv \begin{bmatrix} \eta \\ 1 \end{bmatrix} \Gamma_0.
\]

At order $\epsilon$ we obtain

\[
\{ \mathcal{L}_0 + \mathcal{N}_0 \} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = -\mathcal{N}_1 \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}, \text{ or } \mathcal{M} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

where

\[
\mathcal{M} \equiv \mathcal{L}_0 + \mathcal{N}_0 - 2 \begin{bmatrix} 1 & -\beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} U_0^2 & U_0 V_0 \\ U_0 V_0 & V_0^2 \end{bmatrix}.
\]

The requirement that $\mathcal{M}$ be singular selects the Turing wavenumbers $\pm k_i$ in Eq. (7.16). The general form of the corresponding solution is

\[
\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \xi \\ 1 \end{bmatrix} \left\{ B e^{i k_i x} + \bar{B} e^{-i k_i x} \right\},
\]

where $\xi$ satisfies $\mathcal{M} \begin{bmatrix} \xi \\ 1 \end{bmatrix} = 0$ and $B(X)$ is a complex function of $X$. Proceeding to $O(\epsilon^2)$ we obtain

\[
\{ \mathcal{L}_0 + \mathcal{N}_0 \} \begin{bmatrix} U_2 + u_2 \\ V_2 + v_2 \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \end{bmatrix} = -\{ \mathcal{L}_1 + \mathcal{N}_1 \} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \{ \mathcal{L}_2 + \mathcal{N}_2 \} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}.
\]

We denote by $\mathcal{M}^n$ the operator obtained by replacing $\partial_{xx}$ in $\mathcal{M}$ by $-n^2 k^2_i$. The $X$-independent terms can be written in the form

\[
\mathcal{M}^0 \begin{bmatrix} U_2 \\ V_2 \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \end{bmatrix} = 0.
\]
It follows that the solution for $U_2$ and $V_2$ takes the form
\[
\begin{bmatrix}
U_2 \\
V_2
\end{bmatrix} = -\delta (\mathcal{M}^0)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \delta \begin{bmatrix} \tau \\ 1 \end{bmatrix} \Gamma_2.
\]

The remaining $X$-dependent terms yield
\[
\mathcal{M} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = -\{\mathcal{L}_1 + \mathcal{N}_1\} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + (u_1^2 + v_1^2) \begin{bmatrix} 1 \\ -\beta \end{bmatrix} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}
- 2 \begin{bmatrix} 1 \\ \beta \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}
+ \left\{ 2 \begin{bmatrix} 1 \\ -\beta \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_1 v_1 \end{bmatrix} + (u_1^2 + v_1^2) \begin{bmatrix} 1 \\ -\beta \end{bmatrix}\right\} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}.
\]

The general solution takes the form
\[
\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = s_0 \begin{bmatrix} \chi_0 \\ 1 \end{bmatrix} |B|^2 + s_1 \begin{bmatrix} \chi_1 \\ 1 \end{bmatrix} (iB_X e^{ikx} - i\bar{B}_X e^{-ikx})
+ s_2 \begin{bmatrix} \chi_2 \\ 1 \end{bmatrix} (B^2 e^{2ikx} + \bar{B}^2 e^{-2ikx}),
\]
where
\[
s_0 \begin{bmatrix} \chi_0 \\ 1 \end{bmatrix} = 2 (\mathcal{M}^0)^{-1} \left\{ 2 \begin{bmatrix} 1 \\ \beta \end{bmatrix} \begin{bmatrix} \xi^2 \\ \xi \end{bmatrix} + (\xi^2 + 1) \begin{bmatrix} 1 \\ -\beta \end{bmatrix}\right\} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}
\]
and
\[
s_2 \begin{bmatrix} \chi_2 \\ 1 \end{bmatrix} = (\mathcal{M}^2)^{-1} \left\{ 2 \begin{bmatrix} 1 \\ \beta \end{bmatrix} \begin{bmatrix} \xi^2 \\ \xi \end{bmatrix} + (\xi^2 + 1) \begin{bmatrix} 1 \\ -\beta \end{bmatrix}\right\} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}.
\]

For $n = 1$, $\mathcal{M}_1$ is singular resulting in a family of solutions parametrized by $\chi_1$:
\[
s_1 \mathcal{M}_1 \begin{bmatrix} \chi_1 \\ 1 \end{bmatrix} = -2k_i \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \Rightarrow s_1 = -2k_i \frac{\begin{bmatrix} 1 \\ \alpha \\ -\alpha \\ 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathcal{M}_1 \begin{bmatrix} \chi_1 \\ 1 \end{bmatrix}}.
\]

Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ could have been replaced by any $1 \times 2$ vector without changing the result. Finally at $O(\epsilon^3)$ we obtain
\[
\begin{bmatrix} \mathcal{L}_0 + \mathcal{N}_0 \end{bmatrix} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = -\{\mathcal{L}_1 + \mathcal{N}_1\} \begin{bmatrix} U_2 + u_2 \\ V_2 + v_2 \end{bmatrix} - \{\mathcal{L}_2 + \mathcal{N}_2\} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \mathcal{N}_3 \begin{bmatrix} U_0 \\ V_0 \end{bmatrix},
\]
or, equivalently,
\[
\mathcal{M} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \begin{bmatrix} \xi^2 \\ \xi \end{bmatrix} \partial_x \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} - \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \begin{bmatrix} \xi^2 \\ \xi \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}
+ 2 \begin{bmatrix} 1 \\ \beta \end{bmatrix} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \begin{bmatrix} U_2 + u_2 \\ V_2 + v_2 \end{bmatrix} + \begin{bmatrix} U_2 + u_2 \\ V_2 + v_2 \end{bmatrix} \begin{bmatrix} U_0 \\ V_0 \end{bmatrix}
+ (U_0 (U_2 + u_2) + V_0 (V_2 + v_2)) I_2 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + (u_1^2 + v_1^2) \begin{bmatrix} 1 \\ \beta \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}.
\]
where $I_2$ denotes the $2 \times 2$ identity matrix. The solvability condition is obtained on multiplying this equation by the adjoint null eigenvector of $M^1$ and integrating over $x$ (to select the singular Fourier mode). The adjoint null eigenvector is $[ \Xi \ 1 \ e^{-ik_1x},$ where $\Xi$ satisfies $[ \Xi \ 1 \ ] M^1 = 0.$ The resulting condition can be written

$$a_T B_{XX} = \delta B - b_T B |B|^2,$$

where

$$a_T = -f_1/f_2, \quad b_T = -f_3/f_2$$

and

$$f_1 = [\Xi \ 1 \ ] \begin{bmatrix} \frac{1}{\alpha} & -\alpha \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} - 2k_1 s_1 [\Xi \ 1 \ ] \begin{bmatrix} \frac{1}{\alpha} & -\alpha \\ 1 & 1 \end{bmatrix} [\chi_1 \ 1 \ ],$$

$$f_2 = -2\Gamma_0 \Gamma_2 [\Xi \ 1 \ ] \begin{bmatrix} 1 \beta \\ \beta \ 1 \end{bmatrix} \left\{ \begin{bmatrix} \tau \eta \\ \eta \tau \\ \eta \ 1 \end{bmatrix} + \begin{bmatrix} \eta \tau \\ \tau \eta \\ \eta \ 1 \end{bmatrix} + (\eta \tau + 1) I_2 \right\} \begin{bmatrix} \xi \\ 1 \end{bmatrix},$$

$$f_3 = -2\Gamma_0 [\Xi \ 1 \ ] \begin{bmatrix} 1 \beta \\ \beta \ 1 \end{bmatrix} \left\{ s_0 [\chi_0 \ 1 \ ] + s_2 [\chi_2 \ 1 \ ] \right\} \begin{bmatrix} \xi \\ 1 \end{bmatrix} + \left\{ s_0 (1 + \xi \chi_0) + s_2 (1 + \xi \chi_2) \right\} I_2 \begin{bmatrix} \eta \\ 1 \end{bmatrix} - 3 (1 + \xi^2) [\Xi \ 1 \ ] \begin{bmatrix} 1 \beta \\ \beta \ 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix}.$$

The solutions depend on the signs of the coefficients $a_T$ and $b_T$. Regardless of the sign of $a_T$ there is always a solution of the form $B = \sqrt{\delta/b_T} e^{ib\phi},$ corresponding to the spatially periodic state

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}_i + 2 \begin{bmatrix} \xi \\ 1 \end{bmatrix} \sqrt{\frac{\gamma - \gamma_T}{b_T}} \cos [k_1 x + \phi].$$

When $b_T > 0$ ($b_T < 0$) these periodic states bifurcate toward $\gamma > \gamma_T$ ($\gamma < \gamma_T$). The equation also has localized solutions of the form

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}_i + 2 \begin{bmatrix} \xi \\ 1 \end{bmatrix} \sqrt{\frac{\gamma - \gamma_T}{b_T/2}} \text{sech} \left( \sqrt{\frac{\gamma - \gamma_T}{a_T}} x \right) \cos [k_1 x + \phi].$$

These correspond to the localized solutions with oscillatory tails as reviewed in §2.3.3. When both $a_T$ and $b_T$ are positive (negative), these solutions bifurcate toward $\gamma > \gamma_T$ ($\gamma < \gamma_T$). Finally, we must require the bifurcation to go from type 4 to type 1 equilibrium, because we expect localized solutions asymptotic to type 1 equilibria only, based on the spatial eigenvalue analysis.

The algebraic expressions for localized solutions near bifurcation points, as derived in these two Appendices, can be used as starting points for numerical continuations into strongly nonlinear regimes.
Chapter 8

2D localized states

8.1 Introduction

The 1:1 FCGLE (1.11) exhibits bistability between two equilibria $A^\pm$ that belong to the upper and lower branches of an S-shaped bifurcation diagram. In the previous chapters we have studied in detail the bifurcation structures and temporal dynamics of 1D spatially localized states that result from this bistability interacting with a Turing instability on $A^\pm$. In brief, in Chapter 3 we introduced the bifurcation diagram of localized periodic patterns known as defect-mediated snaking (DMS) [129], and in Chapter 4 we related the depinning dynamics of localized states to traveling fronts between $A^\pm$. These solutions are all localized in the spatial coordinate denoted by $x$. In this chapter, we will explore 2D localized states that exist on the $(x,y)$ plane where $y$ denotes the second spatial coordinate, using the same parameter combination $\alpha = -1.5$, $\beta = 6$ and $\mu = -1$, as in Chapters 3–4.

The first category of 2D localized states are localized in $x$ and infinite in $y$. Two types of such states, localized stripe patterns and planar traveling fronts, can be obtained by translating localized periodic patterns and 1D traveling fronts in the $y$-direction. These possess different inner patterns, the former periodic in $x$ and the latter flat. Because both are translation invariant in $y$, they solve the same PDE as their 1D counterparts and will not be discussed here. The other possible inner patterns can be either periodic in $y$ or crystalline (e.g. square or hexagonal). The preferred pattern created from the Turing instability on $A^\pm$ is hexagonal, and the temporal dynamics of these planar localized hexagons will be discussed in §8.2. The second category of localized states are fully localized in both $x$ and $y$. In this case the interface separating the inner pattern from the $A^-$ background is a closed curve in $\mathbb{R}^2$, and the only shape that respects isotropy in $(x,y)$ is a perfect circle of arbitrary radius. The inner pattern can be either flat or periodic in the radial direction, leading respectively to circular traveling fronts and localized ring patterns studied using numerical continuation techniques in §8.3 and §8.4. Finally, when the inner pattern is hexagonal, the temporal dynamics of these circular localized hexagons will be discussed in §8.5. The generalization of 1D localized states in Chapters 3–4 to 2D is then relatively complete, and we conclude in §8.6.
Figure 8.1: Snapshot of $V(x, y)$ at (b) $t = 100$, starting from a localized stripe pattern at (a) $t = 0$. The domain size is $[-100, 100] \times [-100, 100]$ throughout this chapter. Parameters: $\nu = 7, \gamma = 2.8972$.

### 8.2 Planar localized hexagons

At $\nu = 7$ the Turing instability on $A^+$ leads to a hexagonal pattern in 2D, and planar localized hexagons result from bistability between this pattern and $A^-$. In practice, planar localized hexagons can be found by time evolving a localized stripe pattern (Fig. 8.1(a)) with a small-amplitude random noise. The 1D localized state on the DMS branch that generates the localized stripe pattern has been taken at $\gamma = 2.897$, but it is generally expected that the asymptotic dynamics of the PDE are insensitive to the details of the initial condition. As shown in Fig. 8.1(b) the stripe pattern initially evolves into a hexagonal pattern with possibly multiple defects. The subsequent dynamics of this hexagonal pattern in the $A^-$ background depend on the value of $\gamma$ used in the time evolution.

As shown in Fig. 8.2, at $\gamma = 2.8955$ the planar localized hexagons shrink into a planar symmetric pulse. In this process the hexagonal cells are annihilated by shocks traveling along the pair of bounding fronts (Fig. 8.2(a)). The final state (Fig. 8.2(b)) resembles a 1D symmetric pulse translated in the $y$-direction. As shown in Fig. 8.3, at $\gamma = 2.899$ the planar localized hexagons expand into a hexagonal pattern that fills the 2D domain. In this process new hexagonal cells are created in the interior by two lines of defects that result from the pair of bounding fronts moving apart (Fig. 8.3(a)). The final state (Fig. 8.3(b)) may possess several stable penta-hepta defects.

Despite the outward (inward) motion of the bounding fronts at larger (smaller) $\gamma$, we have not found steady planar localized hexagons at any intermediate $\gamma$. As shown in Fig. 8.4(a), at $\gamma = 2.8972$ the boundary shocks and interior defects compete with each other, leading to non-unidirectional front motions. As a result the final state (Fig. 8.4(b)) is considerably displaced from the centerline $x = 0$ due to the stochastic creation and annihilation of hexagonal cells. More remarkably, this final state provides an example of a stable
Figure 8.2: Snapshots of $V(x,y)$ at (a) $t = 400$ and (b) $t = 1000$ showing the shrinkage of planar localized hexagons. Parameters: $\nu = 7$, $\gamma = 2.8955$. A video can be viewed at http://www.youtube.com/watch?v=5OkFRs1Sczw.

Figure 8.3: Snapshots of $V(x,y)$ at (a) $t = 1500$ and (b) $t = 4000$ showing the expansion of planar localized hexagons. Parameters: $\nu = 7$, $\gamma = 2.899$. A video can be viewed at http://www.youtube.com/watch?v=gfsXKo1AQ0g.
Figure 8.4: Snapshots of $V(x, y)$ at (a) $t = 10000$ and (b) $t = 30000$ showing the competition between shrinkage and expansion of planar localized hexagons. Parameters: $\nu = 7$, $\gamma = 2.8972$. A video can be viewed at \url{http://www.youtube.com/watch?v=6A0_ZDc0c_8}.

(or at least meta-stable) steady localized state with a complicated inner structure, which seems rather uncommon in pattern forming PDEs. The inner structure consists of a few rows of hexagonal cells interspersed with defects arranged in a frustrated configuration that overall neither shrinks nor expands, although the competition between these two tendencies still manifests itself occasionally as local pulsations.

The coordination number of each hexagon can be determined by Delaunay triangulation. In this way the locations of penta-hepta defects could be unambiguously computed and their dynamics could be better understood. This is left as future work but here we conjecture that the tendency to simultaneously shrink from front and expand from center may be due to hysteresis induced by the periodic boundary condition in the $y$-direction.

### 8.3 Circular traveling fronts

To study radially symmetric solutions, we denote

$$f(A) \equiv (\mu + i\nu)A - (1 + i\beta)|A|^2A + \gamma,$$

and rewrite the 1:1 FCGLE (1.11) in the polar coordinate $(r, \theta)$ as

$$A_t = (1 + i\alpha)\left( \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} \right) A + f(A).$$

To predict the instantaneous speed $c$ of a circular front with an inner equilibrium $A^+$ embedded in an outer equilibrium $A^-$, we look for a front solution to (8.2) of the form $A(\xi \equiv r - ct)$ at $t = 0$, i.e. when $\xi = r$. Eq. (8.2) then leads to the non-autonomous ODE

$$-cA_r = (1 + i\alpha)(\partial_r r + \frac{1}{r}\partial_r)A + f(A),$$
with the boundary conditions

\[ A(0) = A^+, \quad A(\infty) = A^- . \]  

(8.4)

In the actual continuation using AUTO, we impose Neumann boundary conditions \( U_r = V_r = 0 \) at both \( r = 0 \) and \( r = \infty \).

The family of steady circular fronts satisfy Eq. (8.3) with \( c = 0 \), i.e.

\[ (1 + i\alpha)(\partial_{rr} + \frac{1}{r}\partial_r)A + f(A) = 0 . \]  

(8.5)

This family can be found by following a weakly nonlinear solution near the lower saddle-node as in Fig. 3.2. This starting point has been computed using the following ODE

\[ (1 + i\alpha)(\partial_{rr} + \frac{d - 1}{r}\partial_r)A + f(A) = 0 . \]  

(8.6)

describing radially symmetric steady states in \( d \) dimensions. Following Ref. [135], we have treated the dimension \( d \) as a continuous parameter and continued the 1D \( (d = 1) \) weakly nonlinear solutions derived in Appendix 7.6 to 2D \( (d = 2) \). Fig. 8.5(a) shows the branch of 2D radially symmetric steady states thus computed at \( \nu = 5 \). The lower part of the bifurcation diagram undergoes a series of folds as the oscillatory tail is generated from \( r = 0 \), but the upper part approaches the collapsing point \( \gamma = \gamma^{CS} \) monotonically. This is because as the front radius \( \rho \to \infty \), the oscillatory decrease of \( \gamma - \gamma^{CS} \) due to oscillations in the front profile, proportional to \( \exp(\lambda \rho) \) where \( \lambda < 0 \) denotes the real part of the stable eigenvalue of \( A^+ \), is asymptotically smaller than the monotonic decay due to the curvature \( \kappa \equiv 1/\rho \).
The traveling circular fronts can be obtained similarly to 1D by time evolving the steady circular front in Fig. 8.5(b) at a different $\gamma$. As shown in Fig. 8.6, for smaller (larger) $\gamma$ the circular front shrinks (expands). In either case we expect the speed $c$ to be determined by Eq. (8.3) whenever the solution approaches $A^+$ ($A^-$) as $r \to 0$ ($r \to \infty$). In the case of expansion, the circular front will merge into the periodic boundaries and evolve into a reciprocal circular front centered at the corner (Fig. 8.7(a)). This reciprocal circular front is expected to satisfy Eq. (8.3) with the boundary conditions
\begin{equation}
A(0) = A^-, \quad A(\infty) = A^+.
\end{equation}

The subsequent elimination of $A^-$ at the corner produces a localized planar spot (Fig. 8.7(b); compare with Fig. 2 in Ref. [135]). The solution eventually becomes flat after the elimination of either $A^+$ at the center or $A^-$ at the corner.

### 8.4 Localized ring patterns

The branch of 2D radially symmetric steady states similar to Fig. 8.5(a) but computed at $\nu = 7$ is shown in Fig. 8.8(a). Overall this branch takes a very similar shape to Fig. 8.5(a), where the curvature of the circular front causes the upper part to converge monotonically. However, in contrast to Fig. 8.5(a) this branch undergoes $\xi$-shaped snaking on narrow intervals of $\gamma$. The separations between successive intervals decrease as the $L^2$-norm $N$ increases. The solutions high up the branch take the form of localized ring patterns (Fig. 8.8(b)).

As shown in the video accompanying Fig. 8.8, the localized ring pattern grows through a central defect at $r = 0$ analogous to Fig. 3.4. To relate this $\xi$-shaped snaking
Figure 8.7: The sequel to Fig. 8.6(b) at (a) $t = 3240$ and (b) $t = 3600$.

Figure 8.8: (a) The branch of 2D radially symmetric steady states followed from the lower saddle-node at $\nu = 7$. A video can be viewed at http://www.youtube.com/watch?v=Z_MGpnmOuHk. (b) A sample solution profile $V(x,y)$. 
Figure 8.9: The branch of $d$-dimensional radially symmetric steady states followed in $d$ from a 1D localized state ($d = 1$) on the DMS branch at $\nu = 7, \gamma = 2.897$.

to DMS, we have continued solutions to Eq. (8.6) in the dimension $d$. The result of this continuation at $\gamma = 2.897$ is shown in Fig. 8.9. In order that the snaking in $d$ corresponds to the snaking in $\gamma$ at nearby values of $d$ and $\gamma$, in the following we shall work with the former reflected relative to the vertical axis (i.e. on the $(−d, N)$ plane), such that the sharper folds are located on the right. In this way a pair of adjacent C-shaped segments would later form the body of the letter $\xi$. Near $d = 1$ the snaking in $d$ mostly takes place in $d > 1$ and resembles DMS except that the successive folds further from $d = 1$ that correspond to maxima and minima in $V(r)$ at $r = 0$ no longer align with each other. As $d$ increases a hysteresis loop appears on the bottom segment for every pair of adjacent C-shaped segments, while the snaking interval in $d$ becomes narrower. At $d = 2$, the two additional segments forming this hysteresis loop can be regarded as the (short) tail of the current and the (long) head of the previous letter $\xi$. Thus by continuity, the localized ring pattern grows one extra ring every time the branch passes through a $\xi$-shaped segment. At large $d$ (e.g. $d = 3$), the body of $\xi$ disappears via two hysteresis bifurcations, although the head and tail of $\xi$ remain. We have not continued the branch to even larger $d$ and thus cannot ascertain whether another hysteresis bifurcation would lead to a bifurcation curve without fold, but from Fig. 8.9 the branch indeed appears to approach a straight line on the $(d, N)$ plane as $d \to \infty$. Apart from the above dimensional continuation for $\gamma = 2.897$ near $\gamma^{DMS}_{2}$, we have also taken other slices of constant $\gamma$ but these continuations do not lead to radially symmetric steady states in higher dimensions. Specifically, for smaller $\gamma$ (near $\gamma^{DMS}_{1}$) the branch snakes towards $d < 1$ while for intermediate $\gamma$ (about halfway between $\gamma^{DMS}_{1}$ and $\gamma^{DMS}_{2}$) the branch snakes around $d = 1$.

### 8.5 Circular localized hexagons

At $\nu = 7$, circular localized hexagons result from bistability between $A^-$ and the hexagonal pattern created by the Turing instability on $A^+$ and can be found by time evolving, for example, the localized ring pattern in Fig. 8.8(b) at a different $\gamma$. In contrast
to the planar case studied in §8.2, in the circular case we no longer need to impose random noise on the initial condition since the discretization error already has a nontrivial projection in the unstable eigenspace responsible for the subsequent formation of hexagons. Fig. 8.10 shows snapshots of this time evolution at $\gamma = 2.8989$. The hexagonal pattern that initially develops roughly aligns itself along the circular rings and contains multiple defects arranged into a square symmetric configuration (Fig. 8.10(a)). In particular, four lines of defects form a cross shape at the center and rapidly break up into two parabolas of defects (Fig. 8.10(b)).

For localized hexagons (either planar or circular), the motion of the bounding front(s) and the response of the hexagonal pattern can be classified into three types. The first type is analogous to slow depinning in 1D (§4.3.3), where the bounding front(s) expand (shrink) slowly and hexagons can be created (annihilated) anywhere within their existence region. The second type is analogous to fast depinning in 1D (§4.3.2), where the bounding front(s) expand (shrink) rapidly and hexagons tend to be created (annihilated) at a fixed distance from the bounding front(s). The third type is intrinsically 2D and may be called ultra-fast, where shock(s) propagate along the bounding front(s) to create (annihilate) hexagons. The dynamics of planar localized hexagons (§8.2) take the form of fast expansion for larger $\gamma$, ultra-fast shrinkage for smaller $\gamma$, and competition between these two types of motion for intermediate $\gamma$. Thus an average front speed $c$ can be defined, which is expected to be a monotonic function of $\gamma$. On the other hand, the front speed $c$ for circular localized hexagons appears to be independent of the inhomogeneities in the hexagonal pattern; its dependence on $\gamma$ and the radius of the bounding front is similar to circular traveling fronts between $A^\pm$ (§8.3). Thus for any given initial condition, there exists a Maxwell point $\gamma_M$ such that the bounding front shrinks (expands) for $\gamma < \gamma_M$ ($\gamma > \gamma_M$).

In Fig. 8.10 the value of $\gamma$ used slightly exceeds $\gamma_M$. As a result, the circular localized hexagons expand first slowly then rapidly before reaching the boundaries. Afterwards the system evolves into reciprocal circular localized hexagons center at the corner, with a circular disk of $A^-$ surrounded by a hexagonal pattern (Fig. 8.10(c)). Eventually the hexagonal pattern fills the domain as shown in Fig. 8.10(d), but in contrast to Fig. 8.3(b) this hexagonal pattern observed shortly after the domain is filled contains multiple lines of defects. The shrinkage of circular localized hexagons for $\gamma < \gamma_M$, though not shown, proceeds similarly to Fig. 8.10 with slow shrinkage followed by fast shrinkage. However, we have also observed ultra-fast shrinkage when the front radius becomes sufficiently small. The final stages of time evolution after the elimination of hexagons are qualitatively the same as those for Fig. 8.6(a).

Finally we observe that although localized ring patterns and circular localized hexagons both possess circular fronts, the former is stationary in a finite snaking interval in $\gamma$ while the latter is stationary only at the Maxwell point $\gamma_M$. A plausible reason is that for the former the circular front is always pinned to a periodic pattern in the direction perpendicular to itself (i.e. the radial direction), while for the latter the circular front cannot be pinned to a hexagonal pattern that only looks periodic from a discrete set of angles.
Figure 8.10: Snapshots of $V(x,y)$ at (a) $t = 300$, (b) $t = 450$, (c) $t = 4080$, and (d) $t = 5000$ showing the expansion of circular localized hexagons. Parameters: $\nu = 7$, $\gamma = 2.8989$. A video can be viewed at http://www.youtube.com/watch?v=-9H_6-lQHAY.
8.6 Discussion

In this chapter we have explored 2D localized states in the 1:1 FCGLE with different inner patterns and front shapes. To conclude, we compare these with previous studies on 2D localized states in other pattern-forming PDEs. In a broad class of such PDEs, a stable pattern can be created by the Turing instability on an equilibrium $A_0$. This instability can be either supercritical or subcritical. In the supercritical case, localized patterns can be formed by the invasion of the stable pattern into the unstable $A_0$ via a pulled front [175]. In 1D we have already seen an example of such an invasion of a periodic pattern into an unstable $A_0$ between the pair of bounding fronts in Fig. 4.7(b). In 2D the invasion of a hexagonal pattern into an unstable $A_0$ has also been investigated (cf. e.g. [68]). In the sub-critical case, $A_0$ and the pattern can be bistable and the front between them can be pinned to the pattern, as pointed out in Ref. [101]. This pinning mechanism underlies much work on spatially localized states in both 1D and 2D. The 1D case is relatively well understood and has been reviewed in §2.4.1. The 2D case still presents many open questions, but some generic behavior can be learned from the following two recent studies.

In Ref. [124], steady localized hexagons in 2D quadratic-cubic Swift-Hohenberg equation (SH23) are studied using numerical continuation techniques. The existence of these solutions follows from a 2D conservation law that generalizes the 1D conserved Hamiltonian. There are two classes of solutions, planar localized hexagons and fully localized hexagons, that respectively possess planar fronts and hexagonal fronts. In either case the hexagonal pattern is periodic in the direction perpendicular to the front(s) and the bifurcation diagram exhibits homoclinic snaking. Moreover, the snaking behavior is similar in nature to standard homoclinic snaking in 1D in that new hexagons are grown from the exterior by the front. A novel feature is that in the planar case, there are steady localized hexagons with almost planar fronts that exist on secondary snaking curves. Along these curves each new row of hexagons is formed by adding hexagons one-by-one along the almost planar front. This growth mechanism is also observed along the snaking curve of fully localized hexagons. Thus in the spectrum of an almost planar solution computed near a saddle-node of a secondary snaking curve, there must be an amplitude mode localized at the locations where new hexagons are added. As a result, even outside the snaking region, this almost planar solution still evolves by losing or gaining hexagons one-by-one similarly to its behavior along the snaking curve. This type of dynamics, which looks similar to ultra-fast shrinkage shown in Fig. 8.2, is typical for systems like SH23 in the sense that only the front evolves in time while the inner pattern stays strictly stationary. Hence an important difference between these systems and the 1:1 FCGLE is that the former cannot exhibit the fast and slow flavors of expansion (shrinkage) defined in §8.5. In addition to comparing temporal dynamics, it would be very useful to compute the bifurcation diagrams for steady localized hexagons in the 1:1 FCGLE and compare these with the results on 2D SH23 reviewed above. However at this stage we can neither prove the existence of steady localized hexagons, nor ascertain whether the apparently steady solutions with frozen defects (Fig. 8.4(b) and Fig. 8.10(a)) can successfully serve as starting points for numerical continuation.

In Ref. [135], localized ring patterns in multi-dimensional SH23 are studied using numerical continuation techniques reviewed in §8.3. A localized ring pattern in $d$ dimensions always possesses a spike at $r = 0$ for $d > 1$, for either SH23 or the 1:1 FCGLE. However for
\[ d = 1, \] the fronts grow new rolls on the snaking curve for the former while the defect grows new rolls on the snaking curve for the latter. As a result, for SH23 the competition between growing from the spike and the fronts leads to complicated snaking behavior for \( d > 1, \) including stacks of isolas and what may be called spike-mediated snaking. In contrast, for the 1:1 FCGLE the defect and the spike are both located at \( r = 0 \) and can smoothly deform into each other. Hence the snaking mechanism in the 1:1 FCGLE remains qualitatively unchanged as \( d \) increases from 1, although a hysteresis bifurcation deforms DMS at \( d = 1 \) into \( \xi \)-shaped snaking at \( d = 2. \) Aside from presenting bifurcation diagrams, Ref. [135] also analyzes the fold accumulation for the snaking curves of localized ring patterns and found algebraic scaling with somewhat unexpected exponents. A geometric theory for this scaling behavior has been formulated by the same authors in an unpublished manuscript, but we shall leave such studies for the 1:1 FCGLE as future work.

These novel features of 2D localized states in the 1:1 FCGLE are caused by the bistability between two equilibria \( A^\pm \) interacting with a supercritical Turing bifurcation on \( A^+ \). This bistability between equilibria is also present in the 2:1 FCGLE, which possesses the up-down symmetry \( A \rightarrow -A. \) Much previous work on this PDE (e.g. [41]) has focused on parameter regimes where two “equivalent” equilibria \( \pm A_0 \) related by this up-down symmetry are bistable. In this case there is generically a codimension-0 family of Ising fronts between \( \pm A_0 \) in 1D. In Ref. [89], a dynamical theory for the domain wall between \( \pm A_0 \) in 2D leads to characterization of different growth regimes and prediction of the existence of the so-called stable droplets. In the 1:1 FCGLE the two equilibria \( A^\pm \) are inequivalent so this theory does not directly apply, but we expect the speed \( c \) of a circular front between \( A^\pm \) to be well predicted by Eq. (8.3), assuming that the radial profile does not change significantly when the front radius is sufficiently large.
Chapter 9

Localized spatiotemporal chaos

9.1 Introduction

In this final chapter on the 1:1 FCGLE, we introduce yet another type of localized structures that results from the bistability between the upper and lower equilibria $A^\pm$, referred to as localized spatiotemporal chaos (LSTC). Here we use the term “spatiotemporal chaos” (STC) in a broad sense to represent solutions to nonlinear PDEs without any overall symmetry in either space or time. STC generalize “classical” chaos for finite-dimensional flows or maps to infinite-dimensional dynamical systems (e.g. PDEs). A somewhat counterintuitive numerical observation is that STC is intrinsically low-dimensional and thus resembles classical chaos rather than 3D turbulence at high Reynolds numbers. In particular, many flavors of STC including precursors to turbulence (e.g. [178]) appear to be built upon a skeleton of coherent structures. For the 1:1 FCGLE, some of these coherent structures have been extensively studied in the previous chapters.

The best known instability that generates STC in the neighborhood of an equilibrium is the Hopf instability. In the 1:1 FCGLE this instability corresponds to a Hopf bifurcation on the equilibrium in the temporal ODE (7.6). The condition for Hopf bifurcation discussed in Chapter 7, i.e. $\mu > 0$ and the norm of the equilibrium $|A| < \sqrt{\mu/2}$, is satisfied on the lower equilibrium $A^-$ over a wider range of parameters. The criticality of this Hopf bifurcation on $A^-$ determines the dynamical behavior in the neighborhood of $A^-$. In the subcritical case this neighborhood is repulsive in the temporal ODE but can contain an STC attractor in the original PDE. In the supercritical case this neighborhood contains a stable limit cycle in the temporal ODE that corresponds to a uniform oscillation (UO) in the original PDE, but this UO can be unstable to spatially nonuniform perturbations.

This chapter is organized as follows. In §9.2 we analyze the linear stability of an arbitrary UO without requiring its proximity to a supercritical Hopf bifurcation, and present an instance of STC that results from a long-wave instability. In §9.3 we construct a relatively simple LSTC by embedding a Hopf unstable $A^-$ in an $A^+$ background. In §9.4 we present a more complicated LSTC where bistability between spatially periodic states plays an important role. In §9.5 we briefly compare LSTC with localized spatiotemporal chaotic patterns in other physical systems.
9.2 Instabilities of uniform oscillations

The linear stability of UO is best analyzed by writing (8.1) as

\[
f(A) = \begin{bmatrix} \mu U - \nu V - (U^2 + V^2)(U - \beta V) + \gamma \\ \mu V + \nu U - (U^2 + V^2)(V + \beta U) \end{bmatrix}, \quad A \equiv \begin{bmatrix} U \\ V \end{bmatrix},
\]

(9.1)
such that the temporal ODE and the original PDE take the form

\[
\frac{\partial}{\partial t} A = f(A),
\]

(9.2)
\[
\frac{\partial}{\partial t} A = \frac{\partial^2}{\partial x^2} MA + f(A), \quad \text{where} \quad M \equiv \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix}.
\]

(9.3)

Let \( A_0(t) \) denote a stable limit cycle in Eq. (9.2) with a finite period \( T \). To analyze the linear stability of \( A_0(t) \) in Eq. (9.3), we perturb \( A_0(t) \) by \( \epsilon \tilde{A}_k(t) e^{ikx} \) \( (0 < \epsilon \ll 1) \) and linearize to get

\[
\frac{\partial}{\partial t} \tilde{A}_k(t) = (-k^2 M + Df(A_0(t))) \tilde{A}_k(t).
\]

(9.4)

To solve Eq. (9.4), we divide a single period \([t_0, t_0 + T]\) into \( N \) equally spaced intervals \( t_i \equiv t_0 + iT/N \) \( (0 \leq i \leq N) \), and take the limit \( N \to \infty \) in the end. The Floquet matrix \( Q_k \) then takes the form

\[
Q_k = \prod_{i=0}^{N-1} \left( I_2 + \frac{T}{N} (-k^2 M + Df_i) \right),
\]

(9.5)

where \( Df_i \equiv Df(A_0(t_i)) \) and \( I_2 \) denotes the \( 2 \times 2 \) identity matrix.

In the long-wave limit \( 0 < |k| \ll 1 \) we can expand Eq. (9.5) as

\[
Q_k = Q_0 + \frac{dQ_k}{d(k^2)} k^2 + \mathcal{O}(k^4)
\]

(9.6)

where

\[
\frac{dQ_k}{d(k^2)} = \sum_{j=0}^{N-1} \prod_{i=0}^{j-1} \left( I_2 + \frac{T}{N} (-k^2 M + Df_i) \right) \frac{T}{N} (-M) \prod_{i=j+1}^{N-1} \left( I_2 + \frac{T}{N} (-k^2 M + Df_i) \right).
\]

(9.7)

The matrix \( Q_0 \) determines the stability of \( A_0(t) \) in the temporal ODE (9.2) via its two eigenvalues known as the Floquet multipliers. One of them is the marginal phase multiplier \( \Lambda_p^0 = 1 \), and the other is the stable amplitude multiplier \( 0 < \Lambda_a^0 < 1 \). Eq. (9.6) then implies that the perturbed Floquet matrix \( Q_k \) also has two multipliers \( \Lambda_k^p \) and \( \Lambda_k^a \) that differ from \( \Lambda_p^0 \) and \( \Lambda_a^0 \) by \( \mathcal{O}(k^2) \). Thus near \( k = 0 \), the amplitude multiplier \( \Lambda_k^a \) remains stable but the phase multiplier \( \Lambda_k^p \) can become unstable. This long-wave instability has been known as the Kuramoto phase instability [116].

In the short-wave limit \( |k| \gg 1 \), Eq. (9.5) simplifies to

\[
Q_k \approx \exp(-k^2 MT),
\]

(9.8)
which yields a complex conjugate pair of stable multipliers $\Lambda_k^\pm \approx \exp(- (1 \pm i\alpha)k^2T)$. For $k \sim \mathcal{O}(1)$, Eq. (9.5) must be evaluated numerically and instability arises whenever a computed multiplier crosses the unit circle $|\Lambda_k| = 1$. The onset mechanism where one multiplier crosses $\Lambda_k = 1$ ($\Lambda_k = -1$) at $k = k_c \sim \mathcal{O}(1)$ corresponds to a saddle-node (period-doubling) bifurcation for maps reviewed in §2.1.4. However the third mechanism that would correspond to a Neimark-Sacker bifurcation is absent because the product of the two multipliers must satisfy

$$\det Q_k = \det \left[ \exp\left(-k^2MT\right) \right] \det Q_0 < 1.$$  \hspace{1cm} (9.9)

Near the onset of the Kuramoto phase instability, the phase multiplier takes the form

$$\Lambda_k^p = 1 + \Lambda^{(2)}k^2 + \Lambda^{(4)}k^4 + O(k^6)$$  \hspace{1cm} (9.10)

where $0 < |k| \ll 1$, $0 < \Lambda^{(2)} \ll 1$, and $\Lambda^{(4)} < 0$. This expression is reminiscent of the dispersion relation for the phase mode in an Eckhaus instability (cf. Eq. (10.160)). In such cases the system often evolves away from the neighborhood of the base state such that a weakly nonlinear analysis around the base state breaks down asymptotically in time. In 1D this leads to phase slips in the spatially periodic wavetrain for the Eckhaus instability (cf. Fig. 3.13). As an example of STC produced from the Kuramoto phase instability, we consider the same parameter combination $\alpha = 4$, $\beta = 1$, and $\mu = 1$ as in Chapter 6, and choose $\nu = 1.5$ and $\gamma = 0.385$ where there exist an unstable lower equilibrium $A^{-}$ and a stable limit cycle in the temporal ODE (9.2). Both 1D and 2D time evolutions have been performed using $A^{-}$ with a small-amplitude random noise as the initial condition. In 1D the Kuramoto phase instability produces phase dislocations that propagate at a constant speed and can backfire (Fig. 9.1(a)). In 2D the Kuramoto phase instability has been known to produce the so-called defect-mediated turbulence [58] in the CGLE, where phase dislocations (known as defects in Ref. [58]) propagate as a spiral wave. For the 1:1 FCGLE, however, phase dislocations appear to propagate as rings or 0-arm spirals (Fig. 9.1(b)).

### 9.3 LSTC from a Hopf instability

In this section we fix $\alpha = 2$, $\beta = 1$, and $\mu = 1$. For the choices of $\nu$ and $\gamma$ below, the upper equilibrium $A^{+}$ is always stable while the lower equilibrium $A^{-}$ can become Hopf unstable. The initial condition for a typical time evolution in this section consists of a region of $A^{-}$ with a small-amplitude random noise embedded in an $A^{+}$ background. Asymptotically in time, the Maxwell point $\gamma_M$ is then defined as the value of $\gamma$ such that the (average) speed of their interface vanishes identically. The Maxwell point for 1D and 2D time evolutions will be denoted respectively by $\gamma^{1D}_M(\nu)$ and $\gamma^{2D}_M(\nu, \kappa)$, where the latter depends on the curvature $\kappa \equiv 1/R$ ($R$ denotes the radius) of the $A^{-}$ region. In either 1D or 2D, the $A^{-}$ region shrinks (expands) for $\gamma > \gamma_M$ ($\gamma < \gamma_M$). Hence in what follows we shall mainly focus on the spatiotemporal patterns along $\gamma_M(\nu)$. As $\nu$ varies there exists $\nu = \nu_H$ (which depends on $\kappa$ in the 2D case) such that $A^{-}$ becomes Hopf unstable for $\nu < \nu_H$.

The Hopf bifurcation is supercritical and creates a stable UO near the onset. As shown in Fig. 9.2(a) for $\nu = 2.68 < \nu_H$, the 1D Maxwell point between $A^{+}$ and a stable
Figure 9.1: An example of STC produced from the Kuramoto phase instability \((\alpha = 4, \beta = 1, \mu = 1, \nu = 1.5, \gamma = 0.385)\). (a) 1D time evolution shown as a space-time plot of \(V(x,t)\). (b) 2D time evolution shown as a snapshot of \(V(x,y)\) at \(t = 1000\), where \((x,y) \in [-400,400] \times [-400,400]\). A video can be viewed at \url{http://www.youtube.com/watch?v=TKSA9VNdwcY}.

UO is located at \(\gamma_{1D}^M(\nu) \approx 1.5731\). Since the stability of UO is the same in 1D and 2D, there exist 2D Maxwell points that satisfy \(\gamma_{2D}^M(\nu,\kappa = 0) = \gamma_{1D}^M(\nu)\). Though not shown here, these Maxwell points represent steady circular fronts with the stable UO embedded in \(A^+\) \((A^+\) embedded in the stable UO) for \(\kappa > 0\) \((\kappa < 0)\). On the other hand, we have found a stable cross-shaped solution homoclinic to \(A^+\) (Fig. 9.2(b)) at \(\nu = 2.7 > \nu_H\) and \(\gamma = 1.6 > \gamma_{2D}^M(\nu,\kappa = 0)\) from an initial condition of \(A^+\) embedded in \(A^-\). The existence of such a solution implies that an initial condition consisting of random patches of \(A^\pm\) generally evolves into a pulse network embedded in \(A^+\).

As \(\nu\) decreases further below \(\nu_H\), we have observed from DNS a finite-wavelength instability on UO but have not determined the nature of this instability (i.e. saddle-node or period-doubling) via an explicit computation of the Floquet multipliers as a function of the perturbation wavenumber. In other systems such a finite-wavelength instability has been known to create the so-called oscillatory Turing patterns [184], which are essentially standing waves in 1D. Although the existence of such stable standing waves remains uncertain in our system, there does exist a weak STC that resembles a mixture of UO and standing waves near the onset of this finite-wavelength instability. Fig. 9.3(a) shows the associated LSTC at \(\nu = 2.65\) and \(\gamma_{1D}^M(\nu) \approx 1.5409\). This weak STC remains in the neighborhood of the base state UO and should be well described by a weakly nonlinear theory. For even smaller \(\nu\) STC becomes stronger and does not return to the neighborhood of UO as frequently. Fig. 9.3(b) shows the associated LSTC at \(\nu = 2.4\) and \(\gamma_{1D}^M(\nu) \approx 1.277\).

This type of 1D LSTC is similar to those found in the study of chaotic nucleation of metastable domains [9]. In both cases the finite-wavelength instability couples to the translational degree of freedom of the front and leads to its stochastic motion. In Ref. [9]
Figure 9.2: (a) Space-time plot of $V(x,t)$ at $\nu = 2.68$ and $\gamma = 1.5731$ showing an almost steady front between $A^+$ and a stable UO. (b) Snapshot of $V(x,y)$ at $\nu = 2.7$ and $\gamma = 1.6$ showing a stable cross-shaped solution homoclinic to $A^+$. The domain size is $[-100, 100] \times [-100, 100]$. A video can be viewed at http://www.youtube.com/watch?v=kR-310zxwP8.

Figure 9.3: Space-time plots of $V(x,t)$ showing weak and strong LSTC. (a) $\nu = 2.65$, $\gamma = 1.5409$; (b) $\nu = 2.4$, $\gamma = 1.277$. 
the front speed has been calculated analytically near the Maxwell point and close to the onset of instability. Although a similar analysis has not been attempted in our case, the stochastic motion of the front for strong LSTC is clearly seen in Fig. 9.4(a).

Since STC in 2D are in general not directly related to their 1D counterparts, we should expect the Maxwell points to differ in these two cases, i.e. $\gamma_{2D}^M(\nu, \kappa = 0) \neq \gamma_{1D}^M(\nu)$. Indeed as shown in Fig. 9.4(b) for $\gamma = 1.277$, a rectangular region of STC with initial width $x \in [-100, 100]$ has visibly expanded at $t = 1000$, thus implying $\gamma_{2D}^M(\nu, \kappa = 0) > \gamma_{1D}^M(\nu) \approx 1.277$. By continuity argument, for small $|\kappa|$ STC always invades $A^+$ as seen from the solution profiles at $t = 1000$ starting from STC with radius $R = 100$ embedded in $A^+$ (Fig. 9.5(a)) or $A^+$ with radius $R = 100$ embedded in STC (Fig. 9.5(b)). Numerically we find that there exists $40 < R_c < 50$ such that circular LSTC with radius $R > R_c$ ($R < R_c$) expands (shrinks) at $\gamma = 1.277$.

### 9.4 LSTC from DMS branches

In this section we consider the large forcing limit (7.22) of the 1:1 FCLE at fixed $\alpha = -2$. At each fixed $\beta$ a branch of localized states with an $A^-$ background bifurcates from the lower saddle-node $\gamma_{SN}^-$ and can be followed in $\gamma$. As $\beta$ increases, this branch transitions from collapsed snaking to two-limit defect-mediated snaking (DMS) and then to three-limit DMS much as described in Chapter 3. As shown in Fig. 3.11 the extra snaking limit in three-limit DMS results from the multivaluedness of spatially periodic states. As shown in Fig. 3.17 a localized state on the three-limit DMS branch can depin intermittently and form a spatiotemporal pattern called a Hanoi tower. In this section we present more spatiotemporal patterns, including LSTC, that are found by time evolving steady states on
Figure 9.5: Snapshots of $V(x,y)$ at $t = 1000$ showing (a) a circular LSTC; (b) a reciprocal circular LSTC. The parameters and the domain size are the same as in Fig. 9.4(b).

three-limit DMS branches.

Fig. 9.6 shows spatiotemporal patterns at $\beta = 4$ for two values of $\gamma$ inside the DMS region $\gamma \in [0.1778, 0.18495]$. The Hanoi tower at $\gamma = 0.18$ in Fig. 9.6(a) reveals certain key processes underlying the complicated LSTC at $\gamma = 0.181$ in Fig. 9.6(b). Specifically, in Fig. 9.6(a) we observe a bistability between two types of periodic states, the one in the initial condition denoted by $P_1$ and the one in the metastable transient denoted by $P_2$. In the initial phase the central part of the $P_1$ wavetrain evolves into a metastable $P_2$ wavetrain, while the remaining parts shrink and leave behind either $A^-$ or pulses homoclinic to $A^-$. Subsequently the metastable $P_2$ wavetrain evolves back into a $P_1$ wavetrain, which then rapidly collapses precisely as in the initial phase.

The complicated LSTC in Fig. 9.6(b) resembles a mixture between $P_1$ and a Hopf unstable version of $P_2$, denoted by $\tilde{P}_2$. At this value of $\gamma$, $P_1$ alone collapses while $P_2$ alone expands in an $A^-$ background. It is the competition between these two motions that leads to the numerous cavitations observed in the interior of this LSTC. Although these interior cavitations do not directly affect the front speed, the “half” cavitations observed along the left edge of this LSTC do cause the left front to shrink. This effect certainly complicates the determination of the Maxwell point $\gamma_M$.

Fig. 9.7 shows spatiotemporal patterns at $\beta = 6$ for two values of $\gamma$ inside the DMS region $\gamma \in [0.13343, 0.15742]$. As shown in Fig. 9.7(a) for $\gamma = 0.152 \approx \gamma_M$, interior cavitations still appear much as in Fig. 9.6(b), but this time the STC cannot be readily related to steady periodic states. An interesting phenomenon happens at $\gamma = 0.15742 > \gamma_M$, i.e. at the right limit of the DMS region, where the periodic wavetrain forming the steady localized state on the DMS branch has a large wavelength and thus resembles an array of pulses homoclinic to $A^-$. As shown in Fig. 9.7(b), with this initial condition the pulses in the array interact weakly with each other and remain stable for a long time, but in the meantime the front triggers the formation of STC that invades $A^-$ towards the boundaries.
and the pulse array towards the center. Since these two invasion fronts differ in shape, we expect their speeds to differ as well.

9.5 Discussion

Among the best known examples in physical systems that closely resemble LSTC studied in this chapter are localized turbulent regions during the transition to turbulence in shear flows [92]. In these systems the turbulent fraction increases as the Reynolds number $Re$ increases, while laminar and turbulent flows coexist for any $Re$ in the transitional regime. Hence it would be oversimplified to regard the above LSTC with a well-defined Maxwell point as a phenomenological model for transitional shear flow. Nonetheless, we believe that a better understanding of LSTC in 1D or 2D pattern forming PDEs can be relevant to the study of similar phenomena in 3D Navier-Stokes equations. See Ref. [16] for a recently proposed simplified model for pipe flows.

In this chapter we have not attempted to find all possible LSTC in the 1:1 FCGLE. In particular, the possibility for LSTC based on STC in §9.2 (or localized defect-mediated turbulence) remains open. On the other hand, STC in §9.4 are much less understood than those in §9.3. In 1D, localized patterns embedded in a spatiotemporal periodic background have been discovered in discrete systems known as cellular automata [181] (e.g. rule 110), but their connection with pattern-forming PDEs is unclear. The 2D version of LSTC in §9.4 is also left as future work.
Figure 9.7: Space-time plots of $V(x,t)$ at $\beta = 6$ and (a) $\gamma = 0.152$; (b) $\gamma = 0.15742$. 
Chapter 10

Derivation of pattern equations

10.1 Introduction

In many well studied pattern-forming processes, the governing equations are known and the search for a pattern equation amounts to an attempt at dimensional reduction, that is, the search for equations that describe structures that develop in subspaces of the full configuration space of the original system. The general forms of the pattern equations may often be found fairly readily if the basic equations are covariant under the action of a symmetry group (e.g. \([98]\)). The use of symmetries to derive pattern equations of the right form is a powerful approach but it has the drawback noted by J.A. Wheeler [179] in another context that “The considerations of symmetry that reveal law hide the mechanism that underlies law.” For this reason, the derivation of pattern equations by other means such as singular perturbation theory is sometimes favored. If the subject ever reaches a stage where theory and experiment may be compared quantitatively, then the latter method may play a greater role than heretofore since it has the advantage of permitting the control parameters of the reduced system to be expressed in terms of the parameters of the original problem.

In the present work, we study the specific example of Rayleigh-Bénard convection, a phenomenon that is paradigmatic in the subject of pattern theory. The beginning of serious interest in convective patterns may be traced to the experimental studies of Bénard and his student Dusan Avsec [13]. Though Bénard’s early convection papers showed a regular array of hexagonal planforms (later understood to be driven by Marangoni effect), he soon appreciated that the patterns of convection represent a rich and complicated problem: “The laboratory problem that I have myself has been to define and to measure in a horizontal liquid layer heated from below, the convection currents that prevail, considered as near as possible to their state of greatest stability.” Though Bénard died in 1939, Avsec’s dissertation [13], completed in the same year, went some way toward fulfilling those aims. Most notably, Avsec experimentally produced a striking example of what would now be called the zig-zag instability of rolls (Fig. 10.1). In his work, the ratio of the horizontal scale of the fluid layer to its depth — its aspect ratio — was large. For that case, the spectrum of horizontal wavenumbers of the normal modes of the linearized fluid dynamical equations for a plane-parallel layer is very closely spaced, so closely that it is best regarded
Figure 10.1: The “zig-zag” instability of rolls as seen in Avsec’s experiments [13]. The arrow represents the prevailing wind that selects the rolls. To see how well this is reproduced by the S-H equation go to the web site of M.C. Cross http://www.cmp.caltech.edu/~mcc/Patients/Demo6_3.html.

as a continuum for present purposes. Thus the equation describing the structures Avsec observed is inevitably a partial differential equation (PDE).

In the study of convection in a fluid layer heated from below, the symmetry most commonly introduced into the problem is (essentially) an up-down symmetry — the Boussinesq symmetry — under which the governing equations are covariant. These approximate equations may be derived from the full Navier-Stokes-Fourier equations for a layer whose vertical extent is very small compared to the scale heights of its material properties. We shall refer to that approximation as the Boussinesq equations, following Lord Rayleigh’s attribution [148] of an approximation that yields them (and in keeping with Arnold’s law of misattribution). The pattern equation in this case describes a nonlinear, time-dependent extension of the planform function arising in the separable solutions of linear convection theory (for which see [50]).

The importance of the convection problem as a guide to pattern theory emerged when Newell and Whitehead [137] and Segel [157] introduced asymptotic methods like those used in nonlinear wave theory [21] to derive convective pattern equations. Their simplest version was the mean-field equation of superconductivity, a time-dependent Ginzburg-Landau equation. The realization that the G-L equation could be derived from a more basic theory attracted the attention of fans of that equation [6] to the subject of convection and brought a new population of scientists into its study.

The Newell-Whitehead-Segel theory, based on singular perturbation methods, went beyond the confines of two-dimensional convection of the strict G-L equation by including in the modal decomposition modes with wave vectors in a narrow wedge around the basic wave vector of the development. Swift and Hohenberg [165] introduced a fully two-dimensional pattern model, with a nod to a previous study of the solidification of a layer of molten metal [33]. Their equation has had great success in producing patterns like those found
in convection experiments (both laboratory and numerical) and patterns studied in other disciplines. The S-H equation captures a remarkable share of the results obtainable from the full equations; for example, the NWS equation may be recovered from it asymptotically just as from the full equations. A physicist’s derivation of the S-H equation has been given by Cross [65] followed by other treatments in a similar vein such as [27].

Other derivations of the S-H equation have been proposed, based on the Bogoliubov method of dimensional reduction like that used in kinetic theory. The Bogoliubov method had previously been used in deriving the nonlinear equations for amplitudes of the slow modes of a system in cases where the spectra of the modes are discrete [61]. The virtue of that method is that it allows a revealing treatment of cases where the onset of instability of several modes may arise in the same small neighborhood of the parameter space of the system. Those equations are called amplitude equations [157] and the modal amplitude vector is sometimes called an order parameter.

When there is only a single real, slightly unstable mode, its amplitude, \( A(t) \), satisfies the Landau equation

\[
\dot{A} = \mu A - A^3
\]  
(10.1)

where the equation has been scaled so that the coefficient of its cubic term is unity. The growth rate, \( \mu \), of the mode passes from negative to positive as the control parameters are modified appropriately. This transition marks the onset of instability that first happens for a mode whose horizontal wavenumber \( k = k_c \). When the aspect ratio of the convecting layer becomes large, several of the modes allowed by the conditions on the horizontal boundaries (not mentioned explicitly here) have wavenumbers quite close to \( k_c \). These are nearly marginal at the onset of convective instability. In the limit of infinite aspect ratio, a continuous band of slowly evolving \( A \)'s is nearly marginal. To allow for this change in the modal spectrum, \( \mu \) in (10.1) should be replaced by an appropriate linear differential operator. All the slow modes must now be included in the description of the dynamics of convection. Qualitatively, we may think of the modal amplitude \( A \) (or its Fourier transform) as a descriptor of convective activity that describes the variations in space and time that comprise the patterns produced by the instability.

In this chapter, our aims are (a) to describe the use of the Bogoliubov method in deriving pattern-forming PDEs and (b) to introduce the use of diagrams into this derivation. As usual, when solving a relatively simple problem to explain a general procedure, we make that problem appear more difficult than it really is. But our hope in doing this is that those who take the trouble to learn the diagrammatic method may find their efforts repaid by the gain in ease of tackling the more difficult problems we have in mind. Those more difficult problems are the derivations of pattern equations in situations with multiple bifurcations where the different branches of the spectrum are continuous (or nearly so). Though we shall not present such derivations here, we shall conclude with a brief outline of the nature of these problems at the end.

We add here that we find the Bogoliubov approach to be very useful in clarifying what lies behind the derivation of pattern equations. Though this has already been brought out in related references ([62], [45], [166], [128]), none of these has appeared in a regular journal. An interesting outcome from these is that, for three-dimensional convection, the two-dimensional pattern equation is nonlocal as has been noted in other discussions of this
Having obtained the nonlocal pattern equation, we shall also mention a few of its features that are of interest. Like the S-H equation, the nonlocal extension possesses a Lyapunov functional, $\mathcal{F}$, and thus exhibits features of a gradient dynamical system, namely that the system always evolves towards a local minimum of $\mathcal{F}$. We find a set of conditions that any solution at a local minimum must satisfy. We review the implications of the variational principle for the behavior of steady roll solutions near the boundary as found by direct calculation in [187]. We also discuss a conservation law that follows from Noether’s theorem ([102], §9.2.1).

At the end of this chapter, we shall present three additional applications of the Bogoliubov method. The first generalizes the nonlocal pattern equation for Rayleigh-Bénard convection to include a larger spectrum of marginal modes. The second addresses the issue of normal form reduction for Hopf bifurcations in spatially extended systems. The third derives a new amplitude equation for the Eckhaus instability that contains an additional saturating term. But before turning to these issues, we next provide an introduction to the basic aspects of convection theory to lay the groundwork and bring out the necessary notation for what follows.

10.2 Rayleigh-Bénard Convection

In this section, we review the formulation of the Rayleigh-Bénard convection problem with fixed-temperature boundary conditions to provide the background needed for the subsequent derivations.

10.2.1 Boussinesq Equations

In the Boussinesq (or thin-layer) approximation, the convection problem is described by the following standard set of non-dimensionalized hydrodynamic equations (cf. [50] II.§7)

\begin{align}
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_1) \mathbf{v} + w \partial_z \mathbf{v} &= -\nabla_1 \Pi + \varpi (\Delta + \partial_z^2) \mathbf{v}, \\
\partial_t w + (\mathbf{v} \cdot \nabla_1) w + w \partial_z w &= -\partial_z \Pi + \varpi (\Delta + \partial_z^2) w + R \varpi \theta, \\
\partial_t \theta + (\mathbf{v} \cdot \nabla_1) \theta + w \partial_z \theta &= (\Delta + \partial_z^2) \theta + w, \\
\nabla_1 \cdot \mathbf{v} + \partial_z w &= 0.
\end{align}

Here $\mathbf{v}$ is the horizontal velocity, $w \mathbf{\hat{z}}$ is the vertical velocity, where $\mathbf{\hat{z}}$ is the vertical unit vector, $\theta$ is the deviation of the temperature from its local static equilibrium value and $\Pi$ is the pressure deviation from its local hydrostatic value divided by the (constant) unperturbed density. A linear relation between density perturbation and temperature perturbation has been introduced. Further, $\nabla_1 = \hat{x} \partial_x + \hat{y} \partial_y$, the horizontal Laplacian is $\Delta = \nabla_1^2$, while $\nabla = \hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z$ with $\hat{x}$ and $\hat{y}$ being two horizontal orthonormal vectors.

The unit of length is the layer thickness and the unit of time is a thermal time based on the diffusivity of heat. The Prandtl number $\varpi$ and the Rayleigh number $R$ are nondimensional parameters, the latter measuring the degree of instability. The simplest
imposed kinetic boundary conditions are usually called (stress) free and rigid; these are
\[ w = 0, \quad \theta = 0; \quad \partial^2_z w = 0 \text{ (free)} \quad \text{or} \quad \partial_z w = 0 \text{ (rigid)} \quad \text{on} \quad z = \pm \frac{1}{2}. \quad (10.6) \]

We denote the full velocity as \( \mathbf{u} = (v, w) \), and the vorticity as \( \mathbf{\omega} = \nabla \times \mathbf{u} \). The vorticity equation is then (cf. [115] §5.5)
\[ \frac{D\mathbf{\omega}}{Dt} = (\mathbf{\omega} \cdot \nabla) \mathbf{u} + \nabla^2 \mathbf{\omega} + \nabla \times (R\mathbf{\omega} \theta \hat{\mathbf{z}}). \quad (10.7) \]

The boundary conditions on the vertical vorticity \( \zeta \equiv \mathbf{\omega} \cdot \hat{\mathbf{z}} \) are
\[ \partial_z \zeta = 0 \text{ (free)} \quad \text{or} \quad \zeta = 0 \text{ (rigid)} \quad \text{on} \quad z = \pm \frac{1}{2}. \quad (10.8) \]

Given \( w \) and \( \zeta \), we can determine \( \mathbf{v} \) up to a gauge using the relations \( \nabla_1 \cdot v = -\partial_z w \) and \( \nabla_1 \times \mathbf{v} = \zeta \hat{\mathbf{z}} \). Therefore, the triplet \((\zeta, w, \theta)\) provides a complete description of the perturbed state of the fluid. We may formulate linear theory based on this set of variables.

### 10.2.2 Linear Theory

Taking the \( \hat{z} \)-component of (10.7) and linearizing, we get an uncoupled equation for \( \zeta \)
\[ \partial_t \zeta = \omega \nabla^2 \zeta. \quad (10.9) \]

In view of (10.8), if we take the boundaries at \( z = \pm 1/2 \) to be rigid surfaces, then all the vertical vorticity modes are strongly damped in linear theory. However, for two free boundaries, there is a linear solution with \( \zeta \) constant in space and time corresponding to a rigid, undamped rotation of the whole fluid. Apart from vertical vorticity modes of motion on the very largest scales, all the other vertical vorticity modes are viscously damped quite effectively. If we rule out the possibility of the large-scale global rotation, as for a finite system, the vertical vorticity modes are slavish modes that may be excited only by nonlinear coupling to the convective modes ([143] and [158]). However, we shall not allow for their nonlinear excitation here and so we leave \( \zeta \) out of the account, that is, we make the approximation \( \zeta = 0 \). For studies of the inclusion of the vertical vorticity see [158], [25] and [140].

With the neglect of vertical vorticity, the fields \( w \) and \( \theta \) specify the state of the fluid. To write the linearized Boussinesq equations in terms of only \( w \) and \( \theta \), we eliminate \( \Pi \) from the linearized version of (10.2) by writing
\[ -\partial_z \nabla_1 \cdot (10.2) + \Delta (10.3) \]
in the linear approximation. Then the linear problem reduces to
\[ \partial_t (\partial^2_z + \Delta) w = \omega (\partial^2_z + \Delta) w + R\omega \Delta \theta \quad (10.10) \]
and
\[ \partial_t \theta = w + (\partial^2_z + \Delta) \theta \quad (10.11) \]
where \( \mathbf{v} \) is readily computed from the solutions. It is convenient to work in Fourier space with
\[ \hat{\phi}(\mathbf{x}, z, t) = \int \hat{\phi}_k(z, t) e^{ik \cdot x} dk, \quad (10.12) \]
where \( \dot{\omega} = w, \theta \) or \( v \). We define \( U_k \equiv (w_k, \theta_k)^T \) and write (10.10) compactly as

\[
\partial_t M_k U_k = L_k U_k, \tag{10.13}
\]

where

\[
M_k = \begin{pmatrix}
\partial_z^2 - k^2 & 0 \\
0 & 1
\end{pmatrix}, \quad L_k = \begin{pmatrix}
\varpi (\partial_z^2 - k^2) & -k^2 \varpi w \\
-k^2 \varpi w & \partial_z^2 - k^2
\end{pmatrix}, \tag{10.14}
\]

and the boundary conditions that follow from (10.6) are

\[
U_k = 0, \quad (1, 0) \partial_z^2 U_k = 0 \text{ (free)} \quad \text{or} \quad (1, 0) \partial_z U_k = 0 \text{ (rigid) on } z = \pm \frac{1}{2}. \tag{10.15}
\]

This linear problem is separable, and we may seek solutions of the form \( U_k = \Phi_k(z) e^{\sigma t} \). For either free or rigid boundaries, there are two linear modes \( \Phi_k(z) \) for each pair \((\ell, k)\) with \( \ell = 0, 1, 2, \ldots \), each with \( \ell \) internal nodes in its vertical structure. The gravest vertical modes (with \( \ell = 0 \)) play a key role near the onset of instability since one of them, denoted \( \phi_k(z) \), may become unstable in a certain band of wavenumbers \( k = |k| \); its growth rate is real and denoted as \( \sigma_k \). The other gravest mode, here called \( \varphi_k(z) \), is always stable; its (real) growth rate \( \xi_k \) is always negative. Though these two modes have similar spatial structures, the reason for their different stability characteristics lies in the signs of \( w \theta \); for \( \phi_k(z) \) this quantity is positive in the unstable wavenumber band so that buoyancy does work on the mode. For the other \( \ell = 0 \) mode, that sign is always negative.

The higher modes with \( \ell > 0 \) are denoted as \( \chi_{k, \ell}^\pm \) and have growth rates \( \gamma_{k, \ell}^\pm \). The functions \( \phi, \varphi \) and \( \chi_{k, \ell}^\pm \) form an orthogonal basis with the inner product

\[
\langle U_k, \tilde{U}_k \rangle \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( w_k^* (k^2 - \partial_z^2) \tilde{w}_k + R \varpi \theta_k^* k^2 \tilde{\theta}_k \right) dz, \tag{10.16}
\]

where \( * \) denotes complex conjugate.

It follows from (10.13) that the linear problems for \( \phi \), and its adjoint \( \phi^\dagger \), are

\[
L_k \phi_k = \sigma_k M_k \phi_k, \quad L_k^\dagger \phi_k^\dagger = \sigma_k M_k^\dagger \phi_k^\dagger, \tag{10.17}
\]

where the adjoint operators are the transposes of the original ones, and the adjoint boundary conditions coincide with the original ones. (\( \sigma_k \) can be shown to be real for our range of \( R \).

The critical point is \((R_c, k_c) = (27\pi^4/4, \pi/\sqrt{2})\) for free boundaries, and \((R_c, k_c) = (1708, 3.116)\) for rigid boundaries (cf. [50] II.§15(b)). For \((R, k)\) near \((R_c, k_c)\), we have the expansion [65] for free boundaries

\[
\sigma_k = \frac{k_c^2 \varpi}{\varpi + 1} \left( 3 \left( \frac{R}{R_c} - 1 \right) - \frac{(k^2 - k_c^2)^2}{k_c^4} \right), \tag{10.18}
\]

and for rigid boundaries

\[
\sigma_k = \frac{19.65 \varpi}{\varpi + 0.5117} \left( \left( \frac{R}{R_c} - 1 \right) - 0.3593 \frac{(k^2 - k_c^2)^2}{k_c^4} \right). \tag{10.19}
\]
The solution at \((R_c, k_c)\) and its adjoint are, for free boundaries,

\[
\phi_{k_c}(z) = \hat{\phi}_{k_c} \cos(\pi z), \quad \phi^\dagger_{k_c} = \left( \frac{1}{1/(3k_c^2)} \right); \tag{10.20}
\]

\[
\phi_{k_c}^\dagger(z) = \hat{\phi}_{k_c}^\dagger \cos(\pi z), \quad \phi^\dagger_{k_c} = \left( \frac{1}{-9k_c^4 \omega} \right); \tag{10.21}
\]

For rigid boundaries,

\[
\phi_{k_c}(z) = \left( \frac{W_{k_c}(z)}{(k_c^2 R)^{-1/3} \Theta_{k_c}(z)} \right), \quad \phi_{k_c}^\dagger(z) = \left( \frac{W_{k_c}(z)}{-\omega (k_c^2 R)^{2/3} \Theta_{k_c}(z)} \right), \tag{10.22}
\]

where the functions \(W_{k_c}\) and \(\Theta_{k_c}\) are defined by

\[
W_{k_c}(z) = \cos q_0 z + (A_1 + iA_2) \cos i(q_1 + iq_2)z + (A_1 - iA_2) \cos i(q_1 - iq_2)z, \tag{10.23}
\]

\[
\Theta_{k_c}(z) = \cos q_0 z + (B_1 + iB_2) \cos i(q_1 + iq_2)z + (B_1 - iB_2) \cos i(q_1 - iq_2)z. \tag{10.24}
\]

With \(q_0 = 3.974, q_1 = 5.194, q_2 = 2.126, A_1 = -0.03076, A_2 = -0.05196, B_1 = 0.06038\) and \(B_2 = -0.0006647\).

### 10.2.3 Nonlinear Terms

In Fourier space, \(v_k\) can be expressed in terms of \(w_k\) by (10.5) as

\[
v_k = i\frac{k}{k^2} \partial_z w_k. \tag{10.25}
\]

With this expression, we can work out the nonlinear terms in (10.2)-(10.5). First, we transform (10.4) into the form

\[
\partial_t \theta_k = w_k + (\partial^2_{\omega} - k^2) \theta_k + \int \int \mathcal{N}_\theta(k - p - q) dp dq. \tag{10.26}
\]

Then, to determine the nonlinear part \(\mathcal{N}_\theta\), we note that \(-w \partial_\omega \theta\) yields \(-w_q \partial_{\omega} \theta_p\), and \(-(v \cdot \nabla_1) \theta\) yields

\[
-(v \cdot \nabla_1) \theta = -\int (iv \cdot p) \theta_p e^{ip \cdot x} dp,
\]

where, by (10.25),

\[
-(iv \cdot p) = \int \frac{p \cdot q}{q^2} \partial_{\omega} w_q e^{iq \cdot x} dq.
\]

We then obtain

\[
\mathcal{N}_\theta = \frac{p \cdot q}{q^2} \theta_p \partial_{\omega} w_q - w_q \partial_{\omega} \theta_p. \tag{10.27}
\]

As for (10.2) and (10.3), we eliminate \(\Pi\) by \(-\partial_\omega \nabla_1 \cdot (10.2) + \Delta(10.3)\). The resulting equation has these four nonlinear terms on the right:

\[
(A) \partial_\omega \nabla_1 \cdot ((v \cdot \nabla_1) v); (B) \partial_\omega \nabla_1 \cdot (w \partial_\omega v); (C) - \Delta((v \cdot \nabla_1) w); (D) - \Delta(w \partial_\omega w). \tag{10.28}
\]
The incarnation of this equation in the Fourier space is
\[
\partial_t (\partial_z^2 - k^2) w_k = \varpi (\partial_z^2 - k^2)^2 w_k - k^2 R \varpi \theta_k + \int \int \mathcal{N}_w \delta(k - p - q) dp dq. \tag{10.29}
\]

The contributions of (10.28) to \( \mathcal{N}_w \) can be worked out by switching to index notation, doing the Fourier transform, and finally returning to vector notation. The results are

\[
(A) \left( \frac{(p \cdot q)^2}{p^2 q^2} + \frac{p \cdot q}{p^2} \right) \partial_z (\partial_z w_p \partial_z w_q); \quad (B) - \left( \frac{p \cdot q}{q^2} + 1 \right) \partial_z (w_p \partial_z^2 w_q);
\]
\[
(C) - \left( p \cdot q + 2 \frac{(p \cdot q)^2}{p^2} + \frac{q^2 p \cdot q}{p^2} \right) w_q \partial_z w_p; \quad (D) \left( p^2 + 2p \cdot q + q^2 \right) w_p \partial_z w_q.
\]

Since we can exchange \( p \) and \( q \) in any term it is possible to simplify this expression. The overall sum is

\[
\mathcal{N}_w = \left( 2 \left( \frac{(p \cdot q)^2}{p^2 q^2} + \frac{p \cdot q}{p^2} - 1 \right) \partial_z w_p \partial_z w_q - \left( \frac{p \cdot q}{q^2} + 1 \right) w_p \partial_z^2 w_q \right.
\]
\[
- \left( 2 \left( \frac{(p \cdot q)^2}{q^2} + \frac{p^2 p \cdot q}{q^2} - 1 \right) p \cdot q - p^2 - q^2 \right) w_p \partial_z w_q. \tag{10.30}
\]

In terms of \( U_k \equiv (w_k, \theta_k)^T \), we may write (10.26) and (10.29) succinctly as
\[
\partial_t M_k U_k = L_k U_k + \int \int \mathcal{N}(k - p - q) dp dq, \tag{10.31}
\]

where \( \mathcal{N} \equiv (\mathcal{N}_w, \mathcal{N}_\theta)^T \), and the boundary conditions are given in (10.15).

As a vector-valued function with vector arguments, \( \mathcal{N} \) is bilinear in \( U_p \) and \( U_q \) with coefficients dependent on \( p \) and \( q \). We have used the freedom to interchange \( p \) and \( q \) to simplify \( \mathcal{N} \) algebraically, but the expression is now not unique. The remedy is to symmetrize \( \mathcal{N} \) by defining
\[
\tilde{\mathcal{N}}(U_p, U_q; p, q) \equiv \frac{1}{2} \left[ \mathcal{N}(U_p, U_q; p, q) + \mathcal{N}(U_q, U_p; q, p) \right] \tag{10.32}
\]

and then remove the tilde. The expression for \( \tilde{\mathcal{N}} \) is twice as complex as before, but (10.32) has a conceptual advantage as we see further on.

### 10.3 The Expansion Procedure

#### 10.3.1 The Bogoliubov Approach

We are here concerned with the convection problem when the parameters are chosen so that the convective instability is weak. In cases where the growth rates form a discrete spectrum, one may proceed in a way that is sanctioned by the center manifold theory [44] and seek solutions near a point in parameter space where linear theory predicts that there are one or more marginal modes with zero growth rate and modes with negative...
growth rates bounded away from zero. (The study of cases with finite positive growth rates at onset can lead into turbulence theory and we shall exclude such cases here.)

The finitely damped modes are slavish modes that may be eliminated in favor of the marginally stable modes that span an invariant subspace in the space of solutions. The possibility of doing this is the essential content of the center manifold theory. Those slowly evolving modes — the slow modes — rule the dynamics of the system much as slowly reacting constituents control the chemical evolution of a reacting mixture.

When the aspect ratio is large, the growth rate spectrum is (in effect) continuous and the main results of center manifold theory are not available. However, we proceed in a qualitatively analogous way and perturbatively eliminate the rapidly decaying modes so that they do not appear explicitly in the reduced system or pattern equation. Our approach is in the spirit of the Bogoliubov method that has been used to derive amplitude equations for multiple bifurcation problems when the spectra of the modes are discrete. In the standard problem of RB convection, only the mode with \( \ell = 0 \) and \( k = k_c \) is marginal at onset, but now there are modes with \( \ell = 0 \) and small \( |k - k_c| \) that are nearly marginal. These slow modes must be kept in the approximate representation of the solution to the basic equation.

The fast modes with \( \ell > 0 \) do not pose a particular difficulty and their elimination proceeds much as for the finitely damped modes of problems with discrete spectra. Those are slaved modes, in the jargon of the subject. They do not appear in the final result explicitly though they do contribute to the coefficients of the nonlinear terms in the reduced equation. But the modes with \( \ell = 0 \) are more troublesome. The ones with large \( |k - k_c| \) are slavish and may be eliminated but those with small \( |k - k_c| \) must be retained. We then face the problem of where in \( k \) to make the cut that separates out the modes with finite decay rates.

The procedure we follow here is to include in the starting representation all of the modes, \( \phi_k \), across the full spectrum of \( k \), thus including highly stable modes in the initial stages of the treatment but avoiding sharp transitions in temporal behavior. But the fast modes retained in this way will resonate among themselves and with the slow modes. Such interactions produce singularities (as has been noted previously [61], [45]). The way we shall cope with this difficulty (within the present approach) is to eliminate the fast modes with \( \ell = 0 \) at a later stage in the development by restricting the final outcome to a narrow band of wavenumbers in the last reductions. This is an \textit{ad hoc} procedure whose justification lies only in the results. The procedure itself will be seen clearly in the illustrative example to be presented below.

\subsection{10.3.2 Diagrammatic Notation}

The Bogoliubov procedure entails expansions in the amplitudes of the modes, most importantly the slow ones. Though the procedure is not overly burdensome in the case of basic Boussinesq convection, it can become so when more than one wavenumber continuum of slow modes is active as can happen in doubly diffusive convection, for example. In anticipation of the need for methods to facilitate the treatment of such more demanding problems, we propose the use of diagrammatic notation for keeping track of terms in the consequent perturbative developments. The derivation of a convective pattern equation by Graham and Pleiner [91] was already considered to be suggestive of a diagrammatic
approach by Swift and Hohenberg [165]. Of course, such diagrams are inspired by the Feynman diagrams of quantum field theory. In the case of QFT, the diagrams also suggest a direct physical interpretation in terms of particles and their interactions. If one becomes practiced in the use of diagrams in the present context, it similarly becomes possible to develop an intuition for reading meaning into the diagrams. This is the experience in many branches of physics including particle physics [141], condensed matter physics [134], the theory of turbulence in fluids [183] and wave propagation in random media [83].

For well designed diagrams, each corresponds to a unique term in the perturbation series. At any given order, one first draws all possible (Feynman) diagrams consistent with the relevant physical laws, and then translates these diagrams back into algebraic expressions with a prescribed set of (Feynman) rules. Thus we can express (10.31) diagrammatically, with the replacement $U \rightarrow \circ$, as

$$\partial_t M = L + N.$$

(10.33)

Whereas in QFT the lines carry momenta (typically not labelled), here they (equivalently) carry wave vectors. Interactions among modes occur at the vertices at which momenta are conserved, as implied by the $\delta$-function of wave vectors in (10.31). Once one learns a given diagrammatic language, the diagrams become largely self-explanatory if certain conventions are observed. (See the book of Mattuck for a readable general introduction [134].) Below, we shall apply the combination of the ideas of Bogoliubov and of Feynman to the derivation of a pattern equation for the convection problem. But first we shall illustrate the procedure with a very simple problem devised for just this purpose. Even before undertaking an explicit calculation involving the details of the use of diagrams, we add a few more preparatory remarks.

In particle physics there are many kinds of particles and accordingly several kinds of line segments — wavy, dashed, straight — but, in the present simple considerations, we require only the straight line segments just introduced. All the other symbols in the diagrams used here label vertices except for the operator $\partial_t$. For any vertex, the lines immediately to its left (right) are called incoming (outgoing) lines, and the number of incoming (outgoing) lines is called the indegree (outdegree) of that vertex. All the vertices herein have outdegree one, but the indegree can be one (e.g. $M$, $L$) or more (e.g. $N$). If only such vertices are present, the diagrams necessarily take the forms of trees, consisting of the root connected to the leaves via the stems. These diagrams are generally topologically simpler than those in quantum field theory, where particles can be either created or annihilated, thereby making loops possible. Note that the leftmost incoming line vector entering the leaves and the rightmost outgoing line exiting the root are not shown explicitly. The total number of leaves is the order of the diagram, as in conventional diagrammatic practice.

A diagram can be translated to an algebraic expression (and vice versa) as follows. (The case of a first order diagram requires no further explanation.) For an $n$-th order diagram ($n \geq 2$), we first label the wave vectors entering the leaves as, say, $p_1, \cdots, p_n$, and the wave vector exiting the root as, say, $k$. Then we label the wave vectors on the stems in terms of $p_1, \cdots, p_n$, using conservation of wave vectors at the vertices. Any vertex corresponds to a function parametrically dependent on the wave vectors carried by its incoming lines and linear in each sub-diagram connected to it through these lines ($N$ is
a rather complicated example). With these guidelines, for any tree diagram, we can start from its root, traverse its stems and finally reach its leaves. The expression that we get, multiplied by $\delta(k - p_1 - \cdots - p_n)$ and integrated over $p_1, \cdots, p_n$, is the final result. The wave vectors carried by the intermediate stems are automatically integrated out, so only the $\delta$-function describing the overall wave vector conservation remains.

For an example of a third-order diagram, we reach into the calculation for the R-B convection problem below and consider the last term in (10.68), reproduced here as the lhs of (10.34).

\[ 2N = \phi^p_q + \phi^p_r + \phi^p_s. \]  

(10.34)

This diagram consists of the root $N$ connected to the leaves $\bullet = A$ through the intermediate vertices $\phi$ and $\mathcal{U}$. Two points should be noted in its interpretation. First, the factor of 2 (known as the symmetry factor in quantum field theory) makes allowance for the two distinct ways to permute the incoming edges of $N$. In terms of diagrams these two permutations can be combined, but algebraically they represent two different terms involving the two arguments of $N$, as spelled out on the rhs of (10.34). However, if we use the symmetrized version of the vertex $N$ in (10.32), then these two terms are also identical algebraically. This can be helpful for harder problems, but in this paper we will stick to the unsymmetrized version. Second, the vertex $\phi$ has indegree and outdegree both one, so it simply propagates the wave vector through itself. At any such vertex, the line is deliberately bent to signify the presence of the vertex. In contrast, for a vertex with indegree two (or more) and outdegree one (e.g. $\mathcal{U}$), its presence is already apparent from the topology of the diagram.

To continue with this example, we have labelled the wave vectors entering the leaves as $p$, $q$, and $r$, from which the wave vectors on all the stems are then uniquely labelled. The algebraic expression corresponding to the rhs of (10.34) is therefore

\[ \int \int \int N(\phi(A_p; p), \mathcal{U}(A_q, A_r; q, r); p, q + r)\delta(k - p - q - r)dpdqdr + \int \int \int N(\mathcal{U}(A_q, A_r; q, r), \phi(A_p; p); q + r, p)\delta(k - p - q - r)dqdr. \]

(10.35)

where $k$ denotes the wave vector exiting the root. For later reference, we use

\[ \phi(A_p; p) = \phi_p A_p, \quad \mathcal{U}(A_q, A_r; q, r) = \mathcal{U}_{q,r} A_q A_r \]

where $\phi_p$ and $\mathcal{U}_{q,r}$ are functions of $p$ and $(q, r)$, to rewrite (10.35) as

\[ \int \int \int [N(\phi_p, \mathcal{U}_{q,r}; p, q + r) + N(\mathcal{U}_{q,r}, \phi_p; q + r, p)] A_p A_q A_r \delta(k - p - q - r)dpdqdr. \]

(10.37)

Now we are ready to derive an amplitude equation. For this purpose, we need to eliminate the fast modes and keep the slow modes, much as in the process of center manifold reduction in bifurcation theory for ODEs. However, before dealing with the full convection problem, we next carry out the reduction to a pattern equation of a simple system contrived to be used as an example.
10.3.3 An Illustrative Calculation

As an illustrative example we consider the pair of equations [166] for two functions of time $t$ and of wave vector $k$:

\[ \partial_t f_k = \sigma_k f_k + \int \int f_p g_q \delta(k-p-q)dpdq, \tag{10.38} \]

\[ \partial_t g_k = \gamma_k g_k + \int \int f_p f_q \delta(k-p-q)dpdq, \tag{10.39} \]

where

\[ \gamma_k = \gamma - (k^2 + k_2^2)^2, \quad \sigma_k = \sigma - (k^2 + k_1^2)^2 \tag{10.40} \]

with $\gamma < 0$ and $|\sigma| \ll |\gamma|$. With the replacements $f \to \circ$, $g \to \circ$, these equations can be represented diagrammatically as

\[ \partial_t \circ = \circ \sigma + \circ \mathcal{N}, \tag{10.41} \]

\[ \partial_t \circ = \circ \gamma + \circ \mathcal{N}. \tag{10.42} \]

The vertex $\mathcal{N}$ here simply takes the product of the two sub-diagrams connected to it.

We seek a near-identity transformation to new coordinates $F_k$ and $G_k$ analogous to the transformation usually made in the discrete case to new coordinates (on and normal to a center manifold) in the function space of solutions:

\[ f_k = F_k + \int \int I_k(p,q)F_p F_q dpdq + \cdots, \tag{10.43} \]

\[ g_k = G_k + \int \int J_k(p,q)F_p F_q dpdq + \cdots. \tag{10.44} \]

With $F \to \bullet$, $G \to \bullet$, the transformation equations are expressed diagrammatically as

\[ \circ = \bullet + \bullet I + \cdots, \tag{10.45} \]

\[ \circ = \bullet + \bullet J + \cdots. \tag{10.46} \]

As will be seen later on, the coordinate transformation (10.45) does not simplify the final result, so we may choose to identify $\circ$ with $\bullet$. Inserting (10.46) into (10.41-10.42), we get

\[ \partial_t \bullet = \bullet \sigma + \bullet \mathcal{N} + \bullet \mathcal{N} + O(\bullet^4), \tag{10.47} \]

\[ \partial_t \bullet + 2 \partial_t \bullet \mathcal{J} + O(\bullet^2 \partial_t \bullet) = \bullet \gamma + \bullet \mathcal{J} \gamma + \bullet \mathcal{N} + O(\bullet^3). \tag{10.48} \]

Inserting $\partial_t \bullet$ from (10.47) into (10.48) yields a single evolution equation for $\bullet$. In order that $\bullet$ be transversal to the center manifold, $\partial_t \bullet$ cannot contain any term of zeroth order in $\bullet$. At second order in $\bullet$, this requirement leads to

\[ 2 \bullet \sigma \mathcal{J} = \bullet \mathcal{J} \gamma + \bullet \mathcal{N}, \tag{10.49} \]
Figure 10.2: The curve of resonances. We plot $\sigma + \sigma_k + k_1 \hat{x}$ in red and $\gamma_k$ in blue, with their intersection implying resonances. Parameters: $k_1 = 1$, $k_2 = 2$, $\sigma = 0.1$, $\gamma = -5$.

which translates to the condition

$$\int\int (D_k(p, q)J_k(p, q) - \delta(p + q - k)) F_p F_q dp dq = 0,$$  \hspace{1cm} (10.50)

where $D_k(p, q) = \sigma_p + \sigma_q - \gamma_k$. There are resonances when

$$p + q - k = 0, \quad D_k(p, q) = 0.$$  \hspace{1cm} (10.51)

These resonance conditions cannot be met if both $|p|$ and $|q|$ are near $k_1$, but resonances are possible when we take, say, only $p = k_1 \hat{x}$. In this case, we can locate the curve of resonances (Fig. 10.2) with a graphical procedure (cf. [14] and [188]).

These resonances involve damped modes that are not slow. Such modes are normally eliminated in the standard singular perturbation approaches to nonlinear bifurcation theory. We have kept them in order to keep things working smoothly but they should be filtered out. We do this as follows.

Once (10.49) is satisfied, $\bullet$ decays exponentially to 0 for large $t$. Then from (10.47), the pattern equation truncated to third order is

$$\partial_t \bullet = \sigma \bullet + \mathcal{J} N,$$  \hspace{1cm} (10.52)

which translates to an equation of the form

$$\partial_t F_k = \sigma_k F_k + \int\int\int \mathcal{J}(q, r)\delta(k - p - q - r) F_p F_q F_r dp dq dr.$$  \hspace{1cm} (10.53)

If $F_k$ is peaked in a band containing $k_1$, we may simplify this result using the method of steepest descent, which filters out the fast modes and removes the singularities. In this way, we obtain the 1-D Swift-Hohenberg equation [166].
10.4 A Convective Pattern Equation

Now we turn to the reduction of the full Boussinesq convection equations to a pattern equation. We presume that the full solution of the problem may be well approximated by an amplitude function that depends on time and horizontal coordinates or horizontal wave vector (together with the appropriate modal structure). This is again in the spirit of the Bogoliubov method.

Since the growth rate $\sigma_k$ depends continuously on $k$, we need to filter out the fast modes at some stage in the derivation. Here, we shall use a slightly different device than we did in the illustrative example. But no choice of procedure is rigorously justified since there is no spectral gap that separates the slow modes from the enslaved fast ones. One might even wonder whether there may be some sort of virtual resonances that show up in higher order than is usually considered in pattern studies. However, at the present level of the subject, these procedures seem to work well.

10.4.1 Setup

A key idea in this calculation is the expressibility of $U_k$ (defined in §10.2.2) as the functional power series

$$
U_k = A_k \phi_k + \int \int d\Phi dq A_p A_q U^{(2)}_k(p, q; z) + \cdots,
$$

and the presumption that $A_k$ satisfies an equation of its own of the form

$$
\partial_t A_k = \sigma_k A_k + \int \int d\Phi dq A_p A_q \Gamma^{(2)}_k(p, q) + \cdots.
$$

The notation in these two equations is as before except for the introduction of the kernels that are to be determined. We can express (10.54) and (10.55) diagrammatically with

$$
\begin{align*}
\phi & = \phi + U + \cdots, \\
\sigma & = \sigma + \Gamma + \cdots.
\end{align*}
$$

We substitute into (10.33) and gather terms of the same order in $A_k$. The linear condition

$$
\sigma \phi = \phi L
$$

is satisfied for a suitable choice of $\phi_k$. At each higher order, the kernels on either side must balance because $A_k$ is arbitrary. Let $\mathcal{N}^{(m)}$ denote the coefficient of the $\delta$-function in the kernel of $\mathcal{N}$ at the $m$-th order. To filter out the fast modes, we admit only slow modes into the representation by restricting the support of $A_k$ to a compact region in $k$-space — a thin annulus containing the critical circle $|k| = k_c$. In that confined region, we have only slow modes whose growth rates are close to zero.
10.4.2 Second Order

\[
\Gamma \phi = L \mathcal{U} + \mathcal{N} \phi.
\]

(10.59)

Setting the combined kernel of the gathered second order terms to 0, we get

\[
-L_k \mathcal{U}^{(2)}_k = \mathcal{N}^{(2)}(p, q) \delta(k - p - q) - \Gamma^{(2)}_k(p, q) M_k \phi_k.
\]

(10.60)

Since \( \mathcal{N}^{(2)} \) is odd, it does not contain the lowest vertical mode and so the solvability condition,

\[
0 = -\Gamma^{(2)}_k(p, q) \langle (\phi^\dagger_k) | M_k \phi_k \rangle,
\]

(10.61)
determines that \( \Gamma^{(2)}_k(p, q) = 0 \). Here \( \langle \cdot | \cdot \rangle \) denotes the usual inner product. We are left with the always solvable equation

\[
-L_k \mathcal{U}^{(2)}_k = \mathcal{N}^{(2)}(p, q) \delta(k - p - q).
\]

(10.62)

Here, \( p \) and \( q \) are constrained to lie close to the critical circle but \( k \) is not. If we introduce the shorthand \( c_{p, q} \equiv p \cdot q / \sqrt{p^2 q^2} \), then

\[
\mathcal{N}_\theta = c_{p, q} \theta_p \partial_z w_q - w_q \partial_z \theta_pacent{
\mathcal{N}_w = (2c^2_{p, q} + c_{p, q} - 1) \partial_z w_p \partial^2_z w_q - (c_{p, q} + 1) w_p \partial^2_z w_q - 2k_c^2 (c^2_{p, q} - 1) w_p \partial_z w_q.
\]

(10.64)

The matrix \( L_k = L_{p+q} \) also depends on \( c_{p, q} \). These lead to

\[
\mathcal{U}^{(2)}_k(p, q; z) = \hat{\mathcal{U}}^{(2)}(\varpi, z, c_{p, q}) \delta(k - p - q).
\]

(10.65)

For free boundaries, we have the relatively simple expression

\[
\hat{\mathcal{U}}^{(2)}(\varpi, z, c_{p, q}) = \frac{1 - c_{p, q}}{(5 + c_{p, q})^3 - \frac{2c_{p, q}}{4} (1 + c_{p, q}) \left( \frac{3(1+c_{p, q})(3+2(5+c_{p, q})\varpi)}{2(5+c_{p, q})^2 + 2(1+c_{p, q})\varpi} \right)^{-1} \sin(2\pi z)}.
\]

(10.66)

For rigid boundaries, it turns out to be easier to evaluate \( \hat{\mathcal{U}}^{(2)} \) numerically rather than analytically. The result can be written as

\[
\hat{\mathcal{U}}^{(2)}(\varpi, z, c_{p, q}) = \begin{pmatrix}
\hat{\mathcal{U}}^{(2)}_{w, 0}(z, c_{p, q}) \\
\hat{\mathcal{U}}^{(2)}_{\theta, 0}(z, c_{p, q})
\end{pmatrix} + \begin{pmatrix}
\hat{\mathcal{U}}^{(2)}_{w, -1}(z, c_{p, q}) \\
\hat{\mathcal{U}}^{(2)}_{\theta, -1}(z, c_{p, q})
\end{pmatrix} \varpi^{-1},
\]

(10.67)

where the functions \( \hat{\mathcal{U}}^{(2)}_{w, 0}, \hat{\mathcal{U}}^{(2)}_{\theta, 0}, \hat{\mathcal{U}}^{(2)}_{w, -1} \) and \( \hat{\mathcal{U}}^{(2)}_{\theta, -1} \) are plotted in Fig. 10.3.
10.4.3 Third Order

\[ \Gamma_k \phi + 2 \Gamma_k U = L U + 2 N U \phi \, . \quad (10.68) \]

Setting the kernel of the gathered third order terms to 0, we get

\[ -L_k U^{(3)}_k = N^{(3)}(p, q, r) \delta(k - p - q - r) - \Gamma_k^{(3)}(p, q, r) M_k \phi_k. \quad (10.69) \]

Because \( k, p, q \) and \( r \) are all restricted to (what we may call) the critical annulus, we have only three possibilities

(1) \( k = p, q = -r \); (II) \( k = q, r = -p \); (III) \( k = r, p = -q \). \quad (10.70)

Now we let \( p \) (resp. \( q \) and \( r \)) denote the wave vector(s) in the first (resp. second) order (see (10.34)). If we suitably permute \( p, q \) and \( r \), we can transform all three cases to the geometry of case (II). Thus the contribution from case (III) is identical to case (II), but differs from case (I). We find, with some algebra, that the bracketed kernel in (10.37) evaluates to

\[ N^{(3)} = 2 \left( -\phi''_w U''_w - \frac{1}{2} \phi_u U''_w + \frac{3}{2} k^2 \phi_u \hat{U}'_w + (c_q r - \frac{1}{2} \hat{U}'_w \phi'_w + c_q r \hat{U}'_w \phi'_w + (2 + c_q r) k^2 \hat{U}'_w \phi'_w \right) + N^{(3)}_{\text{(I)}}, \quad (10.71) \]

where \( ' \) denotes differentiation with respect to \( z \), and subscript (I) denotes case (I). The following solvability condition must be satisfied:

\[ \langle (\phi_k^\dagger)^T | N^{(3)}(p, q, r) \delta(k - p - q - r) - \Gamma_k^{(3)}(p, q, r) M_k \phi_k \rangle = 0. \quad (10.72) \]

For either free or rigid boundaries, we obtain the general expression

\[ \Gamma_k^{(3)}(p, q, r) = \hat{\Gamma}^{(3)}(q, c_q r) \delta(k - p - q - r). \quad (10.73) \]
In the free case, we have the relatively simple result,

\[ \hat{\Gamma}^{(3)}(\varpi, c_{q,r}) = 2\hat{\Gamma}^{(3)}_{(II)} + \hat{\Gamma}^{(3)}_{(I)}, \]

where

\[ \hat{\Gamma}^{(3)}_{(II)} = -\frac{\varpi}{4(1 + \varpi)} \frac{(1 - c_{q,r})^2((5 + c_{q,r})^2 + 9(1 + c_{q,r})\varpi^{-1} + 3(1 + c_{q,r})(5 + c_{q,r})\varpi^{-2})}{(5 + c_{q,r})^3 - \frac{27}{4}(1 + c_{q,r})}\]  

and

\[ \hat{\Gamma}^{(3)}_{(I)} = -\frac{\varpi}{4(1 + \varpi)}. \]

The expression for \( \hat{\Gamma}^{(3)}_{(II)} \) agrees with that given in [65] (which is based on the calculations done in [155]) up to a constant factor due to normalization convention, and the expression for \( \hat{\Gamma}^{(3)}_{(I)} \) agrees with \( \hat{\Gamma}^{(3)}_{(II)} \) when \( c_{q,r} = -1 \). In the rigid case, we find numerically that

\[ \hat{\Gamma}^{(3)}_{(II)} = -\frac{L_+(c_{q,r})\varpi + L_0(c_{q,r}) + L_-(c_{q,r})\varpi^{-1}}{7.781 + 15.21\varpi}\] 

where the functions \( L_+ \), \( L_0 \) and \( L_- \) are plotted in Fig. 10.4. Again, they have been verified to be a constant times their counterparts listed in Table I of [65].

### 10.4.4 The Evolution Equation

The evolution equation truncated to leading order is

\[ \partial_t A_k = [\sigma_0 + \sigma_2(k^2 - k_c^2)]A_k + \iiint (\varpi, c_{q,r})e^{i(p+q+r)\cdot x} u_p u_q u_r d\psi dq dr, \]

where (10.18)-(10.19) have been used, the kernel is given in (10.73), and the integration is done in case (II) of (10.70). With the deviation from the marginal Rayleigh number scaling as \( \sigma_0 \sim \epsilon^2 \), consistent scalings near marginality are \( \partial_t \sim \epsilon^2 \), \( k^2 - k_c^2 \sim \epsilon \), \( A \sim \epsilon \). In the physical space, with additional re-scaling of \( t \) and \( x \), we finally have the nonlocal pattern equation

\[ \partial_t u = ru - (\Delta + k_0^2)u + \iiint (\varpi, c_{q,r})e^{i(p+q+r)\cdot x} u_p u_q u_r d\psi dq dr, \]
where all the coefficients are \(O(1)\) and the notation has been changed to facilitate comparison with recent studies of the Swift-Hohenberg equation (e.g. Eq. (2.64)).

For 2-D convection, \(p, q\) and \(r\) are constrained to be colinear, so \(\hat{\Gamma}^{(3)}\) reduces to a constant, and (10.79) reduces to the 1-D Swift-Hohenberg equation [165]

\[
\partial_t u = ru - (\Delta + k_0^2)^2 u + f_3 u^3,
\]

(10.80)

where \(f_3\) is a constant depending on \(\varpi\). The Laplacian in this equation should strictly be a second derivative with respect to \(x\). However, with suitable approximations [62], this form of nonlocal equation can be reduced to the S-H equation in two dimensions as well, so we leave the full Laplacian in place. The Swift-Hohenberg model for the standard Rayleigh-Bénard convection problem has had a remarkable qualitative success in reproducing the gamut of observed patterns in the Boussinesq context. However, it may be that to make detailed quantitative comparison between theory and experiment a nonlocal pattern equation such as the one found here and in earlier discussions may be called for as Roberts [149] has argued.

10.5 The Variational Structure and Its Applications

The Swift-Hohenberg equation is much simpler than the basic Boussinesq equations and so is useful for studying the possible convective patterns. Nevertheless, analytic solutions of it are not easily found and pattern studies, even with the S-H equations, must be performed numerically (e.g. [135]), though that is readily achieved by going to the web site of M.C. Cross, http://www.cmp.caltech.edu/~mcc/Pat terns/index.html. Still, it is worth recalling that the S-H equation has a variational structure that can be useful in the problem of pattern selection.

Recall from §2.4.2 that an evolution equation \(\partial_t u = M[u]\) is called variational if it can be written in the form

\[
\partial_t u = -\frac{\delta F}{\delta u},
\]

(10.81)

where \(\delta F/\delta u\) is the functional (or Fréchet) derivative. It may then be shown that \(dF/dt \leq 0\). If, in addition, \(F[u]\) is bounded from below, it is called a Lyapunov functional and functions \(u\) that minimize it are stable. It is known that (10.80) has a Lyapunov functional (cf. §10.5.2) and so does (10.79), under suitable restrictions, as we next see.

10.5.1 Variational Structure of the Nonlocal Pattern Equation

To find the Lyapunov functional for (10.79), we seek a

\[
\mathcal{G}[u] = \iiint_{\text{domain}} \Lambda(\varpi, c_{q,r}) \delta(p + q + r + s) u_p u_q u_r u_s dp dq dr ds
\]

(10.82)

such that

\[
\frac{\delta \mathcal{G}}{\delta u} = \iiint_{\text{domain}} \hat{\Gamma}^{(3)}(\varpi, c_{q,r}) e^{i(p+q+r) \cdot x} u_p u_q u_r dp dq dr,
\]

(10.83)
where, out of the three equivalent cases,
\[(1) s = -p, q = -r; \quad (II) s = -q, r = -p; \quad (III) s = -r, p = -q. \quad (10.84)\]
we perform the integration for $G$ in case (II).

After some manipulations with functional derivatives and Fourier transforms, we get
\[
\frac{\delta G}{\delta u} = \int \int (\Pi) \frac{1}{\pi^2} \Lambda(\varpi, c_{q,r}) \exp(i(p + q + r) \cdot x) u_p u_q u_r d\varpi d\rho d\sigma, \quad (10.85)
\]
in light of $c_{q,r} = c_{r,q} = c_{p,s} = c_{s,p}$. Therefore we should pick
\[
\Lambda(\varpi, c_{q,r}) = \pi^2 \hat{\Gamma}^{(3)}(\varpi, c_{q,r}). \quad (10.86)
\]

The Lyapunov functional for (10.79) is
\[
F[u] = \int \left( -\frac{1}{2} r u^2 + \frac{1}{2} ((\Delta + k_0^2)u)^2 \right) dx - G[u]. \quad (10.87)
\]

To show that $F[u]$ is bounded from below, we note that $u = u_k^*$ leads to
\[
u_p u_q u_r u_s = |u_p|^2 |u_q|^2 \geq 0. \quad (10.88)
\]
In addition, given that $\hat{\Gamma}^{(3)}$ is never positive,
\[
-\Lambda(\varpi, c_{q,r}) \geq \frac{\pi^2 \varpi}{4(1 + \varpi)} \equiv \pi^2 M > 0. \quad (10.89)
\]
Therefore, we can establish the estimate (now the integration is done over all three cases)
\[
-\int G[u] \geq \frac{\pi^2 M}{3} \int \int \int \delta(p + q + r + s) u_p u_q u_r u_s d\varpi d\rho d\sigma = \frac{M}{12} \int u^4 dx, \quad (10.90)
\]
where the identity $u_s = \frac{1}{(2\pi)^d} \int u e^{-is \cdot x} dx$ is used. It then follows from (10.87) that
\[
F[u] \geq \int \left( -\frac{1}{2} r u^2 + \frac{1}{2} ((\Delta + k_0^2)u)^2 + \frac{M}{12} u^4 \right) dx \equiv \tilde{F}[u], \quad (10.91)
\]
where $\tilde{F}[u]$ is the Lyapunov functional for (10.80) with $f_3 = -M/3$. It is known (cf. [98] §7.3) that $\tilde{F}[u]$ is bounded from below for $f_3 < 0$, so our claim is proved. (A Lyapunov functional for a similar nonlocal pattern equation appears in an unpublished manuscript of Zaleski and Spiegel.)

We expect the nonlocal pattern equation (10.79) to accurately capture the physics, but PDE models like (10.80) enjoy much greater popularity because of their mathematical simplicity. Hence in the following, we derive a few lesser known properties of steady states in dissipative PDEs with Lyapunov functionals, and mention a few examples.

For this discussion, we consider any system in $d$ spatial dimensions with a Lyapunov functional
\[
F[u] = \int L(u, \partial_t u, \Delta u) dx = \int L(q, p_t, r) dx, \quad x \in \mathbb{R}^d.
\]
We shall focus on \( u(x) \) that locally minimize \( \mathcal{F}[u] \), although the equalities that appear below also apply to other stationary points of \( \mathcal{F}[u] \) (saddles and maxima). In our convention, repeated Latin indices are summed over, but Greek indices are not. Also, an independent variable as subscript denotes partial derivative with respect to that variable; for example \( \mathcal{L}_p \) denotes the gradient of \( \mathcal{L} \) with respect to \( p \).

10.5.2 Generalized Clausius Theorem

In the study of dynamics of swarms of particles the virial theorem of Clausius [55] gives relations concerning certain (time averaged) macroscopic properties of the swarm. Clausius' theorem has been found quite useful in studies of stellar systems and other conglomerations of particles. An extension of this theorem to a continuous system has been derived for the study of wave dynamics by Derrick [76]. This idea has also been extended to the study of a particular continuous, dissipative system with a Liapunov functional — the case of thermal instability [80]. These considerations provide macroscopic side constraints on the systems studied that serve to further delineate their properties. We next describe how the same idea applies in the present instance.

If we introduce a parameter \( \lambda \) and define \( \mathcal{F}(\lambda) \equiv \mathcal{F}[u(x+\lambda y(x))] \), then stationarity and minimality require

\[
\mathcal{F}'(\lambda = 0) = 0, \quad (10.92)
\]
\[
\mathcal{F}''(\lambda = 0) \geq 0. \quad (10.93)
\]

Consider a linear function \( y(x) \), for which the Jacobian matrix \( J \) (defined by \( J_{ij} \equiv \partial y_j / \partial x_i \)) is a constant matrix independent of \( x \). For any \( \lambda \), we can transform the coordinate by \( x + \lambda y(x) \to x \) to obtain the exact expression

\[
\mathcal{F}(\lambda) = \int L(u, \partial_i u, \Delta u + \lambda (J_{jk} + J_{kj}) \partial_j u + \lambda^2 J_{ij} J_{jk} \partial_{jk} u) \det(I + \lambda J) \, dx, \quad (10.94)
\]

where \( I \) is the identity matrix. Here the integration domain is left unspecified, but certain conditions near the boundary must be satisfied as we see for the following evident example. For an arbitrary function \( \psi(x) \)

\[
\int_0^L \psi(\lambda x) d(\lambda x) = \int_0^{L/\lambda} \psi(x) dx \neq \int_0^L \psi(x) dx \quad (10.95)
\]

unless \( \psi(x) \to 0 \) at \( x \to L \). Therefore, we may either take a finite domain and require \( \mathcal{L} = 0 \) near the boundary, or take the domain to be \( \mathbb{R}^d \) and require \( \mathcal{L} \in L^1(\mathbb{R}^d) \). In other words, what follows should be applied to steady localized states with bounded total free energy.

We see that in (10.94), \( \lambda \) and \( J \) always appear only in the combination \( \lambda J \), so \( \mathcal{F}^{(n)}(\lambda = 0) \) must be \( n \)-linear in \( J \). For \( |\lambda| \ll 1 \), the denominator can be expanded as

\[
\det(I + \lambda J) = \exp(\text{tr}(\log(I + \lambda J))) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\sum_{j=1}^{\infty} \frac{(-\lambda)^j}{j} \text{tr}(J^j) \right)^k, \quad (10.96)
\]
from which follows the useful expression

\[
\frac{1}{\det(I + \lambda J)} = 1 - \lambda \text{tr} J + \frac{\lambda^2}{2} (\text{tr}^2 J + \text{tr} J^2) + O(\lambda^3). \tag{10.97}
\]

Since \( F'(\lambda = 0) \) is linear in \( J \), we need just to impose (10.92) on the following two classes of transformations. The first is the class of scaling transforms \( J_{ij} = \delta_{ij} \delta_{\mu \nu} \), which leads to

\[
\int (\mathcal{L}_p \partial_\mu u + 2 \mathcal{L}_r \partial_{\mu \nu} u - \mathcal{L}) \, dx = 0, \tag{10.98}
\]

Summation over \( \mu \) yields a global Clausius theorem,

\[
\int (\mathcal{L}_p \cdot p + 2 \mathcal{L}_r r - d \mathcal{L}) \, dx = 0, \tag{10.99}
\]

or equivalently (in fact, more generally)

\[
\sum_i (i - d) \int \mathcal{L}_i \, dx = 0, \tag{10.100}
\]

where \( \mathcal{L}_i \) is the term in \( \mathcal{L} \) with \( i \) spatial derivatives in total. The second is the class of shear transforms \( J_{ij} = \delta_{ij} \delta_{\mu \nu} (\mu \neq \nu) \), which leads to

\[
\int (\mathcal{L}_p \partial_\mu u + 2 \mathcal{L}_r \partial_{\mu \nu} u) \, dx = 0. \tag{10.101}
\]

We may summarize (10.98) and (10.101) as an orthogonality condition (with \( \mu \) and \( \nu \) arbitrary) as

\[
\int (\mathcal{L}_p + 2 \mathcal{L}_r \partial_\mu) \partial_\nu u \, dx = \delta_{\mu \nu} \int \mathcal{L} \, dx. \tag{10.102}
\]

Since \( F''(\lambda = 0) \) is a quadratic form in the matrix elements \( J_{ij} \), (10.93) is equivalent to the requirement that the \( d^2 \times d^2 \) matrix

\[
\mathcal{H}_{\kappa \lambda \mu \nu} = \int (\delta_{\kappa \lambda} \delta_{\mu \nu} + \delta_{\kappa \nu} \delta_{\lambda \mu}) + 2 \mathcal{L}_r (\delta_{\kappa \mu} \partial_{\lambda \nu} u - 2 \delta_{\kappa \lambda} \partial_{\mu \nu} u) - 2 \mathcal{L}_p \partial_{\kappa \lambda} \partial_\nu u \tag{10.103}
+ 4 \mathcal{L}_r \partial_{\kappa \lambda} u \partial_{\mu \nu} u + 4 \mathcal{L}_r \partial_\mu u \partial_{\kappa \lambda} u + \mathcal{L}_p \partial_{\kappa \lambda} u \partial_\nu u) \, dx, \tag{10.104}
\]

defined such that \( F''(\lambda = 0) = J_{kl} \mathcal{H}_{klmn} J_{mn} \), is nonnegative definite. An apparent application of this result is seen in the special case \( J_{ij} = \delta_{ij} \). Then

\[
F''(\lambda = 0) = \int [d(d + 1) \mathcal{L} + 2(1 - 2d) \mathcal{L}_r r - 2d \mathcal{L}_p \cdot p \tag{10.105}
+ 4 \mathcal{L}_r r^2 + 4 \mathcal{L}_r \cdot p r + \mathcal{L}_p \cdot p J_{ij} \cdot J_{ij}] \, dx \geq 0, \tag{10.106}
\]

or equivalently (in fact, more generally)

\[
F''(\lambda = 0) = \sum_i (i - d)(i - d - 1) \int \mathcal{L}_i \, dx \geq 0. \tag{10.107}
\]

Let us introduce the shorthand \( I_i \equiv \int \mathcal{L}_i \, dx \). Sometimes, (10.100) and (10.107) together with the positivity (or negativity) of \( I_i \) can be restated as a linear programming problem. The nonexistence of solutions for \( I_i \) then implies the nonexistence of stable localized states.
Reaction-Diffusion Equation

For example, the one-component reaction-diffusion equation
\[ \partial_t u = D \nabla^2 u + R(u), \quad D > 0 \]
has a Lyapunov functional with free energy density
\[ \mathcal{L}^{RD} = \frac{D}{2} |\nabla u|^2 - V(u), \quad V' = R. \]
The free energy consists of $I_0$ and $I_2$. The conditions (10.100) and (10.107) then become
\[ (2 - d)I_2 + (-d)I_0 = 0, \quad (2 - d)(1 - d)I_2 + (-d)(-1 - d)I_0 \geq 0. \]
For any $d \geq 3$, (10.110) has no nontrivial solutions (those that satisfy $I_2 > 0$). Thus, (10.108) does not admit stable localized solutions for any $d \geq 3$.

Generalized S-H Equation

Another example (most relevant here) is the generalized Swift-Hohenberg equation, that is,
\[ \partial_t u = ru - (\nabla^2 + k_0^2)u^2 + f(u), \]
where the nonlinear function $f$ may be either quadratic-cubic or cubic-quintic in $u$. It has a Lyapunov functional with free energy density
\[ \mathcal{L}^{SH} = -\frac{1}{2}ru^2 + \frac{1}{2}((\nabla^2 + k_0^2)u)^2 - F(u), \quad F' = f. \]
We consider that the nonlinear term takes the form
\[ f = \sum_{i=2}^{\infty} f_i u^i, \quad F = \sum_{i=2}^{\infty} f_i \frac{u^{i+1}}{i+1}. \]
The free energy for (10.111) consists of $I_0$, $I_2$ and $I_4 > 0$. We introduce the ratios $\rho_i = I_i/I_4$ ($i = 0, 2$), and rewrite (10.100) and (10.107) as
\[ (4 - d) + (2 - d)\rho_2 + (-d)\rho_0 = 0, \quad (4 - d)(3 - d) + (2 - d)(1 - d)\rho_2 + (-d)(-1 - d)\rho_0 \geq 0. \]
The segment that they define on the $\rho_0$-$\rho_2$ plane depends on $d$ (Fig. 10.5). In addition
\[ I_2^{SH} = k_0^2 \int (u \nabla^2 u) \, dx = -k_0^2 \int |\nabla u|^2 \, dx < 0. \]
As for $I_0$, we have
\[ I_0^{SH} \geq C \int u^2 \, dx, \]
where $C > 0$ for some choices of parameters. These inequalities imply that (10.111) can possibly have stable localized solutions for any $d$. One can make a better estimate for $\rho_0$ from the Cauchy-Schwarz inequality
\[ \left| \int (u \nabla^2 u) \, dx \right| \leq \left( \int u^2 \, dx \right)^{1/2} \left( \int |\nabla u|^2 \, dx \right)^{1/2}. \]
This further shortens the segment on the $\rho_0$-$\rho_2$ plane, but does not change the conclusion.
Cahn-Hilliard Equation

Though the generalized Clausius theorem is not very constraining for the Swift-Hohenberg equation, it is worth remarking that the case of the Cahn-Hilliard equation \([43]\]

\[\partial_t u = \Delta (u^3 - u - \gamma \Delta u), \quad \gamma > 0 \tag{10.118}\]

is more interesting in this respect. It has the free energy density

\[\mathcal{L}^{CH} = \frac{1}{4}(u^2 - 1)^2 + \frac{\gamma}{2} |\nabla u|^2, \tag{10.119}\]

such that \(u_t = \Delta (\delta \mathcal{F}/\delta u)\) where \(\mathcal{F} = \int \mathcal{L}^{CH} dx\). Stable localized solutions to (10.118) must satisfy \(\delta \mathcal{F}/\delta u = 0\), but they are not necessarily local minima of \(\mathcal{F}[u]\). For example, a spherical bubble of \(u = -1\) with any finite radius, embedded in a \(u = 1\) background, is stable. We cannot decrease \(\mathcal{F}\) by shrinking the bubble radially, as this is forbidden — it violates

\[\frac{d}{dt} \int u dx = 0, \tag{10.120}\]

a conservation law that follows from (10.118) rewritten as

\[\partial_t u = \nabla \cdot j \quad \text{with} \quad j = \nabla (u^3 - u - \gamma \Delta u). \tag{10.121}\]

We refer the reader to [76] for such reasoning applied to nonlinear wave equations.

10.5.3 Least Action Principle

Consider a finite spatial domain \(\mathcal{D}\). The change in free energy under perturbation \(\delta u\) is

\[\delta \mathcal{F} = \mathcal{F}[u + \delta u] - \mathcal{F}[u] = \int_{\partial \mathcal{D}} [\mathcal{L}_r \nabla \delta u + \delta u (\mathcal{L}_p - \nabla \mathcal{L}_r)] \cdot \hat{n} dA - \int_{\mathcal{D}} H \delta u dx, \tag{10.122}\]

where \(H = -\mathcal{L}_q + \partial_i \mathcal{L}_{p_i} - \Delta \mathcal{L}_r, \partial \mathcal{D}\) is the boundary of \(\mathcal{D}\), \(\hat{n}\) is the unit outward normal to \(\partial \mathcal{D}\) and \(dA\) is the surface element on \(\partial \mathcal{D}\). If \(\delta \mathcal{F}\) is to vanish for arbitrary \(\delta u\), \(H = 0\) must be satisfied on \(\mathcal{D}\) and, in addition, the following boundary conditions must be fulfilled:

\[\mathcal{L}_r = 0, \quad (\mathcal{L}_p - \nabla \mathcal{L}_r) \cdot \hat{n} = 0 \quad \text{on} \quad \partial \mathcal{D}. \tag{10.123}\]
Application to the S-H Equation

For the Swift-Hohenberg equation, the boundary conditions are (here $\mathcal{L}_r = (\nabla^2 + k_0^2)u$)

$$\mathcal{L}_r = 0, \quad \nabla \mathcal{L}_r \cdot \hat{n} = 0 \quad \text{on} \quad \partial \mathcal{D}. \quad (10.124)$$

These conditions require that, at the boundary, both $u$ and $\nabla u \cdot \hat{n}$ must have planform wavenumbers $k_0$. However, if the convection rolls meet the boundary perpendicularly, then $\nabla u \cdot \hat{n} = 0$ and only the constraint on $u$ remains. This is a plausible reason why such behavior is most often observed in convection experiments near the onset of instability. (A derivation of this result from the NWS equation is given in [187].)

10.5.4 A Conservation Law

According to [124], for any $\mathcal{L}$ not explicitly dependent on $x$, we have the conservation law

$$\partial_{x_1} \mathcal{L} - \nabla \cdot (u_{x_1} \mathcal{L}_p + \mathcal{L}_r \nabla u_{x_1} - u_{x_1} \nabla \mathcal{L}_r) = 0 \quad \text{for all} \quad x \in \mathbb{R}^d, \quad (10.125)$$

where $\partial_{x_1}$ can be replaced by any other directional derivative. To interpret (10.125) physically, we make a perturbation $\delta u$ proportional to $u_{x_1}$. In view of (10.122), the second term of (10.125) is the local contribution to the bilinear concomitant or equivalently the free energy flux. On the other hand, the perturbation essentially translates $u$ in the $-x_1$ direction, so the free energy flux can also be expressed as the first term of (10.125). Thus, (10.125) is a consequence of the special choice of perturbation by a translational mode.

Application to 2-D S-H

For $d = 2$ in the Swift-Hohenberg equation, (10.125) is directly responsible for selecting the wavelength of hexagonal pattern that connects through a front to the trivial state [124]. For the PDE studied in [80], however, (10.125) simply reduces to the original PDE.

10.6 A generalized nonlocal pattern equation

The Bogoliubov method is fundamentally a generalization of the technique of center manifold reduction for ODE to the PDE case. However, as pointed out in §10.3.3, if we define the center manifold $W$ to cover the entire spectrum of wave vectors $k$, then resonances inevitably arise between the slow modes and certain fast modes. In the example with one slow and one fast mode studied in §10.3.3, this problem has been settled by the Vlasov decree borrowed from plasma physics [166]. However, in all previous approaches to the R-B convection problem including §10.4, these resonances are implicitly avoided by defining $W$ to cover only a thin annulus of wave vectors around $k = k_c$. This choice is unsatisfactory since a sharp cutoff has been introduced in the continuum of wave vectors $k$. Hence in this section, we present an alternative derivation (following Ref. [166]) of the amplitude equation for $A_k$, where $k \in \mathbb{R}^2$. 
In what follows it is convenient to use slightly different notations for the vertical modes. The most unstable mode, which becomes marginal at \((k, R) = (k_c, R_c)\), is denoted by \(\phi_k(z)\) with its growth rate denoted by \(\sigma_k\). The remaining ones, denoted by \(\varphi_{k,\ell}(z)\), \(\ell \in \mathbb{N}\), are all linearly stable at \(R = R_c\) with their growth rates denoted by \(\gamma_{k,\ell}\). For later reference, the defining equations for these modes and their adjoints are

\[
\mathbb{L}_k\phi_k = \sigma_kM_k\phi_k, \quad \mathbb{L}_k^+\phi_k^\dagger = \sigma_kM_k^\dagger\phi_k^\dagger; \\
\mathbb{L}_k\varphi_{k,\ell} = \gamma_{k,\ell}M_k\varphi_{k,\ell}, \quad \mathbb{L}_k^+\varphi_{k,\ell}^\dagger = \gamma_{k,\ell}M_k^\dagger\varphi_{k,\ell}^\dagger.
\] (10.126)

(10.127)

The ansatz of the Bogoliubov method is identical to §10.4, but the detailed calculations outlined below digress from and generalize those in §10.4.

### 10.6.1 Second Order

Setting the combined kernel of the gathered second order terms to 0, we get

\[
((\sigma_p + \sigma_q)M_k - \mathbb{L}_k)U_k^{(2)} = \mathcal{N}^{(2)}(p, q)\delta(k - p - q) - \Gamma_k^{(2)}(p, q)M_k\phi_k.
\] (10.128)

We form inner product with \((\phi_k^\dagger)\) to obtain the solvability condition

\[
(\sigma_p + \sigma_q - \sigma_k)(\phi_k^\dagger)M_kU_k^{(2)} = -\Gamma_k^{(2)}(p, q)(\phi_k^\dagger)M_k\phi_k,
\] (10.129)

where the term \((\phi_k^\dagger)\mathcal{N}^{(2)}(p, q)\delta(k - p - q)\) on the rhs has been omitted since \(\mathcal{N}^{(2)}\) is odd in \(z\) and hence does not contain the lowest vertical mode. It follows that on the set

\[
V_2 := \{(p, q) : \sigma_p + \sigma_q = \sigma_{p+q}\},
\] (10.130)

(10.129) requires that \(\Gamma_k^{(2)}(p, q) = 0\). Otherwise, \(\Gamma_k^{(2)}(p, q)\) is arbitrary, so we may set it to be 0. This arbitrariness results from the fact that highly damped modes \(\phi_k\) where \(|k - k_c| \sim O(1)\) have been included in the center manifold. We are then left with

\[
((\sigma_p + \sigma_q)M_k - \mathbb{L}_k)U_k^{(2)} = \mathcal{N}^{(2)}(p, q)\delta(k - p - q).
\] (10.131)

To solve this, we form inner product with \((\varphi_{k,\ell}^\dagger)\) to obtain

\[
(\sigma_p + \sigma_q - \gamma_{k,\ell})(\varphi_{k,\ell}^\dagger)M_kU_k^{(2)} = \langle(\varphi_{k,\ell}^\dagger)\mathcal{N}^{(2)}(p, q)\rangle\delta(k - p - q).
\] (10.132)

Here we face a situation analogous to [166], namely that resonances occur on the set

\[
V_2' := \{(p, q) : \mathcal{D}_\ell(p, q) \equiv \sigma_p + \sigma_q - \gamma_{p+q,\ell} = 0\}.
\] (10.133)

In such cases, we write the formal solution as

\[
\langle(\mathbb{M}_k^\dagger\varphi_{k,\ell}^\dagger)^\mathcal{N}^{(2)}(p, q)\rangle = \left(\frac{1}{\mathcal{D}_\ell(p, q)} - \lambda\delta(\mathcal{D}_\ell(p, q))\right)\langle(\varphi_{k,\ell}^\dagger)\mathcal{N}^{(2)}(p, q)\rangle\delta(k - p - q).
\] (10.134)
where $\lambda$ is a set of constants and $\mathcal{P}$ means principal value (P.V.). As in [166], we will abide by the Vlasov decree and set $\lambda = 0$ without attempting to justify it. To derive an explicit expression for $U^{(2)}_k$, we expand it as

$$U^{(2)}_k = \sum_\ell U^{(2)}_{k,\ell} \varphi_{k,\ell}. \quad (10.135)$$

By virtue of the following orthogonality condition that results from (10.127)

$$\langle (\mathcal{M}^\dagger_k \varphi_{k,\ell}^\dagger) | \varphi_{k,\ell'} \rangle = 0 \quad \text{for} \quad \ell \neq \ell', \quad (10.136)$$

(10.134) then determines the expansion coefficients. Inserting these into (10.135), we finally have

$$U^{(2)}_k = \sum_\ell \mathcal{P} \frac{1}{\mathcal{D}_\ell(p, q)} \frac{\langle (\varphi_{k,\ell}^\dagger) | \mathcal{N}^{(2)}(p, q) \rangle}{\langle (\mathcal{M}^\dagger_k \varphi_{k,\ell}^\dagger) | \varphi_{k,\ell} \rangle} \varphi_{k,\ell} \delta(k - p - q). \quad (10.137)$$

### 10.6.2 Third Order

Setting the kernel of the gathered third order terms to 0, we get

$$((\sigma_p + \sigma_q + \sigma_r)\mathcal{M}_k - \mathbb{I}_k)U^{(3)}_k = \mathcal{N}^{(3)}(p, q, r) \delta(k - p - q - r) - \Gamma^{(3)}_k(p, q, r)\mathcal{M}_k \varphi_k. \quad (10.138)$$

As before, we form inner product with $(\varphi_k^\dagger)^T$ to obtain the solvability condition. On the set

$$V_3 := \{(p, q, r) : \sigma_p + \sigma_q + \sigma_r = \sigma_{p+q+r}\}, \quad (10.139)$$

this condition requires that

$$\Gamma^{(3)}_k(p, q, r) = \frac{\langle (\varphi_k^\dagger)^T | \mathcal{N}^{(3)}(p, q, r) \rangle}{\langle (\mathcal{M}^\dagger_k \varphi_k^\dagger) | \varphi_k \rangle} \delta(k - p - q - r). \quad (10.140)$$

Otherwise, $\Gamma^{(3)}_k(p, q, r)$ is arbitrary, so we may set it to coincide with (10.140).

### 10.6.3 Result

The evolution equation truncated to leading order is therefore

$$\partial_t A_k = \sigma_k A_k + \int\int\int \Gamma^{(3)}_k(p, q, r) A_p A_q A_r dp dq dr. \quad (10.141)$$

It remains to evaluate $\Gamma^{(3)}_k$ explicitly for free and rigid boundaries. This consists of two steps, though we do not implement them here. First, we compute $\mathcal{N}^{(2)}$ and substitute into (10.137) to obtain $U^{(2)}_k$. Second, we compute $\mathcal{N}^{(3)}$ from $U^{(2)}_k$ and substitute into (10.140) to obtain $\Gamma^{(3)}_k$. The details are less involved in the free case than the rigid case, because the number of summation terms in (10.137) is finite for the former but infinite for the latter.
Let us compare the old and new approaches. In the old approach, the set of dynamically significant wave vectors is
\[ T^2 := \{ (p, q) : |p| = |q| = k_c \} \tag{10.142} \]
at second order, and
\[ T^3 := \{ (p, q, r) : |p| = |q| = |r| = k_c \} \tag{10.143} \]
at third order. In our approach, this set is the locus of "self-resonances" \( V_2 \) (\( V_3 \)) at second (third) order, which contains \( T^2 \) (\( T^3 \)) as a lower-dimensional subset. These two approaches yield identical expressions for \( \Gamma^{(3)}_{k} \) on \( T^3 \), but ours predicts nonzero \( \Gamma^{(3)}_{k} \) on \( V_3 \setminus T^3 \). As a result, the new pattern equation (10.141) is most likely non-variational, and this makes it a plausible candidate for explaining the long transients observed in experiments [30].

### 10.7 Hopf bifurcations in spatially extended systems

As reviewed in §2.1, local bifurcation for ODEs consists of two reduction steps, namely center manifold reduction and normal form theory. In §10.4, the PDE model undergoes a pitchfork bifurcation at finite wavenumber. As reviewed in §2.1.2, normal form theory cannot simplify the nonlinear terms for codimension-1 steady-state bifurcations. In this section, however, we derive a normal form pattern equation for a Hopf bifurcation, the simplest situation where normal form theory is useful.

Consider a physical system with an active control parameter \( \lambda \) whose critical modes are a pair of complex conjugate normal modes \( (\phi_k, \phi_k^*) \) with growth rates \( (\mu_k + i\omega_k, \mu_k - i\omega_k) \), where \( k \) again denotes a wave vector. This pair of critical modes loses stability, that is, \( \mu_k \) crosses through 0, at a critical wavenumber \( |k| = k_c \) as \( \lambda \) crosses through a critical value \( \lambda_c \). We assume that all the other normal modes remain linearly stable when \( \lambda \) is near \( \lambda_c \).

We expect that the dynamics is captured by the amplitudes of the pair of critical modes \( (\phi_k, \phi_k^*) \), which are denoted by \( (\alpha_k, \alpha_k^*) \). Hence we may utilize the Bogoliubov method to derive the following amplitude equation for \( \lambda \) near \( \lambda_c \) and \( |k| \) near \( k_c \)
\[ \partial_t \alpha_k = (\mu_k + i\omega_k)\alpha_k + \Gamma_k[\alpha_p, \alpha_q^*] \tag{10.144} \]
where \( \Gamma_k \) is a purely nonlinear functional power series of \( \alpha_p \) and \( \alpha_q^* \), which probably contains nonlocal interactions. The amplitude \( \alpha_k^* \) satisfies the complex conjugate of (10.144). In normal form theory, we try to simplify (10.144) by the near-identity coordinate change
\[ \alpha_k = A_k + \Psi_k[A_p, A_q^*], \]
so that \( A_k \) satisfies
\[ \partial_t A_k = (\mu_k + i\omega_k)A_k + g_k[A_p, A_q^*]. \]
The equation for \( \Psi_k \) is found by substitution into (10.144) to be
\[ \mathcal{L}_k \Psi_k = T_k - g_k \tag{10.145} \]
where the linear operator is

\[ \mathcal{L}_k \Psi_k = \int \frac{\delta \Psi_k}{\delta A_p} (\mu_p + i\omega_p) A_p d\mathbf{p} + \int \frac{\delta \Psi_k}{\delta A_q} (\mu_q - i\omega_q) A_q^* d\mathbf{q} - (\mu_k + i\omega_k) \Psi_k \]

and the nonlinear term has been abbreviated by

\[ T_k = \Gamma_k [A_p + \Psi_p, A_q^* + \Psi_q^*] - \int \frac{\delta \Psi_k}{\delta A_p} g_p d\mathbf{p} - \int \frac{\delta \Psi_k}{\delta A_q} g_q^* d\mathbf{q}. \tag{10.146} \]

The functional power series above are formed by monomials conveniently written as

\[ |DK\rangle_Q = \left( \prod_{i=1}^{D-K} \int d\mathbf{p}_i \right) \left( \prod_{j=1}^{K} \int d\mathbf{q}_j \right) Q_{DK}(\mathbf{p}_1, \ldots, \mathbf{p}_{D-K}, \mathbf{q}_1, \ldots, \mathbf{q}_K) \times \delta(\sum_{i=1}^{D-K} \mathbf{p}_i + \sum_{j=1}^{K} \mathbf{q}_j - \mathbf{k}) \prod_{i=1}^{D-K} A_{\mathbf{p}_i} \prod_{j=1}^{K} A_{\mathbf{q}_j}^*, \tag{10.147} \]

where \( Q = \Psi, g \) or \( T \). When the kernel \( Q_{DK} \) is constant (denoted by \( \bar{Q}_{DK} \)), the monomial \( |DK\rangle_Q \) reduces to the local expression \( \bar{Q}_{DK}(A^{D-K}(A^*)^K)_k \). It may seem that in (10.147) we need to deal with two fields \( A_k \) and \( A_k^* \), but the relation \( A_k^* = A_{-k} \) suggests that \( A^* \) can be transformed to \( A \) by reversal of wave vector. Therefore, we may choose to represent \( \bar{Q}_{DK}\) by a diagram with \( D - K \) lines with right arrows and \( K \) lines with left arrows, both connected to a vertex labelled \( Q \). We can use these two notations interchangeably, e.g.

\[ |31\rangle \Gamma \Leftrightarrow \Gamma. \]

Regardless of the kernel \( Q_{DK} \), any monomial is an eigenvector of \( \mathcal{L}_k \)

\[ \mathcal{L}_k |DK\rangle_Q = \Lambda_{DK} |DK\rangle_Q \]

if we take \( A_k \) to have infinitesimal support around \( |k| = k_c \). The eigenvalue is

\[ \Lambda_{DK} = i\omega_k(D - 2K - 1) + \mu_k(D - 1). \tag{10.148} \]

Then (10.145) becomes

\[ \Lambda_{DK} |DK\rangle_\Psi = |DK\rangle_T - |DK\rangle_g \tag{10.149} \]

for \( D \geq 2 \). From (10.146) we know \( |2K\rangle_T \), so we can solve (10.149) when \( D = 2 \), once we choose \( |2K\rangle_g \). Thereafter, at each new \( D \), \( |DK\rangle_T \) is known if we solve sequentially.

As long as \( \Lambda_{DK} \neq 0 \), we can always choose \( g_{DK} = 0 \). However, (10.148) suggests that when \( \mu_{k_c} = 0 \) (i.e. \( \lambda = \lambda_c \)), \( \Lambda_{DK} \) vanishes when \( K = (D - 1)/2 \), or equivalently \( \Lambda_{2L+1,L} = 0 \) for \( L = 1, 2, \ldots \). To avoid the small denominator \( \Lambda_{2L+1,L} \) near \( \lambda_c \), we require that \( g_{2L+1,L} = T_{2L+1,L} \). All the other \( g_{DK} \) may be set equal to 0. Therefore the normal form is

\[ \partial_t A_k = (\mu_k + i\omega_k) A_k + |31\rangle_T + |52\rangle_T + \cdots. \tag{10.150} \]
The diagrammatic notation is not needed to get (10.150), but it can be useful for calculating the normal form “coefficients” \( T_{31}, T_{52}, \text{ etc.} \). As an example, to expand the convolution \( \int (\delta \Psi_k / \delta A_p) g_p \delta p \) in (10.146), we first cross out any of the \( D - K \) instances of \( A \) that occur in \( |DK\rangle \psi \), which results in \( D - K \) diagrams with an empty slot each. Then we fill each slot with the functional power series \( g_p \), and sum up these \( D - K \) diagrams in the end.

Overall, the above normal form theory for Hopf bifurcation in nonlocal pattern equation parallels the ODE case (cf. [160] §4). We will not study the properties of (10.150), but mention that when \( T_{31} \) is constant and \( k_c = 0 \), (10.150) truncated to third order is the complex Ginzburg-Landau equation (CGLE), whose solutions have been extensively documented [6]. The CGLE, or more generally (10.150), is invariant under the phase-rotation symmetry \( A \rightarrow Ae^{i\phi} \) where \( \phi \) is a constant, exactly like Hopf bifurcations for ODEs.

10.8 Pattern equation of the Eckhaus instability

In this section, we derive a phase equation near the Eckhaus instability in the real-coefficient Ginzburg-Landau equation using the Bogoliubov method.

10.8.1 The Unreduced Equations

Consider the Ginzburg-Landau equation

\[
A_t = \mu A - |A|^2 A + A_{xx}, \quad x \in (-\infty, \infty)
\]  

(10.151)

which admits as a solution the perfect roll pattern

\[
A = R_0 e^{ik_0 x} \quad \text{where} \quad R_0^2 = \mu - k_0^2.
\]  

(10.152)

The stability of this pattern can be analyzed by writing \( A = R_0(1 + r)e^{i(k_0 x + \phi)} \) where \(|r|, |\phi| \ll 1\). The real and imaginary parts of (10.151) lead to

\[
rt = \mu(1 + r) - R_0^2(1 + r)^3 + r_{xx} - (1 + r)(k_0 + \phi_x)^2,
\]  

(10.153)

\[
(1 + r)\phi_t = 2r_x(k_0 + \phi_x) + (1 + r)\phi_{xx}.
\]  

(10.154)

We let \( \theta = \phi_x \) and organize (10.153-10.154) by powers of \( r \) and \( \theta \) into

\[
\begin{pmatrix}
rt \\
\theta_t
\end{pmatrix} = \begin{pmatrix}
-2R_0^2 r + r_{xx} - 2k_0 \theta \\
2k_0 r_{xx} + \theta_{xx}
\end{pmatrix} + \begin{pmatrix}
-3R_0^2 r^2 - 2k_0 r \theta - \theta^2 \\
2(r_x \theta_x + r_{xx} \theta - k_0 (r_x^2 + r r_{xx}))
\end{pmatrix}
\]  

\[
+ \begin{pmatrix}
-R_0^2 r^3 - r \theta^2 \\
2(-r r_x \theta_x + r_x^2 (2k_0 r - \theta) + r r_{xx} (k_0 r - \theta))
\end{pmatrix} + \cdots.
\]  

(10.155)

It is convenient to work in Fourier space with

\[
\phi(x, t) = \int \phi_k(t) e^{ikx} dk,
\]  

(10.156)
where $\\phi = r$ or $\theta$. We define $U_k \equiv (r_k, \theta_k)$ and write (10.155) compactly as

$$
\partial_t U_k = \mathbb{L}_k U_k + \iint \mathcal{N}^{(2)} \delta(k-p-q)dpdq + \iiint \mathcal{N}^{(3)} \delta(k-p-q-s)dpdqds + \cdots, \tag{10.157}
$$

where

$$
\mathbb{L}_k = \begin{pmatrix}
-2R_0^2 - k^2 & -2k_0 & -k^2 \\
-2k_0 k^2 & -k^2 & \end{pmatrix}, \quad \mathcal{N}^{(2)} = \begin{pmatrix}
-3R_0^2 r_p r_q - 2k_0 r_p \theta_q - \theta_p \theta_q \\
2(pq + p^2)(-r_p \theta_q + k_0 r_p r_q) \\
\end{pmatrix},
$$

$$
\mathcal{N}^{(3)} = \begin{pmatrix}
-R_0^2 r_p r_q r_s - r_p \theta_q \theta_s \\
2(qs + pq + q^2)(r_p r_q \theta_s - k_0 r_p r_q r_s) \\
\end{pmatrix}. \tag{10.158}
$$

### 10.8.2 Linear Theory and the Bogoliubov Method

Linear stability analysis performed on the linear part of (10.157) reveals two normal modes, both parametrically dependent on $k$. One is the strongly damped amplitude mode $\rho_k$ with dispersion relation

$$
\gamma_k = -2R_0^2 - (1 + 2 \frac{k_0^2}{R_0^2})k^2 + O(k^4). \tag{10.159}
$$

The other is the phase mode $f_k$ with dispersion relation

$$
\sigma_k = \left( 2 \frac{k_0^2}{R_0^2} - 1 \right) k^2 - 2 \frac{k_0^4}{R_0^6} k^4 + O(k^6). \tag{10.160}
$$

For $k_0^2 < \mu/3$ ($\mu/3 < k_0^2 < \mu$), namely the Eckhaus stable (unstable) region, the peak $\sigma_k = 0$ ($\sigma_k > 0$) occurs at $k^2 = 0$ ($k^2 > 0$). Near the critical case $(k_0^2, k^2) = (\mu/3, 0)$, we may use the Bogoliubov method to derive an evolution equation for the amplitude of $f_k$. For later reference, the phase and amplitude modes and their adjoints near criticality are

$$
f_k = (-k_0/R_0^2 + \frac{k_0^2}{R_0^2}) + O(k^4), 1^T, \quad \bar{f}_k = (-k_0/k_0 - 1 + O(k^4)) + 1^T, \tag{10.161}
$$

$$
\rho_k = (1, k_0^2/R_0^2 + O(k^4)) + 1^T, \quad \bar{\rho}_k = (1, k_0^2/R_0^2 + O(k^4)) + 1^T. \tag{10.162}
$$

As in §10.4, the ansatz of this method is to express $U_k$ as the functional power series

$$
U_k = \Theta_k f_k + \iint U_k^{(2)}(p, q) \Theta_p \Theta_q \delta(k-p-q)dpdq + \cdots, \tag{10.163}
$$

and to presume that $\Theta_k$ satisfies an equation of its own of the form

$$
\partial_t \Theta_k = \sigma_k \Theta_k + \iint \Gamma_k^{(2)}(p, q) \Theta_p \Theta_q \delta(k-p-q)dpdq + \cdots. \tag{10.164}
$$

We substitute into (10.157) and collect terms at each order in $\Theta_k$. The linear condition is satisfied by construction. At each higher order, the kernels on either side must balance because $\Theta_k$ is arbitrary. In terms of diagrams, the terms involved at each order also coincide with those appearing in the derivation in §10.4, except for an additional term that comes from $\mathcal{N}^{(3)}$ in (10.157) at third order.
10.8.3 Second Order

Setting the combined kernel of the gathered second order terms to 0, we get

$$
(\sigma_p + \sigma_q - \mathbb{L}_k) \mathcal{U}^{(2)}_k(p, q) = \mathcal{N}^{(2)}(f_p, f_q) - \Gamma^{(2)}_k(p, q) f_k. \tag{10.165}
$$

The solvability condition requires

$$
0 = \langle (f_k^\dagger)^T | \mathcal{N}^{(2)}(f_p, f_q) \rangle - \Gamma^{(2)}_k(p, q) \langle (f_k^\dagger)^T | f_k \rangle, \tag{10.166}
$$

which leads to

$$
\Gamma^{(2)}_k(p, q) = \frac{3}{4k_0} (p + q)(3p + q) + O((p, q)^4). \tag{10.167}
$$

Now the solution to (10.165) can be written as

$$
\mathcal{U}^{(2)}_k(p, q) = \frac{1}{\sigma_p + \sigma_q - \gamma_k} \langle (\rho^\dagger)\mathcal{N}^{(2)}(f_p, f_q) \rangle \rho_k = \left(\begin{array}{c}
-\frac{3}{8k_0} + \frac{4p^2 + 6pq + 3q^2}{8k_0^2} + O((p, q)^4) \\
-\frac{3(p + q)^2}{16k_0^2} + O((p, q)^4)
\end{array}\right). \tag{10.168}
$$

10.8.4 Third Order

Setting the combined kernel of the gathered third order terms to 0, we get

$$
(\sigma_p + \sigma_q + \sigma_s - \mathbb{L}_k) \mathcal{U}^{(3)}_k(p, q, s) = 2\mathcal{N}^{(2)}(f_p, \mathcal{U}^{(2)}_k(q, s)) + \mathcal{N}^{(3)}(f_p, f_q, f_s) - 2\mathcal{U}^{(2)}_k(p, q + s) \mathcal{N}^{(2)}(f_p, f_q) - \Gamma^{(3)}_k(p, q, s) f_k, \tag{10.169}
$$

where the overhead tildes imply symmetrization of functional arguments. The solvability condition requires

$$
0 = \langle (f_k^\dagger)^T | 2\mathcal{N}^{(2)}(f_p, \mathcal{U}^{(2)}_k(q, s)) + \mathcal{N}^{(3)}(f_p, f_q, f_s) \rangle - \Gamma^{(3)}_k(p, q, s) \langle (f_k^\dagger)^T | f_k \rangle, \tag{10.170}
$$

where the identity $\langle (f_k^\dagger)^T | \mathcal{U}^{(2)}_k \rangle = 0$ has been used. This leads to

$$
\Gamma^{(3)}_k(p, q, s) = (p + q + s) \left(\frac{3}{4k_0}(p + 3q + 2s) - \frac{3}{8k_0^2}(17p^3 + 66pq^2 + 15pq^2)\right) + O((p, q, s)^6), \tag{10.171}
$$

where $p$, $q$ and $s$ have been permuted in order to simplify the second term on the rhs.

10.8.5 Result

In the physical space, the evolution equation (10.164) for $\Theta$ then takes the form

$$
\Theta_t = -\left(2\frac{k_0^2}{R_0^2} - 1\right)\Theta_{xx} - \frac{3}{2k_0^4} \Theta_{xxxx} + \cdots - \frac{3}{k_0} (\Theta \Theta_x)_x + \cdots
$$

$$
- \frac{9}{2k_0^2} (\Theta^2 \Theta_x)_x - \frac{3}{8k_0^2} (17\Theta^2 \Theta_{xxx} + 66\Theta \Theta_x \Theta_{xx} + 15\Theta^3)_x + \cdots, \tag{10.172}
$$
where ellipsis denotes terms with higher \( x \) derivatives. The coefficient of the quadratic term agrees with earlier studies [98]. The cubic terms are absent in these studies, but if the coefficient of the quadratic term could be turned off by adjusting an extra parameter (the Boussinesq limit), then the cubic terms would show up in the final amplitude equation by taking a distinguished limit. There is, however, no Boussinesq limit in the version of Eckhaus instability that we consider here, as opposed to models of compressible convection [75] or bioconvection [54] where this limit naturally arises.

Nonetheless, (10.172) is a conservation law \( \Theta_t = [F(\Theta)]_x \) for the perturbation wavenumber \( \Theta \). The linear terms and the dominant quadratic term \( \Theta \Theta_x \) in the flux function \( F(\Theta) \) are both perfect \( x \)-derivatives, so these terms are diffusive in the PDE. In fact for any \( N \in \mathbb{N} \), the \( N \) terms quadratic in \( \Theta \) containing \((2N-1)\) \( x \)-derivatives, denoted by \( \Theta^{(n)} \Theta^{(2N-1-n)} \) \((0 \leq n \leq N-1)\), are all perfect \( x \)-derivatives of linear combinations of \( \Theta^{(m)} \Theta^{(2N-2-m)} \) \((0 \leq m \leq N-1)\). Among the terms cubic in \( \Theta \), \( \Theta^2 \Theta_x \) is again a perfect \( x \)-derivative but those containing three \( x \)-derivatives are not, because arbitrary linear combinations of \( \{ \Theta^2 \Theta_{xxx}, \Theta \Theta_x \Theta_{xx}, \Theta^3_x \} \) cannot be derived from only two integrals \( \{ \Theta^2 \Theta_{xx}, \Theta^2_{x} \} \).

We pick the leftover term to be \( \Theta^3_x \) since \( (\Theta^3_x)_x \) in the PDE can be interpreted as a saturating nonlinearity if its coefficient is positive. This can be confirmed after a short calculation, and the resulting truncated PDE for \( \Theta \) is

\[
\Theta_t = \left(- (2 \frac{k_0^2}{R_0^2} - 1) \Theta - 2 \frac{k_0^4}{R_0^6} \Theta_{xx} - \frac{3}{2k_0} \Theta^2 \right)_x + \frac{3}{8k_0} (\Theta^2_x)_x. \tag{10.173}
\]

Upon inclusion of the cubic term \( (\Theta^3_x)_x \), the PDE fails to be quasilinear, so its solutions may not be sufficiently smooth for generic initial conditions. However we expect (10.173) to describe large-scale patterns observed in experimental and numerical studies of the Rayleigh-Bénard convection [177] that may result from an Eckhaus instability similar to the scenario analyzed above.

10.9 Discussion

We have presented several illustrations of the use of Bogoliubov’s method supplemented by (Feynman) diagrams for deriving pattern equations. These were both in the style of singular perturbation theory and they represent problems that are tractable by algebraic manipulation. They were chosen for demonstrating the joint use of the Bogoliubov method and of the proposed diagrammatic approach because of their relative simplicity.

But, for derivations of pattern equations by this combination of methods, we have in mind more difficult problems such as those raised when multiple bifurcations arise and the aspect ratio is large. In finite systems, where the spectra of the normal modes are typically discrete, we normally do not seek pattern equations (PDEs or integro-differential equations) as the reduced equations, but amplitude equations or ODEs. The virtues of the Bogoliubov method for those problems is clear. In such cases, we begin with a development of the solution of the equations in terms of the discrete set of normal modes of the linear stability problem. The modal amplitudes represent coordinates in the function space of solutions. At the onset of instability, there are \( N \) marginal modes whose eigenvalues have zero real parts together with a (possibly infinite) number of stable modes whose eigenvalues have
negative real parts. The amplitudes of the stable modes are functions of the amplitudes of the marginal modes at onset; this relation defines an $N$-dimensional invariant subspace (or center manifold) in which the dynamics of the system is played out. This (loose) statement of the center manifold theorem is an outgrowth of Bogoliubov’s approach to kinetic theory.

When the aspect ratio becomes large, the spectrum of the modes is best treated as continuous and the modes with wave numbers near to the critical wavenumber must also be included in (what we think of as) a functional center manifold. Here we have dealt with the extension of the amplitude equation for $N = 1$ into a pattern equation, the Swift-Hohenberg model. For higher $N$, matters become more complicated and it is here that the use of symmetries comes into its own. However, we suggest that, for direct calculation, the Bogoliubov method buttressed by the diagrammatic approach may provide a tractable means of getting detailed results.

In the Bogoliubov procedure, in the discrete case, we make the ansatz that the temporal variation of the amplitudes of the slow modes — those that are marginal at onset — is governed by a system of ODEs of the form

$$\frac{d\mathbf{A}}{dt} = \mathbf{M}\mathbf{A} + \mathbf{g}(\mathbf{A}).$$

(10.174)

Here $\mathbf{g}(\mathbf{A})$ is strictly nonlinear and the components of $\mathbf{A}$ serve also as coordinates on the center manifold. The $N \times N$ matrix $\mathbf{M}$ is constructed so that its eigenvalues are those of the slow modes, each of whose eigenvalues has real part equal to zero at onset.

This much of the procedure is what we may call nonlinear instability theory or normal form theory. It has been discussed in detail elsewhere from the standpoint of Bogoliubov’s method [62, 64, 160]. When the aspect ratio is large, each single discrete, critical, mode is embedded in a continuous band of slow modes. The transform of each such band describes an individual pattern like the one treated here. The equations describing $N$ of these patterns and their interactions when $N > 1$ form the next level of complexity in pattern theory. In such regimes, though there is no center manifold theorem, we nevertheless proceed under the assumption that something analogous allows the slow modes to control the dynamics.

The method that we have proposed for doing this is designed to facilitate the process of generalizing (10.174) to the case of very large aspect ratio. That generalization is analogous to the one made here of (10.1) into (10.80). The components of $\mathbf{A}$ in (10.174) become functions on the several annuli (more generally, shells) concentrated on (possibly) different critical wavenumbers and (10.174), on Fourier transformation, becomes a set of equations for the nonlinear interaction and evolution of $N$ planform functions. (The shells need not be spherical.)
Chapter 11

Conclusion

The core of this dissertation is dedicated to the studies of localized structures in the 1:1 FCGLE. These localized structures arise from the bistability between two equilibria \( A^\pm \) interacting with an instability on one of them. As a conclusion, we summarize the major discoveries and point out some future directions.

In Chapter 3 we study the bifurcation structures of 1D localized states and discover a new mechanism for their growth called defect-mediated snaking (DMS). This growth mechanism results from fronts between \( A^\pm \) interacting with a Turing bifurcation on either equilibrium, and differs from standard homoclinic snaking by growing the localized state through a hole-like defect in the middle of the wavetrain. As a generic snaking mechanism, DMS is expected to be observed in other pattern-forming PDEs that exhibit bistability between two equilibria interacting with a Turing bifurcation on either.

In Chapter 4 we study the temporal dynamics outside the DMS region and discover a new mechanism of depinning mediated by successive phase slips. This depinning mechanism has been reported in an earlier study of cavity solitons in optical parametric oscillators [159]. Although DMS provides a natural context for depinning by phase slips, this depinning mechanism may be prevalent in any physical system in which the wavetrain forming the localized state is near an Eckhaus instability.

In Chapter 5 we study the interaction between a homoclinic snake and a saddle-center bifurcation of a periodic orbit. The theory formulated in this chapter provides a universal description for the accumulation of folds in DMS and an apparently unrelated bifurcation phenomena referred to as snakes mating. This theory may find applications in other physical systems described by 2D reversible maps such as fluid mixing in 2D. On the other hand, this theory cannot yet explain the connection between the left and right DMS folds into a single snaking curve. This important issue may be addressed by combining the theories on collapsed snaking [110] and the Turing bifurcation [100].

In Chapter 6 we study the bifurcation structures of localized phase-winding states, which are heteroclinic cycles between \( A^- \) and phase-winding states (PWS). In particular we have shown that LPWS are codimension-1 structures and therefore there is no pinning between \( A^- \) and PWS. Although LPWS are temporally unstable in the FCGLE, they may become stable in other physical systems. As mentioned in the end of Chapter 7, LPWS and DMS can convert into each other as control parameters vary; in such cases the phase-
winding states and symmetric periodic states involved both belong to the same invariant torus in the 4D phase space.

In Chapter 7 we outline a classification program of 1D localized states. The classification of uniform states has been greatly facilitated by the use of elementary analytical geometry. This geometric method of analyzing otherwise complicated algebraic expressions may be applicable to other nonlinear problems. The termination of certain branches of localized states has also been studied, though not systematically because such studies usually involve the bifurcation behavior of homoclinic-to-periodic orbits, which are difficult to completely characterize. Many phenomena found in this classification effort, e.g. secondary fronts between $A^\pm$ and nonreversible homoclinic snaking, are not yet well understood and hence have been omitted from this chapter. However these topics will be included in a forthcoming article [130].

In Chapter 8 we study 2D localized states resulting from fronts between $A^\pm$ interacting with a Turing bifurcation on either equilibrium. In particular we have found planar and fully localized hexagons that exhibit very different dynamics from what was known from previous studies. This chapter belongs to the emerging discipline of multi-dimensional localized pattern formation. Most of the recent studies in this field are numerical and require new mathematical techniques to understand.

In Chapter 9 we study localized spatiotemporal chaos (LSTC) resulting from fronts between $A^\pm$ interacting with a supercritical Hopf bifurcation in time. In addition we find complicated LSTC by time evolving steady localized states on the DMS branches. In both cases we have identified a weakly chaotic regime where STC appears to result from a bistability between two coherent structures, a standing wave and a uniform oscillation (in time) for the former, and a standing wave and a spatially periodic state for the latter. These flavors of STC may be fruitfully studied using techniques from the theory of pattern formation. Special techniques from chaos theory such as cycle expansions [69] may be called for if one wishes to compute from first principles certain statistics of STC in the strongly chaotic regime, such as the mean and variance of the velocity of a stochastic front in LSTC.

Finally we mention that spatially localized states, which result from pinning between the bounding front and the periodic pattern, could arise in a broader context than usually considered in the current literature. As an example, pinning between uniform oscillations and steady stripes has been observed in a Brusselator model [72], but the snaking diagram has not been computed.

The 1:1 FCGLE has been originally proposed as “a laboratory to study patterns and defects” [57]. We expect that the new phenomena described in this dissertation, which have been discovered in this particular “laboratory”, can be observed in many other pattern forming systems.
Bibliography


