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Reduced Basis Method for Nanodevices Simulation

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Ballistic transport simulation in nanodevices, which involves self-consistently solving a coupled Schrödinger-Poisson system of equations, is usually computationally intensive. Here, we propose coupling the reduced basis method with the subband decomposition method to improve the overall efficiency of the simulation. By exploiting a posteriori error estimation procedure and greedy sampling algorithm, we are able to design an algorithm where the computational cost is reduced significantly. In addition, the computational cost only grows marginally with the number of grid points in the confined direction.

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I. INTRODUCTION

As size of electronic devices shrinks to nanometer scale, ballistic charge transport becomes increasingly important in describing the transport phenomena in these devices [1]. However, ballistic charge transport simulation is usually computationally intensive — it involves self-consistently solving a coupled Schrödinger – Poisson system of equations [2–4]. Described in greater details in Section II, the iterative procedure involves repetitively solving a Schrödinger equation with open boundary conditions [2] at many different energy states within each iteration. The large number of states required to accurately determine the distribution of the electron density and the number of self-consistent iterations needed to achieve convergence lead to the large computational cost usually associated with ballistic charge transport simulation. A more efficient method to solve the Schrödinger equation can thus greatly improve the overall efficiency of ballistic charge transport simulation. Note that another popular approach to ballistic transport simulation involves solving the non-equilibrium Green’s function equations (NGEF) – Poisson system of equations [5, 6]. In this paper, we will concentrate on the approach based on the Schrödinger equation although the methodology we describe can potentially be applied to the approach based on the NGEF formalism as well.

The finite difference method and the finite element method are most widely used to approximate the Schrödinger equation due to their flexibility [3, 4, 7–9]. However, a direct application of these methods, especially in higher spatial dimensions, can lead to a large algebraic system of equations, of which the solution is computational expensive. The subband decomposition method [10, 11] or more commonly known as the coupled-mode approach [5, 12] attempts to reduce the computational cost by decomposing the Schrödinger equation into two smaller subproblems, resulting in a bounded Schrödinger equation in the confined directions and an open Schrödinger equation in the transport direction. In particular, by first solving the bounded Schrödinger equation at different locations along the transport direction, we are able to obtain a smaller algebraic system of equations for the open Schrödinger equation, which can then be solved more efficiently; the procedure is then effective in the limit where we need to solve the Schrödinger equation at large number of different energy levels. The efficiency can be further improved by a WKB approximation of the open Schrödinger equation [11]. Nevertheless, solving the bounded Schrödinger equation, which involves solving an eigenvalue problem at different locations along the transport direction, can still be potentially ex-

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pensive, especially when strong confinement of the electron demands a finely discretized simulation domain. This paper proposes an efficient method based on the reduced-basis approach to reduce the computational cost of solving the bounded Schrödinger equation.

The reduced basis method is a model-order reduction technique which exploits dimension reduction afforded by the smooth and low-dimensional parametrically induced solution manifold. Instead of using general basis sets such as finite element, an approximation to a solution of an underlying parameterized partial differential equation is obtained by a projection onto a finite and low dimensional vector space spanned by a basis sets consisting of solutions at a number of judiciously selected parameter points. The reduced basis method was first introduced in the late 1970s in the context of nonlinear structural analysis [13, 14] and subsequently abstracted, analyzed, and extended to a much larger class of parameterized partial differential equations [15–19]. In the more recent past the reduced basis approach and in particular associated a posteriori error estimation procedures have been successfully developed for many different types of PDEs that are affine in the parameters [20–26], general nonaffine PDEs [27, 28], and linear eigenvalue problem [21, 29]. We will elaborate further the methodology in Section III. In particular, we extend the methodology described in [29] to eigenvalue problem that is nonaffine in the parameter, and describe how reduced basis methodology can be incorporated into the overall solution procedure for the Schrödinger – Poisson system of equations.

This paper is organized as follows. We first describe the problem that we would like to solve. To simplify the presentation of the methodology, we will use the double-gate MOSFET as a model problem. We then provide the weak formulation of the equations involved and briefly describe the subband decomposition method. This serves as a platform for us to describe the reduced basis methodology, and how it fits into the overall solution procedure. We conclude with some numerical results and comparison to the subband decomposition method.

II. PROBLEM STATEMENT

With the effective mass approximation [30], the electron is described by a wavefunction $\psi(E) \in H^1(\Omega) \subset \mathbb{C}^2$ which for a given $E$, satisfies the following schrödinger equation:

$$-\nabla \cdot \left( \frac{1}{2m^*} \nabla \psi(E) \right) + V_{\text{eff}}(\psi)\psi(E) - E\psi(E) = 0,$$  \hspace{1cm} (1)

with appropriate open boundary conditions [2]. The potential $V_{\text{eff}} \in L^2(\Omega)$ is given by

$$V_{\text{eff}}(\psi) = -\phi(\psi) + V_{\text{xc}}(\psi) + V_b,$$  \hspace{1cm} (2)

where $\phi \in H^1(\Omega)$, $V_{\text{xc}} \in H^1(\Omega)$ and $V_b \in L^2(\Omega)$. We ignore the exchange-correlation term $V_{\text{xc}}$ for simplicity but the methodology described will easily accommodate the $V_{\text{xc}}$ term; and $V_b$ describes the potential gap between the insulator and the semiconductor. The potential $\phi$ in turn satisfies a Poisson equation given by

$$-\nabla \cdot [\epsilon \nabla \phi] = -n(\psi) + N_D,$$  \hspace{1cm} (3)

with appropriate boundary conditions. Here, $\epsilon$ is the dielectric function of the materials, $n(\psi)$ is the density of free electrons, and $N_D$ is the concentration of donor impurities; we ignore contribution of hole and acceptor impurities for simplicity. Equation (1) and (2) are thus coupled through the term $n(\psi)$, which can be defined as

$$n(\psi) = C \int_0^\infty f_{-1/2} \left( \frac{E_f - E}{T} \right) |\psi(E)|^2 dE,$$  \hspace{1cm} (4)

where $C$ is a model-dependent coefficient and $f_{-1/2}(\cdot)$ is the Fermi-Dirac integral of order $-1/2$. Here, $E_f$ is the fermi level of the electrons entering the system.
To solve the above coupled system of equations, an iterative scheme is needed. We utilize a relaxed fixed-point method. Starting from an initial guess $n^0$, we construct the sequence $n^k$ where

$$n^k = \alpha n^{k-1} + (1 - \alpha)n^f,$$

$\alpha$ is an adjustable parameter to improve convergence, and $n^f$ is determined from (4) with $\psi^k$ computed from (1) with $V_{dd} = \phi^k + V_b$. We note that (4) must be evaluated numerically, and thus (1) must be evaluated many times. We then solve (3) for $\phi^{k+1}$ with the new value of $n^k$.

The procedure is repeated until $\|\phi^k - \phi^{k-1}\| \leq \varepsilon_{tol}$, where $\varepsilon_{tol}$ is our desired tolerance. Clearly, even if numerical approximation of (1) is not particularly expensive, the cumulative effects due to the iterative nature of the algorithm and numerical approximation of (4) can be computationally challenging.

For the purpose of this paper, we will consider a 2-dimensional nanodevice (a double-gate MOSFET) shown in Figure II. Given a source potential, $V_S$, a drain potential $V_D$ and a gate potential $V_G$, we would like to determine the current flow $I$ in the $x_1$-direction. The simulation domain $\Omega \equiv [0, a] \times [0, b] \subset \mathbb{R}^2$ can be further divided into 5 subdomains denoted by $\Omega_i$, $i = 1, \ldots, 5$; $(x_1, x_2)$ denotes a point in $\Omega$. The material properties we will be using is that of Si in $\Omega_1$, $\Omega_2$ and $\Omega_3$, and SiO$_2$ in $\Omega_4$ and $\Omega_5$. In addition, $\Omega_2$ and $\Omega_3$ are doped to provide free carriers for the charge transport. We assume the crystal structure of the device is oriented such that $x_1$ is in the $\langle 100 \rangle$ direction and $x_2$ is in the $(001)$ direction. The axes are then aligned with the principal axes of the six equivalent ellipsoids of the conduction band. Based on the effective mass approximation, we then have three configurations for $m^* \equiv (m^*_1, m^*_2, m^*_3)$ and

$$\nabla \cdot \left( \frac{1}{2m^*} \nabla \psi \right) = \frac{\partial}{\partial x_1} \left( \frac{1}{2m^*_1} \frac{\partial}{\partial x_1} \psi \right) + \frac{\partial}{\partial x_2} \left( \frac{1}{2m^*_2} \frac{\partial}{\partial x_2} \psi \right) + \frac{\partial}{\partial x_3} \left( \frac{1}{2m^*_3} \frac{\partial}{\partial x_3} \psi \right).$$

The three configurations of $m^*$ are given by $(m_t, m_t, m_t)$, $(m_t, m_l, m_l)$, $(m_l, m_t, m_l)$; $m_t$ and $m_l$ are the transverse and longitudinal masses of the material. We assume $m_t$ and $m_l$ for Si and SiO$_2$ are the same. Finally, we assume we have a two dimensional electron gas with a parabolic dispersion relation in the $x_3$ direction.

### A. Abstract Formulation

We now derive the weak formulation for (1) and (3) for the model problem described in Section II. For (1), the weak formulation is: given $E \in \mathbb{R}$, find $\psi \in Y \equiv H^1(\Omega)$ such that
\[
\frac{1}{2m_1^2} \int_\Omega \frac{\partial \psi}{\partial x_1} \frac{\partial v^*}{\partial x_1} + \frac{1}{2m_2^2} \int_\Omega \frac{\partial \psi}{\partial x_2} \frac{\partial v^*}{\partial x_2} + \int_\Omega \psi \nabla \nabla v^* - E \int_\Omega \psi v^* = \frac{1}{2m_1^2} \int_{\Gamma_{S,D}} \frac{\partial \psi}{\partial x_1} v^*, \quad \forall v \in Y.
\]

(5)

where \(\Gamma_S\) and \(\Gamma_D\) are respectively the boundaries in contact with source and drain electrodes. Based on the quantum transmitting boundary method [2], we expand the R.H.S of (5): for \(g = S, D\),

\[
\int_{\Gamma_g} \frac{\partial \psi}{\partial x_1} v^* = \sum_{m=1}^{N_g} \sum_{i=1}^\infty 2a_m^g k_m^g \int_{\Gamma_g} \chi_m^g v^* + \int_{\Gamma_g} \chi_m^g v^* \int_{\Gamma_g} \chi_m^g \psi
\]

(6)

where \((\xi_m^g, E_m^g), 1 \leq m \leq \infty\) are the eigenstates along \(\Gamma_g\); \(k_m^g = \sqrt{2m^*(E - E_m^g)}\); \(N_g\) is the largest \(m\) for which \(E > E_m^g\), \(b_m^g, 1 \leq m \leq N_g\) are the coefficients of outgoing traveling-wave states, and \(a_m^g\) is a parameter that we can vary while \(b_m^g\) and \(N_g\) is determined as part of the solution.

To facilitate the variational formulation, We now define the following functional forms: \(\forall w \in Y, v \in Y, v \in L^2, \chi^g \in H_0^1(\mathbb{R})\),

\[
a_0(w; v; \alpha) = \int_\Omega \alpha \nabla w \nabla v^*,
\]

(7)

\[
a_1(w; v; m^*) = \frac{1}{2m_1^2} \int_\Omega \frac{\partial w}{\partial x_1} \frac{\partial v^*}{\partial x_1} + \frac{1}{2m_2^2} \int_\Omega \frac{\partial w}{\partial x_2} \frac{\partial v^*}{\partial x_2},
\]

(8)

\[
a_2(w; v; V) = \int_\Omega V v^*,
\]

(9)

\[
a_3(w; v) = \int_\Omega v^*,
\]

(10)

\[
c(w; v; \chi^g) = \frac{1}{2m_1^2} \int_{\Gamma_g} \chi^g w \int_{\Gamma_g} \chi^g v^*,
\]

(11)

\[
b(v; \chi^g) = \frac{1}{2m_1^2} \int_{\Gamma_g} \chi^g v^*.
\]

(12)

The abstract formulation is then: given \(E \in \mathbb{R}\), find \(\psi \in Y\) that satisfies

\[
-a_1(\psi; v; m^*) + a_2(\psi; v; V_{\text{eff}}) - E a_3(\psi, v)
\]

\[
- \sum_{g=S,D} \sum_{m=1}^{N_g} ik_m^g c(\psi, v; \chi_m^g) + \sum_{g=S,D} \sum_{m=N_g+1}^\infty k_m^g c(\psi, v; \chi_m^g)
\]

\[
= - \sum_{g=S,D} \sum_{m=1}^{N_g} i 2a_m^g k_m^g b(v; \chi_m^g), \quad \forall v \in Y.
\]

(13)

For (3), the weak formulation is: given \(n(\psi)\), the solution \(\phi \in H^1(\Omega)\) is given by

\[
\int_{\Omega/\Gamma_0} \epsilon \nabla \phi \nabla v^* + \int_{\Gamma_0} \epsilon \nabla V_G \nabla v^* = \int_{\Omega} (-n(\psi) + N_D) v^*, \quad \forall v \in Y,
\]
where $\Gamma_0$ is the boundary in contact with gate electrode. We have imposed the following boundary conditions:

$$\phi|_{\Gamma_0} = 0, \quad \text{and} \quad \frac{d\phi}{dx_1}\bigg|_{\Gamma_S \cup \Gamma_D} = 0.$$  

Let $f(v; V) = \int_{\Gamma_0} Vv^*$, and $h(v; V) = \int_{\Gamma_0} eV\nabla v^*$. Then, the abstract formulation is: given $n(\psi)$, the solution $\phi \in Y$ is given by

$$a_0(\phi, v; \epsilon) = f(v; -n(\psi) + N_D) - h(v; V_G), \quad \forall v \in Y. \quad (14)$$

For the current problem where we have assumed a 2-dimensional electron gas, the charge density $n$ is given by [6]

$$n(\psi) = \frac{2}{\pi} \sqrt{\frac{2m^* T}{\pi}} \int_0^\infty f_{-1/2} \left( \frac{E_f - E}{T} \right) |\psi(E)|^2 dE, \quad (15)$$

and this is sum over the three different configuration of $m^*$. We assume $E_f = 0$ at zero bias. Finally, the current intensity $I$ is given by

$$I = \int_0^b j_1(x_1, x_2; \psi)dx_2 \quad (16)$$

where $j_1$, the current density in the $x_1$ direction, is defined as

$$j_1(\psi) = \frac{1}{m^*_1} \frac{2}{\pi} \sqrt{\frac{2m^* T}{\pi}} \int_0^\infty \text{Im} \left( \psi(E) \frac{\partial \psi(E)}{\partial x_1} \right) f_{-1/2} \left( \frac{E_f - E}{T} \right) dE. \quad (17)$$

Numerical approximation of (13) – (14) based on, say, finite element method, can however be computationally very expensive since (13) must be solved many times in a single iteration in order to numerically determine the density $n$. In particular, suppose we substitute the unbounded upper limit in (15) by $E_{\text{max}}$ and subdivide the interval $[0, E_{\text{max}}]$ into $n_E$ intervals. We then use Gauss quadrature formulation within each interval to arrive at the following approximation of $n$:

$$n(\psi) \approx \frac{2}{\pi} \sqrt{\frac{2m^* T}{\pi}} \sum_{g=\epsilon, \epsilon'} \sum_{i=1}^{N_q} \sum_{j=1}^{n_E} \sum_{i=1}^Q f_{-1/2} \left( \frac{E_f - E_i^g}{T} \right) |\psi(E_i^g, a_{m'}^g)|^2 w_q \quad (18)$$

where $N_q$ is the number of modes considered at $\Gamma_g$; $a_{m'}^g = 1$ if $m' = m$ and $g' = g$, and 0 otherwise; $E_i^g$ are the quadrature points in interval $i$; $w_q$ is the quadrature weight; and $Q$ is the number of quadrature points used per interval. Then, in each iteration, the maximum number of times we must solve (13) is $(N_q^* + N_D)n_1Q$. This can be somewhat smaller by excluding $E$ for which $f_{-1/2}((E_f - E)/T)$ is negligibly small.

B. Subband Decomposition Approach

The subband decomposition method is first described in [5, 10]. Assuming that the wavefunction is bounded in the $x_2$-direction, we can write $Y$ as $X^1 \times X^2$ where $X^1 = H^1(\Omega^1 \equiv [0,a])$ and $X^2 = H^1_0(\Omega^2 \equiv [0,b])$. Then, we can express $\psi \in Y$ as

$$\psi(x; E) = \sum_{i=1}^\infty \varphi(x_1; E)\varphi_1(x_2; x_1), \quad \varphi(x_1; E) \in X^1, \quad \varphi_1(x_2; x_1) \in X^2. \quad (19)$$
Here, \( \xi_i(\cdot; \mu \equiv x_1) \in X^2 \), \( i = 1, \ldots, \infty \) are solutions to the following eigenvalue problem:

\[
\tilde{a}_1(\xi_i(\mu), v; m_i^2) + \tilde{a}_2(\xi_i(\mu), v; V_{\text{eff}}(\mu)) = \lambda_i(\mu)\tilde{a}_3(\xi_i(\mu), v),
\]

\[
1 \leq i \leq \infty, \quad \forall v \in X^2,
\]

\[
\tilde{a}_3(\xi_i(\mu), \xi_j(\mu)) = \delta_{ij}, \quad 1 \leq i, j \leq \infty;
\]

where \( V_{\text{eff}}(\mu) = V_{\text{eff}}(x_2; \mu \equiv x_1) \); and

\[
\tilde{a}_1(w, v; \alpha) = \int_{\Omega} \frac{1}{2\alpha} \nabla w \nabla v, \quad \tilde{a}_2(w, v; t) = \int_{\Omega^2} x t v, \quad \tilde{a}_3(w, v) = \int_{\Omega^2} w v,
\]

for \( w \in X^2 \), \( v \in X^2 \) and \( t \in L^2(\Omega^2) \).

Substituting (19) into (13), we obtain a one dimensional problem for \( \varphi_i(E) \):

\[
\sum_{i=1}^{\infty} \frac{1}{2m_i^1} \left\{ \int_{\Omega^1} \frac{d\varphi_i(E)}{dx_1} \frac{dt}{dx_1} \tilde{a}_3(\xi(x_1), \xi_j(x_1)) + \int_{\Omega^1} \frac{d\varphi_i(E)}{dx_1} t(x_1) \tilde{a}_3(\xi(x_1), \frac{\partial \xi_j}{\partial x_1}(x_1)) \\
+ \int_{\Omega^1} \varphi_i(x_1; E) \frac{dt(x_1)}{dx_1} \tilde{a}_3(\frac{\partial \xi_i}{\partial x_1}(x_1), \frac{\partial \xi_j}{\partial x_1}(x_1)) \\
+ \int_{\Omega^1} \varphi_i(x_1; E) t(x_1) \tilde{a}_3(\frac{\partial \xi_i}{\partial x_1}(x_1), \frac{\partial \xi_i}{\partial x_1}(x_1)) \right\}
\]

\[
+ \int_{\Omega^1} (\lambda_i(x_1) - E) \varphi_i(x_1; E) t(x_1) \delta_{ij} - \sum_{g=S,D} \sum_{m=1}^{N_g} \sum_{i,j} \delta_{mi} \frac{k_m^g}{2m_i^1} \frac{\varphi_i(x_g) t(x_g)}{2m_i^1} \delta_{mj}
\]

\[
= - \sum_{g=S,D} \sum_{m=1}^{N_g} 2a_m^g k_m^g \frac{t(x_g)}{2m_i^1} \delta_{mj}, \quad \forall t \in X^1, \quad 1 \leq j \leq \infty.
\]

This is simply the weak form for the following one-dimensional Schrödinger equation [10]:

\[
-\frac{d}{dx_1} \left( \frac{1}{2m_i^1} \frac{d}{dx_1} \varphi_i \right) - \sum_{j=1}^{\infty} \frac{a_{ij}(x_1)}{m_i^1} \frac{d}{dx_1} \varphi_j - \sum_{j=1}^{\infty} \frac{b_{ij}(x_1)}{2m_i^1} - \lambda_i \delta_{ij} + E \right) \varphi_j = 0,
\]

for \( i = 1, \ldots, \infty \) with the appropriate open boundary condition; \( a_{ij}(x_1) = \int_{\Omega^2} \xi_i(x_1) \{ \partial \xi_j(x_1)/\partial x_1 \} \) and \( a_{ij}(x_1) = \int_{\Omega^2} \xi_i(x_1) \{ \partial^2 \xi_j(x_1)/\partial x_1^2 \} \). It is further found that only finite number of \( \xi_i \) is needed, which we denote as \( n_c \). If these \( n_c \) \( \xi_i(x_1) \) are known, this one-dimensional problem can be solved very efficiently.

To achieve self-consistency, each fixed point iteration now consists of three parts: (i) determination of the subbands \( \xi_i(x_2; x_1), 1 \leq i \leq n_c \), for finite points in \( \Omega^1 \), (ii) determination of \( n(\psi) \) by solving (23) for \( (N^S + N^D)n_{xQ} \) different combination of \( E \) and \( a_m^g \), and (iii) determination of \( \phi(\psi) \) by solving (14) given \( n(\psi) \). In [10], finite element method is used to approximate the solutions at all stages of the algorithm. It is hoped that the computational overhead incurred in part (i) will significantly reduce the computational cost of solving the open Schrödinger equation needed to determine the electron density. However, part (i) can be computationally expensive, especially if very fine mesh is needed to resolve the strong confinement of the electrons in the \( x_2 \)-direction or when (20) must be solve at large number of points if finer mesh is needed in the \( x_1 \)-direction. Our goal is to speed-up the determination of \( \xi_i \) for any given \( x_1 \) through the reduced basis method.

Before we proceed, we describe how we can determine \( \partial \xi_i/\partial \mu \). Let \( \partial \xi_i/\partial \mu \in X^2 \). Then, by taking
the derivative of (20) with respect to $\mu$, we obtain
\[
\tilde{a}_1(\frac{\partial \xi_i}{\partial \mu}(:,\mu), v; m^*_2) + \tilde{a}_2(\frac{\partial \xi_i}{\partial \mu}(:,\mu), v; V_{\text{eff}}(:,\mu)) - \lambda_i(\mu)\tilde{a}_3(\frac{\partial \xi_i}{\partial \mu}(:,\mu), v) \\
= - \tilde{a}_2(\xi_i(:,\mu), v; \frac{\partial V_{\text{eff}}(:,\mu)}{\partial \mu}) + \frac{d\lambda_i(\mu)}{d\mu}\tilde{a}_3(\xi_i(:,\mu), v), \\
1 \leq i \leq \infty, \quad \forall v \in X^2.
\] (25)

In addition, by letting $v = \xi_i$, and invoking (20), we obtain
\[
\frac{d\lambda_i(\mu)}{d\mu} = \tilde{a}_2(\xi_i(:,\mu), \xi_i(:,\mu); \frac{\partial V_{\text{eff}}(:,\mu)}{\partial \mu}).
\] (26)

Finally, by substituting (26) into (25), we can solve for $\partial \xi_i/\partial \mu$. At present $\partial V_{\text{eff}}/\partial \mu$ is computed using a difference formula. In Appendix A, we describe a formulation that is more consistent with the finite element approximation space of $\phi$; it however leads to a higher computational cost. We also note that since $V_b$ does not depend on $x_1$, $\partial V_{\text{eff}}/\partial \mu = -\partial \phi/\partial \mu$.

III. REDUCED BASIS METHOD

We now present the reduced basis formulation for (20). Figure 2 shows the variation of $\xi_1$ and $\xi_8$ for several different values of $x_1$. The variation is small but nontrivial, and we can discern a smooth variation of $\xi_i$ with respect to $x_1$. This suggests that reduced basis method can very efficiently approximate $\xi_i$. This section describes how $\xi_i$ and $\partial \xi_i/\partial x_1$ can be approximated by the reduced basis method. Let $\mu \equiv x_1$, and $\mathcal{D} \equiv \Omega^1$. In addition, for notational convenience, we have $\xi_i(\mu) = \xi_i(x_2;\mu)$, $\phi(\mu) = \phi(x_2;\mu)$, $d\xi_i(\mu) = \partial \xi_i(x_2;\mu)/\partial \mu$ and $d\phi(\mu) = \partial \phi(x_2;\mu)/\partial \mu$.
A. Approximation Spaces

We first introduce nested sample sets $S_N = (\mu_1, \ldots, \mu_{N_s})$, $1 \leq N_s \leq N_{s,\text{max}}$ and define the associated nested reduced-basis spaces as

$$W_N = \text{span} \{ \xi_j(\mu), 1 \leq i \leq n_e, 1 \leq j \leq N_s \}, \quad 1 \leq N_s \leq N_{s,\text{max}},$$

where $\xi_j(\mu)$ are the solutions of (20) at $\mu = \mu_j$; and $\zeta_n$ are basis functions obtained after $\xi_j(\mu_j)$, $1 \leq i \leq n_e$, $1 \leq j \leq N_s$ are orthonormalized.

We also construct collateral approximation spaces for $\phi(\mu)$ and $d\phi(\mu)$ based on the empirical interpolation procedure [27, 28, 31]. For $p = \phi(\mu)$ and $d\phi(\mu)$, we construct nested sample sets $S_M^p \equiv \{ \mu_1^p, \ldots, \mu_M^p \}$, $1 \leq M \leq M_{\text{max}}^p$, nested approximation spaces $W_M^p \equiv \text{span} \{ q_1^p, \ldots, q_M^p \}$, $1 \leq M \leq M_{\text{max}}^p$, and nested interpolation points $T_M^p \equiv \{ t_1^p, \ldots, t_M^p \}$, $1 \leq M \leq M_{\text{max}}^p$.

In (27), we have assumed $\xi_j(\mu_j)$ are known exactly. In practice however, $\xi_j(\mu_j)$ must be determined through some form of “truth” approximation — here, we use the finite element method with $P_1$ elements. We build our reduced basis approximation on, and measure the error in the reduced basis approximation relative to this “truth” approximation. Note that since reduced basis approximation is built upon this “truth” approximation, it cannot perform better than this “truth” approximation. Thus, the number of elements used to obtain our “truth” approximation, $N$, must usually be large. Similarly, the $W_M^\phi$ and $W_M^{d\phi}$ are constructed from a “truth” approximation of $\phi$ and $d\phi$, here based on finite element method utilizing $Q_2$ elements.

B. The Approximation

Our reduced basis approximation to (20) is then given by: find $(\xi_{i,N,M}(\mu), \lambda_{i,N,M}(\mu)) \in \mathcal{Y}_N \equiv (W_N \times \mathbb{R})$, $1 \leq i \leq n_e$ such that

$$\tilde{a}_1(\xi_{i,N,M}(\mu), v; m_2^*) + \tilde{a}_2(\xi_{i,N,M}(\mu), v; V_{\text{eff},M}(\mu)) = \lambda_{i,N,M}(\mu) \tilde{a}_3(\xi_{i,N,M}(\mu), v),$$

$$1 \leq i \leq n_e, \quad \forall v \in W_N,$$

$$\tilde{a}_3(\xi_{i,N,M}(\mu), \xi_{j,N,M}(\mu)) = \delta_{ij}, \quad 1 \leq i, j \leq n_e,$$ (28)

where $V_{\text{eff},M} = V_b + \phi_M$.

Similarly, our reduced basis approximation to (25) is given by: find $d\xi_{i,N,M}(\mu) \in W_N$, $1 \leq i \leq n_e$ such that

$$\tilde{a}_1(d\xi_{i,N,M}(\mu), v; m_2^*) + \tilde{a}_2(d\xi_{i,N,M}(\mu), v; V_{\text{eff},M}(\mu)) - \lambda_{i,N,M}(\mu) \tilde{a}_3(d\xi_{i,N,M}(\mu), v)$$

$$= \tilde{a}_2(\xi_{i,N,M}(\mu), v; d\phi_{N,M}(\mu)) + \frac{d\lambda_{i,N,M}(\mu)}{d\mu} \tilde{a}_3(\xi_{i,N,M}(\mu), v),$$

$$1 \leq i \leq n_e, \quad \forall v \in W_N,$$ (30)

where

$$\frac{d\lambda_{i,N,M}(\mu)}{d\mu} = \tilde{a}_3(\xi_{i,N,M}(\mu), \xi_{i,N,M}(\mu); \phi_M(\mu)).$$ (31)

It is not immediately clear that $d\xi_{i,N,M}(\mu)$ can be sufficiently approximated in $W_N$. In Section IV B, we will examine if it is necessary to replace $W_N$ by an enlarged space $W_N'$ given by

$$W_N' = \text{span} \{ \xi(\mu_j), \ldots, \xi_{n_e}(\mu_j), d\xi(\mu_j), \ldots, d\xi_{n_e}(\mu_j), 1 \leq j \leq N_s \},$$

$$= \text{span} \{ \xi_n, 1 \leq n \leq N \equiv 2N_s n_e \}.$$ (32)
C. Offline-online Decomposition

We first expand our reduced basis approximation as

$$\xi_{n,N,M}(\mu) = \sum_{j=1}^{N} \xi_{n,N,M,j}(\mu) \zeta_j, \quad 1 \leq n \leq n_e,$$

(33)

where $\zeta_j \in W_N$, and $\xi_{n,N,M,j}(\mu) \in \mathbb{R}$.

We then expand our empirical interpolation approximation for $\phi(\cdot; \mu)$ as

$$\phi_M(\cdot; \mu) = \sum_{m=1}^{M} \beta_{M,m}(\mu) q_m(\cdot),$$

(34)

where $\beta_M(\mu) \in \mathbb{R}^M$ is given by

$$\sum_{k=1}^{M} B_{M,\phi}^{m,k} \beta_{M,k}(\mu) = \phi(t_{m}^\phi; \mu), \quad 1 \leq m \leq M;$$

(35)

and $B_{M,\phi} \in \mathbb{R}^{M \times M}$ is given by $B_{M,\phi}^{m,k} = q_{m}^\phi(t_{k}^\phi), 1 \leq m, k \leq M$. We note that $\{q_{m}^\phi, 1 \leq m \leq M\}$ is pre-constructed offline based on the empirical interpolation method, and we compute $\beta_M(\mu)$ given any $\mu \in D$. Inserting the above representations (33) and (34) into (28), we obtain the following discrete equations

$$\sum_{j=1}^{N} \left\{ A_{i,j} + \sum_{m=1}^{M} C_{N,\phi,m}^{N,i,m} \beta_{M,m}(\mu) \right\} \xi_{n,N,M,j}(\mu) =$$

$$\lambda_{n,N,M}(\mu) \sum_{j=1}^{N} M_{i,j}^{N,i} \xi_{n,N,M,j}(\mu), \quad 1 \leq i \leq N, \quad 1 \leq n \leq n_e;$$

(36)

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \xi_{n,N,M,i}(\mu) M_{i,j}^{N} \xi_{m,N,M,j}(\mu) = \delta_{nm}, \quad 1 \leq n, m \leq n_e;$$

(37)

where $A^{N} \in \mathbb{R}^{N \times N}, M^{N} \in \mathbb{R}^{N \times N}, C_{N,\phi,m}^{N,i,m} \in \mathbb{R}^{N \times N}, 1 \leq m \leq M$ are given by $A_{i,j}^{N} = \tilde{a}_1(\zeta_j, \zeta_i; m_1^2) + \tilde{a}_2(\zeta_j, \zeta_i; V_b), 1 \leq i, j \leq N, M_{i,j}^{N} = \tilde{a}_3(\zeta_j, \zeta_i), 1 \leq i, j \leq N$, and $C_{i,j}^{N,\phi,m} = \tilde{a}_2(\zeta_j, \zeta_i; q_{m}^\phi), 1 \leq i, j \leq N, 1 \leq m \leq M$ respectively.

Similarly, for (30), we expand

$$d\xi_{n,N,M}(\mu) = \sum_{j=1}^{N} d\xi_{n,N,M,j}(\mu) \zeta_j,$$

(38)

where $\zeta_j \in W_N$, and $d\xi_{n,N,M,j}(\mu) \in \mathbb{R}$. We also expand our empirical interpolation approximation for $d\phi(\mu)$ as

$$d\phi_M(\cdot; \mu) = \sum_{m=1}^{M^{d\phi}} \gamma_{M,m}(\mu) q_m^{d\phi}(\cdot),$$

(39)
where $\gamma_M(\mu) \in \mathbb{R}^M$ is given by
\[
\sum_{k=1}^{M} B_{m,k}^{M,\phi} \gamma_M k(\mu) = d\phi(t_m^{\phi}; \mu), \quad 1 \leq m \leq M; \tag{40}
\]
and $B_{m,k}^{M,\phi} \in \mathbb{R}^{M,\phi} \times \mathbb{R}^{M,\phi}$ is given by $B_{m,k}^{M,\phi} = q_m^{\phi}(t_k^{\phi})$, $1 \leq m, k \leq M$. Inserting the above representations (33), (38) and (39) into (30), we obtain the following discrete equations
\[
\begin{align*}
\sum_{j=1}^{N} \left\{ A_{i,j}^{N} + \left( \sum_{m=1}^{M^{\phi}} C_{m,N,\phi,\mu}^{N,\phi,\mu} \right) - \lambda_n, N, M(\mu) M_{i,j}^{N} \right\} \, d\xi_{n,N,M} j(\mu) = \\
\sum_{j=1}^{N} \left\{ \frac{d\lambda_{i,N,M}(\mu)}{d\mu} M_{i,j}^{N} - \left( \sum_{m=1}^{M^{\phi}} C_{m,N,\phi,\mu}^{N,\phi,\mu} \right) \right\} \xi_{n,N,M} j(\mu),
\end{align*}
\]
$1 \leq i \leq N$; \tag{41}

where $C_{m,N,\phi,\mu}^{N,\phi,\mu} \in \mathbb{R}^{N \times N}$, $1 \leq m \leq M^{\phi}$ is given by $C_{i,j}^{N,\phi,\mu} = \delta_{2}(\zeta_{i}, \zeta_{j}; q_{m}^{\phi})$, $1 \leq i, j \leq N$, $1 \leq m \leq M^{\phi}$, respectively.

Finally, the linear functional $\delta_{3}$ is simply approximated by
\[
\delta_{3}(w_{n}(\mu), v_{m}(\mu)) \approx \delta_{3}(w_{n,N,M}(\mu), v_{m,N,M}(\mu)) = \sum_{i=1}^{N} \sum_{j=1}^{N} M_{i,j}^{N} w_{n,N,M}(\mu) v_{m,N,M}(\mu) \tag{42}
\]
when we want to compute the $\delta_{3}$ terms in (23).

The computational decomposition is then clear. At the beginning of each inner iteration, we generate nested reduced-basis spaces $W_{N}$, $1 \leq N \leq N_{\text{max}}$, nested approximation spaces $W_{M}^{\phi}$, $1 \leq M \leq M_{\text{max}}^{\phi}$, and the associated nested sets of interpolation points $T_{M}^{\phi}$ and $T_{M}^{\phi}$. For determining $\xi_{n,N,M}(\mu)$, $1 \leq i \leq n_{e}$, we form and store $A_{i,j}^{N}$, $M_{i,j}^{N}$, $B_{i,j}^{M,\phi}$, $C_{i,j}^{N,\phi,\mu}$, $1 \leq m \leq M_{\text{max}}^{\phi}$ and $C_{i,j}^{N,\phi,\mu}$, $1 \leq m \leq M_{\text{max}}^{\phi}$. This is equivalent to the offline stage in a more typical reduced-basis formulation. The computational cost is (to leading order) $O(N N^{2} + n_{e} N^{3} + M^{2} N^{2})$, where $\bullet$ and $\dagger$ depend on the complexity of the eigenvalue solver and linear solver used, $M = \text{max}(M^{\phi}, M^{\phi})$, and $N$ is the dimension of our “truth” approximation.

In the online stage — during construction of discrete matrices for (23) — we solve (36) – (37) for $\xi_{n,N,M,j}(\mu)$, $1 \leq j \leq N$, $1 \leq n \leq n_{e}$, and $d\xi_{n,N,M,j}(\mu), 1 \leq j \leq N^{d_{\phi}}, 1 \leq n \leq n_{e}$, and evaluate (42). The computational costs for each $\mu$ is then $O((n_{e} N)^{3} + n_{e} N^{3} + M^{2} N^{2})$, which is then independent of $N$.

D. A Posteriori Error Estimation

The derivation of the a posteriori error estimator follows [29]. We note that the eigenvalues $\lambda_{i}$ are of multiplicity one but $\tilde{a}(v, v; V_{\text{eff}}(\mu)) = \tilde{a}_{1}(v, v; m_{i}^{2}) + \tilde{a}_{2}(v, v; V_{\text{eff}}(\mu))$ is not strictly positive for all $\mu \in D$.

For $i = 1, \ldots, n_{b}$, we define the residual as
\[
R_{i}(v; \mu) = \tilde{a}(\xi_{i,N,M}(\mu), v; V_{\text{eff}}(\mu)) - \lambda_{i,N,M}(\mu) \tilde{a}_{3}(\xi_{i,N,M}(\mu), v), \tag{43}
\]
for $\forall v \in Y$ where $\tilde{a}(w, v; V_{\text{eff}}(\mu)) = \tilde{a}_{1}(w, v) + \tilde{a}_{2}(w, v; V_{\text{eff}}(\mu))$. We also define a reconstructed error
\[ \hat{a}(\hat{e}_i, v) = R_i(v; \mu), \quad \forall v \in Y, \]  

where

\[ \hat{a}(w, v) = \hat{a}_1(w, v; m^*_2) + \hat{a}_2(w, v; V_b) + (\gamma + \max_{\mu \in \mathcal{D}, x_2 \in \Omega^2} \phi(x_2; \mu))\hat{a}_3(w, v); \]  

\[ \gamma = \left| \min_{\mu \in \mathcal{D}, x_2 \in \Omega^2} \phi(x_2; \mu) \right|; \]  

\[ \| R_i(\cdot; \mu) \| = \sup_{v \in Y} \frac{R_i(v; \mu)}{\hat{a}(v, v)^{1/2}} = \hat{a}(\hat{e}_i, \hat{e}_i)^{1/2}; \]  

and \( \| \cdot \| = \hat{a}(\cdot, \cdot)^{1/2}. \)

We now define \( \hat{a}^+(w, v; \text{eff}(\mu)) = \hat{a}(w, v; \text{eff}(\mu)) + \gamma \hat{a}_3(w, v) \) and introduce the following eigenvalue problem: for \( \mu \in \mathcal{D} \), find \((\xi^+_i(\mu), \lambda^+_i(\mu)) \in Y \times \mathbb{R}, 1 \leq i \leq n_e \) such that

\[ \hat{a}^+(\xi^+_i(\mu), v; \text{eff}(\mu)) = \lambda^+_i(\mu)\hat{a}_3(\xi^+_i(\mu), v), \quad \forall v \in Y, \quad 1 \leq i \leq n_e, \]  

\[ \hat{a}_3(\xi^+_i(\mu), \xi^+_j(\mu)) = \delta_{ij}, \quad 1 \leq i, j \leq n_b. \]  

It is clear that \( \xi^+_i(\mu) = \xi_i(\mu) \) and \( \lambda^+_i = \lambda_i + \gamma. \)

\textbf{Proposition 1.} Given \( \hat{a}(w, v) \) as defined in (45), we have

\[ \hat{a}(v, v) \geq \hat{a}^+(v, v; \text{eff}(\mu)) \geq \hat{a}_3(v, v) \geq 0, \]  

for all \( \mu \in \mathcal{D}. \)

\textbf{Proof.} We first prove left inequality. Let \( f(\cdot) = \max_{\mu \in \mathcal{D}, x_2 \in \Omega^2} \phi(x_2; \mu). \) By expanding \( \hat{a}^+ \), we obtain

\[ \hat{a}^+(v, v; \text{eff}(\mu)) = \hat{a}_1(v, v; m^*_2) + \hat{a}_2(v, v; V_b) + \hat{a}_3(v, v; f) + \gamma \hat{a}_3(v, v), \]  

since \( \hat{a}_1(v, v; m^*_2) \geq 0, \hat{a}_2(v, v; V_b) \geq 0 \) as \( V_b \geq 0, \hat{a}_3(v, v) \geq 0, \) and \( \gamma \geq 0. \) Since the R.H.S of (51) is equivalent to \( \hat{a}(v, v) \), left inequality is proven.

To prove the right inequality, we first note that

\[ \hat{a}(v, v; \text{eff}(\mu)) \geq \lambda_1(\mu)\hat{a}_3(v, v), \]  

and \( \lambda_1(\mu) \geq \min_{x_2 \in \Omega^2} \{ V_b(x_2) + \phi(x_2; \mu) \}. \) Then,

\[ \hat{a}^+(v, v; \text{eff}(\mu)) = \hat{a}(v, v; \text{eff}(\mu)) + \min_{\mu \in \mathcal{D}, x_2 \in \Omega^2} \phi(x_2; \mu)\hat{a}_3(v, v) \]  

\[ \geq (\min_{\mu \in \mathcal{D}} \lambda_1(\mu) + \min_{\mu \in \mathcal{D}, x_2 \in \Omega^2} \phi(x_2; \mu))\hat{a}_3(v, v) \]  

\[ \geq \hat{a}_3(v, v), \]  

since \( \min_{x_2 \in \Omega^2} V_b(x_2) = 0. \) This concludes the proof for Proposition 1. \qed

\textbf{Hypothesis 1.} Assuming our reduced-basis approximation is convergent in the sense that

\[ \lambda_{i,N,M}(\mu) \rightarrow \lambda_i(\mu), \quad 1 \leq i \leq n_e, \quad \text{as} \quad N \rightarrow \infty. \]
Then, for sufficiently large $N$,

$$i = \arg\min_{1 \leq j \leq N} \left| \frac{\lambda_{i,N,M}(\mu) - \lambda_{j}(\mu)}{\lambda_j^+ (\mu)} \right|.$$ 

(54)

**Proposition 2.** Assume our reduced-basis approximation is convergent in the sense given by (53). Then, for large $N$ and $i = 1, \ldots, n_e$,

$$\left| \frac{\lambda_{i,N,M}(\mu) - \lambda_i(\mu)}{\lambda_i(\mu) + \gamma} \right| \leq \frac{\|R_i(\cdot;\mu)\|}{(\lambda_{i,N,M}(\mu) + \gamma)^{1/2}}$$

(55)

In addition, for $\lambda_{i,N,M}(\mu)$ of multiplicity one and associated $u_{N,i}(\mu)$, we have

$$\|u_{i,N,M}(\mu) - u_i(\mu)\| \leq \frac{\|R_i(\cdot;\mu)\|}{d_i},$$

(56)

and

$$|\lambda_{i,N,M}(\mu) - \lambda_i(\mu)| \leq \frac{\|R_i(\cdot;\mu)\|^2}{d_i^2},$$

(57)

where $d_i = \min_{j \neq i} \left| \frac{\lambda_{i,N,M}(\mu) - \lambda_{j,N,M}(\mu)}{\lambda_{j,N,M}(\mu) + \gamma} \right|$.

**Proof.** The proof utilizes Proposition 1 and details of the proof can be found in [29].

We note that (57) will in general be a better bound due to the $\|R_i\|^2$ term. Numerical experiments also indicate this is so. We thus define our error estimators based on (56) and (57):

$$\Delta^\lambda_{N,M}(\mu) = \max_{1 \leq i \leq n_e} \frac{1}{d_i^2} \frac{\|R_i(\cdot;\mu)\|^2}{|\lambda_{i,N,M}(\mu)|},$$

(58)

$$\Delta^\xi_{N,M}(\mu) = \max_{1 \leq i \leq n_e} \frac{1}{d_i} \frac{\|R_i(\cdot;\mu)\|}{|\xi_{i,N,M}(\mu)|}.$$ 

(59)

We can construct efficient offline-online computational strategies for the evaluation of our error estimators (58) – (59). From (45) and our reduced basis approximation, we have

$$\hat{a}(\hat{e}, v) = \tilde{a}_1(\xi_{i,N,M}(\mu), v; m_{\xi}) + \tilde{a}_2(\xi_{i,N,M}(\mu), v; \Phi) + \sum_{m=1}^{M^\phi} \beta_m(\mu) \tilde{a}_2(\xi_{i,N,M}(\mu), v; q_{m})$$

$$+ \bar{\varepsilon}_M + \tilde{a}_3(\xi_{i,N,M}(\mu), v), \quad v \in \mathbb{V}, \quad 1 \leq i \leq n_e.$$ 

(60)

where $\varepsilon_M = \max_{\mu \in D} \hat{\varepsilon}_M(\mu)$ and $\hat{\varepsilon}_M(\mu) = |\phi(t_{M+1}^\phi; \mu) - \phi_M(t_{M+1}^\phi; \mu)|$. It then follows from linear superposition that

$$\hat{e}_i(\mu) = \sum_{n=1}^{N} \xi_{i,N,M}(\mu) \left\{ p_n^{1+} + p_n^{2+} + \sum_{m=1}^{M^\phi} \beta_mp_n^{2+m} + \bar{\varepsilon}_M p_n^{M+3} \right\}$$

$$- \lambda_{i,N,M}(\mu) \sum_{n=1}^{N} \xi_{i,N,M}(\mu)p_n^0,$$ 

(61)
where

\[
\hat{a}(p^1_n, v) = a_1(\zeta_n, v; m_2^o), \quad v \in Y, \quad 1 \leq n \leq N, \\
\hat{a}(p^2_n, v) = a_2(\zeta_n, v; V_0^e), \quad v \in Y, \quad 1 \leq n \leq N, \\
\hat{a}(p^{2+m}_n, v) = a_2(\zeta_n, v; q^{o}_m), \quad v \in Y, \quad 1 \leq m \leq M^o + 1, \\
\hat{a}(p^{0}_n, v) = a_3(\zeta_n, v), \quad v \in Y, \quad 1 \leq n \leq N.
\]

Then, \(\|R_i(\cdot ; \mu)\|\) is given by

\[
\|R_i(\cdot ; \mu)\|^2 = \hat{a}(\hat{e}_i, \hat{e}_i) = \sum_{k=1}^{3+M^o} \sum_{k'=0} \sum_{n=1}^N \sum_{n'=1}^N \Theta_k(\mu)\Theta_{k'}(\mu)\xi_{i,n,M} n(\mu)\xi_{i,N,n'} n'(\mu)\hat{A}_{n,n'}^{k,k'},
\]

\[
+ \sum_{n=1}^N \sum_{n'=1}^N \lambda^2_{i,n,M}(\mu)\xi_{i,n,M} n(\mu)u_{i,n,M,n'} n'(\mu)\hat{A}_{n,n'}^{0,0},
\]

\[
+ \sum_{n=1}^N \sum_{n'=1}^N \sum_{k=1}^{3+M^o} u_{N,i,n}(\mu)\lambda_{N,i}(\mu)\Theta_k(\mu)\hat{A}_{n,n'}^{N,0}.
\]

where \(\hat{A}_{n,n'}^{k,k'} \in \mathbb{R}^{N \times N}, 0 \leq q, q' \leq Q\) are given by \(\hat{A}_{n,n'}^{k,k'} = \hat{a}(p^k_n, p^{k'}_{n'})\), \(0 \leq p, p' \leq M^o + 3, 1 \leq n, n' \leq N\), \(\Theta_1 = 0, \Theta_2 = 0, \Theta_{2+m} = \beta_m, 1 \leq m \leq M^o\), and \(\Theta_{M^o+3} = \varepsilon_M\). We now see that the dual norm of the residual is the sum of products of parameter-dependent functions and parameter-independent functionals. The offline-online decomposition is now clear.

In the offline stage, we compute \(p^k_0\), \(0 \leq k \leq M^o + 3, 1 \leq n \leq N\), based on (60) at the cost of \(O((4 + M)NN^o)\), where the \(\bullet\) denotes computational complexity of the linear solver used to obtain \(p^k_0\). We then evaluate \(\hat{A}_{n,n'}^{k,k'}\) at the cost of \(O(4 + M)N^2N^2\). We store the matrices \(\hat{A}_{n,n'}^{k,k'}\) at a total cost of \((4 + M)N^2\).

In the online stage, we simply evaluate the sum (61) for a given \(\xi_{i,n,M}(\mu)\) and \(\lambda_{i,n,M}(\mu)\), \(1 \leq i \leq n_e\). The operation count is only \(O(n_eM^2N^2)\). The online complexity is thus independent of \(N\). Unless \(M\) is large, the online cost to compute the error estimator is then a fraction of the cost required to obtain \(\xi_{i,n,M}(\mu)\) and \(\lambda_{i,n,M}(\mu)\).

### E. Solution Method

In each fixed point iteration, part (i) of the solution method described in Section II-B will now consist of (a) constructing the reduced basis machinery required to approximate \(\xi(x_2; x_1)\) and \(\lambda(x_1)\), and their derivatives to a required level of accuracy, and (b) approximating \(\xi(x_2; x_1)\) and \(\lambda(x_1)\) for finite points on \(\Omega_1\) by \(\xi_{i,n,M}(x_2; x_1)\) and \(\lambda_{i,n,M}(x_1)\). In part (a), we must construct the reduced basis approximation space \(W_N\) and the relevant reduced basis matrices described in earlier sections; we construct \(W_N\) based on an adaptive greedy algorithm [23, 26] that utilize the error estimator to very efficiently choose a good set of \(S_N\). Note that we do not need to reconstruct the reduced basis machinery at each fixed point iteration. Armed with the \textit{a posteriori} error estimators, we only reconstruct the reduced basis machinery when the estimated errors of the solutions based on \(W_N\) of the previous iteration are above the tolerance we desired. This significantly reduces the cost of reduced basis method by limiting the number of times we need to perform the expensive "offline" computation. The procedure is summarized in Figure 3.

Several variations to the above procedure. For example, a more frequent reconstruction may lead to smaller \(N\), thus reducing the cost of "online" calculation. Thus, one could impose compulsory reconstruction of \(W_N\) at fixed intervals; at present we do not impose this as \(N\) required is generally small. In addition, we do not expect \(N\) to change drastically since \(\phi\) only changes slightly for each
iteration. We could also reduce the offline computational cost by reconstructing the $W_N$ based on existing $S_N$. While this removes the cost associated with greedy sampling procedure, we are less certain that the approximation space will be optimal and the solutions within the tolerance we desired.

IV. NUMERICAL RESULTS

We consider a domain $\Omega = [0, 340] \times [0, 120]$, which can be divided into subdomains detailed in Table I, which also gives the relative dielectric constant, $\epsilon_r$, and donor concentration, $N_D$ in each subdomain. The source and gate voltages applied are $V_S = 0$ and $V_G = 0.015$ while drain voltage, $V_D$ is allowed to vary. We consider a temperature of $10^{-4}$, and $P_{S\text{ max}} = P_{D\text{ max}} = 8$. To evaluate (18), we use $E_{\text{max}} = 20T$ since $f_{1/2}(-E_{\text{max}}/T) < 10^{-8}$. In addition, $n_E = 50$ and $Q = 3$.

In Section IV A and IV B, we will first look at the accuracy of reduced basis method in approx-
TABLE I: Definition of $\Omega_1 - \Omega_5$, and $\epsilon_r$ and $N_D$ used in the model problem; $\epsilon_0 = 1/4\pi$.

<table>
<thead>
<tr>
<th>extent</th>
<th>$\epsilon_r = \epsilon/\epsilon_0$</th>
<th>$N_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_1$ $[80, 260] \times [20, 100]$</td>
<td>11.7</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_2$ $[0, 80] \times [20, 100]$</td>
<td>11.7</td>
<td>$2.96 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\Omega_3$ $[26, 340] \times [20, 100]$</td>
<td>11.7</td>
<td>$2.96 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\Omega_4$ $[0, 340] \times [100, 120]$</td>
<td>3.9</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_5$ $[0, 340] \times [0, 20]$</td>
<td>3.9</td>
<td>0</td>
</tr>
</tbody>
</table>

FIG. 4: $\phi^N$ (Left) and $\partial \phi^N/\partial \mu$ (Right) for $V_D = 0.015$. The superscript $N$ indicates that it is a finite element approximation of $\phi$.

A. Empirical interpolation approximation of $\phi$ and $\partial \phi/\partial \mu$

We first examine the approximation of $\phi$ and $\partial \phi/\partial \mu$ based on the empirical interpolation method. Figure 4 shows the solutions of $\phi$ and $\partial \phi / \partial \mu$ at convergence for the case $V_D = 0.015$. We note that the variation of $\phi(x_2; \mu)$ with respect to $\mu$ is nontrivial. The empirical interpolation errors of $\phi_M$ and $\partial \phi_M / \partial \mu$, denoted by $\bar{\varepsilon}^\phi_M$ and $\bar{\varepsilon}^{d\phi}_M$ respectively, are shown in Figure 5. The figure shows that we have a rapidly converging approximation — with $M^\phi = 20$ and $M^{d\phi} = 21$, the error $\bar{\varepsilon}^\phi_M$ and $\bar{\varepsilon}^{d\phi}_M$ are less than $10^{-8}$.

B. Convergence of the reduced basis approximation

For our convergence analysis, the test sample $\Xi_\mu$ is given by the number of grid points in the $x_1$ direction — for the test problem, the size of $\Xi_\mu$ is 68. We will also define the following error
FIG. 5: $\varepsilon_M$ versus $M$ for $\phi_M$ (Left) and $\frac{\partial \phi_M}{\partial \mu}$ (Right).

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$N$</th>
<th>$\varepsilon^{\lambda}_{N,M}$</th>
<th>$\varepsilon^{\xi}_{N,M}$</th>
<th>$\max_{\mu \in \Xi_{\mu}} \Delta^{\lambda}_{N,M}$</th>
<th>$\max_{\mu \in \Xi_{\mu}} \Delta^{\xi}_{N,M}$</th>
<th>$\max_{\mu \in \Xi_{\mu}} \eta^{\lambda}_{N,M}$</th>
<th>$\max_{\mu \in \Xi_{\mu}} \eta^{\xi}_{N,M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>4.48E+3 1.02E+1</td>
<td>1.74E+2 8.85E+1</td>
<td>4.38E+1 5.87E+0</td>
<td>1.48E+1 7.31E+0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>1.42E+5 1.66E+3</td>
<td>1.79E+4 8.10E+3</td>
<td>4.99E+1 6.69E+0</td>
<td>1.48E+1 7.31E+0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>24</td>
<td>2.02E+9 3.53E+5</td>
<td>9.10E+8 2.58E+4</td>
<td>1.48E+1 7.31E+0</td>
<td>1.48E+1 7.31E+0</td>
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<td></td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>5.69E+11 6.09E+6</td>
<td>2.09E+9 3.49E+5</td>
<td>6.31E+2 4.27E+1</td>
<td>6.31E+2 4.27E+1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE II: Convergence of the reduced basis approximation for $V_D = 0.015$.

measures:

$$
\varepsilon^{\lambda}_{N,M} = \max_{\mu \in \Xi_{\mu}} \epsilon^{\lambda}_{N,M}(\mu), \quad \varepsilon^{\xi}_{N,M} = \max_{\mu \in \Xi_{\mu}} \epsilon^{\xi}_{N,M}(\mu),
$$

where

$$
\epsilon^{\lambda}_{N,M}(\mu) = \max_{1 \leq i \leq n_e} \frac{|\lambda_i,N,M(\mu) - \lambda_i(\mu)|}{|\lambda_i(\mu)|},
$$

$$
\epsilon^{\xi}_{N}(\mu) = \max_{1 \leq i \leq n_e} \frac{||\xi_i,N,M(\mu) - \xi_i(\mu)||}{||\xi_i(\mu)||}.
$$

We also define the effectivity measures as

$$
\eta^{\lambda}_{N,M}(\mu) = \frac{\Delta^{\lambda}_{N,M}(\mu)}{\epsilon^{\lambda}_{N,M}(\mu)}, \quad \eta^{\xi}_{N,M}(\mu) = \frac{\Delta^{\xi}_{N,M}(\mu)}{\epsilon^{\xi}_{N,M}(\mu)}.
$$

Table II show that our reduced basis approximation is rapidly convergent. We require only 24 basis functions to reduce the relative errors $\varepsilon^{\lambda}_{N,M}$ to below $10^{-8}$ and $\varepsilon^{\xi}_{N,M}$ to below $10^{-4}$ for the case where $V_D = 0.015$ and $M^\phi = M^d\phi = 25$. In addition, the effectivity measures are small, indicating that our error estimators are good surrogate to the actual errors. Although $\eta^{\lambda}_{N,M}$ and $\eta^{\xi}_{N,M}$ increase with $N$, $\Delta^{\lambda}_{N,M}$ and $\Delta^{\xi}_{N,M}$ also decrease — thus the absolute difference between the actual errors and the error estimators is small.

We now look at the reduced basis error in $d\xi_i,N,M(\cdot)$, $d\lambda_i,N,M(\cdot)$, $1 \leq i \leq n_e$ and
The computational time is reduced by a factor of 5. In addition, the reduced basis approximation spaces only the computational time spent in part (i), the computational savings is significant higher — the approximation of part (i) is implemented using FEM and reduced basis method as SDM/FEM and SDM/RBM respectively; part (ii) and (iii) are approximated by finite element method for both approaches. The finite element approximation of part (i) is implemented using $P_1$ elements with $N = 71$ while the reduced basis approximation uses the accuracy criteria given by $\Delta d_{N,M} = 0$, $\Delta d_{N,M} = 0$, $\Delta d_{N,M} = 0$, $\Delta d_{N,M} = 0$, $\Delta d_{N,M} = 0$. For higher accuracy, $W_N$ can indeed be a better approximation space although for the current purpose, $W_N$ appears to be sufficient and leads to a smaller $N$.

### C. Effects of reduced basis approach on efficiency of subband decomposition method

We denote the methods where we approximate part (i) of the subband decomposition method by finite element method and reduced basis method as SDM/FEM and SDM/RBM respectively; part (ii) and (iii) are approximated by finite element method for both approaches. The finite element approximation of part (i) is implemented using $P_1$ elements with $N = 71$ while the reduced basis approximation uses the accuracy criteria given by $\Delta d_{N,M} < 10^{-8}$.

In Table V, we compare the computational cost of the two methods for $V_D = 0$ and $V_D = 0.015$. For $V_D = 0$, the total computational time is reduced by a factor of 2, without any appreciable loss of accuracy — the $L_2$ norm error of $n_{N,M}$ obtained through SDM/RBM is of order $10^{-11}$. If we consider only the computational time spent in part (i), the computational savings is significant higher — the computational time is reduced by a factor of 5. In addition, the reduced basis approximation spaces are only reconstructed 3 times, out of the 22 fixed point iterations needed for convergence. We see similar results for $V_D = 0.015$. Note that the computational cost of part (i) includes the cost of

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$d_{N,M}$</th>
<th>$d_{N,M}$</th>
<th>$a_{N,M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>3.94E−1</td>
<td>9.99E−1</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
<td>3.77E−4</td>
<td>1.15E−2</td>
</tr>
<tr>
<td>3</td>
<td>24</td>
<td>8.83E−7</td>
<td>7.03E−6</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>3.12E−9</td>
<td>7.98E−8</td>
</tr>
</tbody>
</table>

Table III: Convergence of the reduced basis approximation of $d\lambda_{i,N,M}$ and $d\xi_{i,N,M}$, $1 \leq i \leq n_e$ for $V_D = 0.015$.

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$d_{N,M}$</th>
<th>$d_{N,M}$</th>
<th>$a_{N,M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
<td>1.32E−3</td>
<td>1.54E−2</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>5.40E−10</td>
<td>5.29E−9</td>
</tr>
</tbody>
</table>

Table IV: Convergence of the reduced basis approximation of $d\lambda_{i,N,M}$ and $d\xi_{i,N,M}$, $1 \leq i \leq n_e$ for $V_D = 0.015$ with $W_N$. 

We define 

\[
\epsilon_{d_{N,M}}^{d_{N,M}} = \max_{\mu \in \Xi} \max_{1 \leq i \leq n_e} \frac{|d\lambda_{i,N,M}(\mu) - d\lambda_i(\mu)|}{|d\lambda_i(\mu)|}, 
\]

\[
\epsilon_{d_{N,M}}^{d_{N,M}} = \max_{\mu \in \Xi} \max_{1 \leq i \leq n_e} \frac{|d\xi_{i,N,M}(\mu) - d\xi_i(\mu)|}{|d\xi_i(\mu)|}, 
\]

\[
\epsilon_{a_{N,M}}^{a_{N,M}} = \max_{\mu \in \Xi} \max_{1 \leq i,j \leq n_e} \frac{|\tilde{a}_3(d\xi_{i,N,M}(\mu), \xi_j, N,M(\mu)) - \tilde{a}_3(d\xi_i(\mu), \xi_j(\mu))|}{|\tilde{a}_3(d\xi_i(\mu), \xi_j(\mu))|} 
\]

From Table III, we again see the rapid convergence in the errors defined by (67) – (69). In particular, the error in $\tilde{a}_3(\cdot, \cdot)$, which determines the effects of reduced basis approximation on the subband decomposition method, decreases rapidly with $N$. For a relative error of $10^{-5}$, $N = 24$. Since the magnitude of $\tilde{a}_3(d\xi_i(\mu), \xi_j(\mu))$ is of order $10^{-4}$, the absolute error in the approximation is actually very small.

As indicated in Section III B, we now examine the approximation of $d\xi_i(\mu)$ in $W_N^d$ given by (32). We note that the solutions $(\xi_{i,N,M}, \lambda_{i,N,M})$ must also be determined in $W_N^d \times \mathbb{R}$. From Table IV, we indeed see a faster convergence in the errors with respect to $N_s$. However, the total number of basis, $N$, also increases with $N_s$ at a rate double that of $W_N$. As such, for higher accuracy, $W_N^d$ can indeed be a better approximation space although for the current purpose, $W_N$ appears to be sufficient and leads to a smaller $N$. 

We denote the methods where we approximate part (i) of the subband decomposition method by finite element method and reduced basis method as SDM/FEM and SDM/RBM respectively; part (ii) and (iii) are approximated by finite element method for both approaches. The finite element approximation of part (i) is implemented using $P_1$ elements with $N = 71$ while the reduced basis approximation uses the accuracy criteria given by $\Delta d_{N,M} < 10^{-8}$.
<table>
<thead>
<tr>
<th>Case</th>
<th>Subband Time, s</th>
<th>Reduced Basis Time, s</th>
<th>Total Part (i) N_{\text{offline}}</th>
<th>k_{\text{max}}</th>
<th>| n_N - n_{N,M} |_{L_2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_D = 0 )</td>
<td>730 546</td>
<td>305 112</td>
<td>3</td>
<td>22</td>
<td>4.03 E – 11</td>
</tr>
<tr>
<td>( V_D = 0.015 )</td>
<td>926 717</td>
<td>378 165</td>
<td>5</td>
<td>28</td>
<td>2.28 E – 11</td>
</tr>
</tbody>
</table>

TABLE V: Comparison of the computational cost for the subband decomposition method and the reduced basis method. Here \( N_{\text{offline}} \) is the number of times \( W_N \) is reconstructed, and \( k_{\text{max}} \) is the maximum number of fixed-point iteration.

FIG. 6: Comparison of (a) the total computational time and (b) the computational time for part time, for the subband decomposition method and the reduced basis method with increasing mesh size. The time is scaled with respect to total time for the SDM/RBM method at \( h_2 = 4 \).

We next examine how the computational time scales with respect to mesh size in the \( x_2 \)-direction. Figure 6(a) compares the computational time of SDM/FEM and SDM/RBM for increasingly fine mesh; the reported time has been scaled with respect to the total computational time of SDM/RBM at \( h_2 = 4 \), where \( h_2 \) is the mesh spacing in the \( x_2 \)-direction. We see that the total computational time of SDM/FEM scales exponentially with the mesh size. In particular, this exponential growth originates from the computational time of part (i) of the algorithm, as shown in Figure 6(b), while the computational time in part (ii) and (iii) only contribute slightly to the increase of the total computational time. On the other hand, the increase in the total computational time for SDM/RBM is slow. In particular, computational time of part (i) only increases very slightly, due to marginal increase in the computational cost of the offline stage; there should be little or no increase in the computational cost of the online stage. This observation strongly suggests that the reduced basis approach is particularly suited for situations where computational cost of part (i) dominates the total computational cost. For example, fine resolution may be needed in the \( x_2 \)-direction due to strong confinement of the electrons. In nanowires and nanotubes where we have a 2-dimensional confinement, the higher dimension will also lead to larger mesh size, thus increasing the computational cost of part (i).

Finally, we look at a quantity of interest, the drain current, \( I_{SD} \). Figure 7 shows that we have a typical current-voltage relation for a MOSFET, where the rate of increase in \( I_{SD} \) decreases as the applied voltage \( V_{SD} \) increases. We further note that SDM/RBM method gives comparable result to SDM/FEM method.
FIG. 7: Comparison of the computed drain current for SDM/FEM and SDM/RBM.

V. CONCLUSION

We have described how reduced basis method can improve the efficiency of the subband decomposition approach to ballistic transport simulation in nanodevices. In particular, the novel use of a posteriori error estimator and adaptive sampling procedure leads to a very efficient solution procedure. Numerical results based on a double-gate MOSFET show that the computational cost is reduced by 50% – 60% for a reasonably-sized problem and depends very weakly on the mesh size in the confined direction. We expect the computational savings to increase in cases of 2D confinement, such as those encountered in nanowires.

APPENDIX A: DERIVATIVE OF $\phi$

To solve (23), we must evaluate $\partial \xi_n/\partial x_1$; in [10], $\xi_n$, $\partial \xi_n/\partial x_1$ and $\tilde{a}_3(\cdot, \cdot)$ are evaluated at the nodes $(i, j)$ of the rectangular mesh, and interpolated to the quadrature points when evaluating the functionals in (23). In addition, $\partial \xi_n/\partial x_1$ are evaluated by difference formula. In our approach, $\partial \xi_n/\partial x_1$ are determined from (25), and this involves determining $\partial \phi/\partial x_1$ at the nodes $(i, j)$. However, as we have used $Q_2$ elements to solve for $\phi$, its derivative is discontinuous, and thus not defined at the nodes. So, we compute the $\partial \phi/\partial x_1$ based on a difference formula. We then compute $\xi_n$, $\partial \xi_n/\partial x_1$ and $\tilde{a}_3(\cdot, \cdot)$ at the nodes $(i, j)$ of the rectangular mesh, and interpolate to the quadrature points when evaluating the functionals in (23).

To avoid evaluating $\partial \phi/\partial x_1$ at the nodes, we can choose to compute $\partial \xi_n/\partial x_1$ directly at the quadrature points used to evaluate the functionals in (23). The reduced basis approximation procedure is as follows:

1. Compute $\partial \phi/\partial x_1$ at $(i + 1/2, j)$, where $i + 1/2$ is the midpoint between $i$ and $i + 1$.

2. Construct a magic point approximation for $\partial \phi/\partial x_1$, and the reduced basis machinery for $\partial \xi_n/\partial x_1$.

3. Evaluate the terms $\xi_{n,N,M}$, $\partial \xi_{n,N,M}/\partial x_1$ and $\tilde{a}_3$ at the quadrature points. To evaluate $\xi_{n,N,M}$ and $\partial \xi_{n,N,M}/\partial x_1$, values of $\phi_M$ and $\partial \phi_M/\partial x_1$ at the magic points for a given quadrature point must first be determined. For $\phi_M$, these are obtained by the interpolation of the $Q_2$ elements. For $\partial \phi_M/\partial x_1$, since the gradient between node $(i, t_M^\phi)$ and $(i + 1, t_M^\phi)$ is a constant, the values...
at the magic points for quadrature point falling between \((i, t^d_M)\) and \((i + 1, t^d_M)\) is given by the
value at node \((i + 1/2, t^d_M)\); \(t^d_M\) are the magic points for \(\partial \phi_M / \partial x_1\).

The above formulation should then be consistent with the \(\mathbb{Q}_2\) elements we use. It is however
more expensive: the computational cost of part (i) is increased by 66\%. Determining the accuracy
of the two approaches is also tricky. A comparison to, say, a full finite element approximation may
be necessary although approximation error of subband decomposition method may dominate. In
addition, the convergence criteria used in the fixed point iteration is not stringent, and any difference
between the two approaches may not be discernible.

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