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Deformations of Manifolds of Calabi-Yau Type

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Nicholas Brandon Reale

Dissertation Committee:
Professor Zhiqin Lu, Chair
Professor Li-Sheng Tseng
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2015
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ABSTRACT OF THE DISSERTATION

Deformations of Manifolds of Calabi-Yau Type

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2015

Professor Zhiqin Lu, Chair

In 2011, A. Iliev and L. Manivel proposed the class of manifolds of Calabi-Yau type as a generalization of Calabi-Yau threefolds that would include Kähler mirror pairs even for threefolds with \( h^{2,1} = 0 \). We construct sufficient conditions for such manifolds to have unobstructed deformations even in the non-Fano case.

In such cases where the deformations are unobstructed, we consider two metrics on the moduli space of such manifolds, one equivalent to the Weil-Petersson metric and the other the resulting partial Hodge metrics. Curvature tensors for both metrics are produced, and bounds are found for holomorphic bisectional curvature and holomorphic sectional curvature for a specific partial Hodge metric on the moduli space.

Finally, a formula is found relating these metrics, the Euler characteristic, and the BCOV torsion on manifolds of Calabi-Yau type.
Chapter 1

Manifolds of Calabi-Yau Type

1.1 A Brief History of Mirror Symmetry

This section is adapted from the introductions of [19] and [40].

The path to manifolds of Calabi-Yau type began in 1985, with the publication of "Vacuum configurations for superstrings" by Philip Candelas, Gary Horowitz, Andrew Strominger, and Edward Witten[6], which demonstrated that if the extra physical dimensions posited by superstring theory were compactified as a three-dimensional Calabi-Yau manifold, the existing models could incorporate supersymmetry of supatomic particles. Both [8] and [25] showed this compactification need not be unique for a particular quantum field theory (QFT); rather, there could be two possible Calabi-Yau threefolds that would suffice. The hunt then began for these so-called mirror pairs, with the resulting theory of mirror symmetry developed in [7] [30] [13], with many non-trivial early examples demonstrated in [12].

The aspect of the mirror pairs most relevant to manifolds of Calabi-Yau type is the Hodge numbers as arranged in the Hodge diamond, so we digress from historical background to
cover these necessary mathematics in brief.

1.2 Mathematical Background

It is first necessary to establish some notational conventions.

1.2.1 Complex Manifolds

Unless otherwise specified $X$ will be a complex manifold of complex dimension $n$. The local coordinates on $X$ will be denoted by $z = (z^1, \ldots, z^n)$. $T^{(1,0)}(X)$ is the holomorphic component of the complexified tangent bundle.

The hermitian metric tensor on $X$ will be denoted

$$g = \sum_{i,j=1}^{n} g_{ij} dz^i \otimes d\overline{z}^j$$

Additionally, through slight abuse of notation, $\det g$ will denote $\det(g_{ij})$, considered as the standard determinant of an $n \times n$ matrix. This matrix $(g_{ij})$ is positive definite since $g$ is Hermitian. Unless otherwise specified, then $\omega$ will denote the Kähler form

$$\frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{ij} dz^i \wedge d\overline{z}^j$$

defined by $g$ (whether or not $g$ is the Hermitian metric on the manifold $X$).

If $(X, \omega)$ is a Kähler manifold (i.e., $d\omega = 0$) and only a single metric is being considered on $X$, “$X$ is Kähler” will be understood to mean that $(X, \omega)$ is Kähler, in a standard abuse of notation. With such notational abuse established, $X$ will be assumed to be Kähler with respect to any implicitly established metric unless otherwise specified.
Let $\mathcal{A}^{p,q}(X)$ denote the space of $C^\infty$ differential forms on $X$ with $p$ holomorphic components and $q$ anti-holomorphic components. Additionally, let $\mathcal{A}^k(X)$ denote the space of $C^\infty$ differential forms with $k$ components. It follows immediately that

$$\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X).$$

The operator $d : \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X)$ linearly splits into holomorphic and anti-holomorphic components: $\partial : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X)$ and $\bar{\partial} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)$. When more complicated situations arise, $\partial$ and $\bar{\partial}$ will still denote these operators on the original manifold $X$.

Similarly, if $E$ is a holomorphic vector bundle on $X$ with Hermitian metric $h$, then $\mathcal{A}^{p,q}(E)$ and $\mathcal{A}^k(E)$ denote the $E$-valued $(p, q)$-forms and $k$-forms on $X$, respectively. That is,

$$\mathcal{A}^{p,q}(E) = \mathcal{A}^{p,q}(X) \otimes E$$

and

$$\mathcal{A}^k(E) = \mathcal{A}^k(X) \otimes E.$$ 

There is a similar decomposition:

$$\mathcal{A}^k(E) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(E).$$

Similarly, let $\Omega^p(X)$ (or just $\Omega^p$) denote the sheaf of holomorphic $p$-forms on $X$, and let $\Omega^p(E)$ denote the sheaf of $E$-valued holomorphic $p$-forms. Finally, for any sheaf $S$ over $X$, let $\Gamma(X, S)$ denote the group of sections of the sheaf.

On $\mathcal{A}^{p,q}(E)$, the metric generates a point-wise inner product $\langle \cdot , \cdot \rangle$. For fixed $\phi_1, \phi_2 \in \mathcal{A}^{p,q}(E)$,
\[ \langle \phi_1, \phi_2 \rangle \] is a bounded smooth function on \( X \). This allows the definition of a global inner product:

\[
(\phi_1, \phi_2) := \int_X \langle \phi_1, \phi_2 \rangle \, dV_g,
\]
called the \( L^2 \) inner product on \( \mathcal{A}^{p,q}(E) \). The induced norm is of course called the \( L^2 \) norm.

Let \( \partial^* \) denote the adjoint of the \( \partial \) operator on \( \mathcal{A}^{p,q}(E) \) with respect to the \( L^2 \) metric. Then the Hodge Laplacian is defined as

\[
\Delta_{\nabla} = \partial \partial^* + \partial^* \partial.
\]

When it is unambiguous from context, this will simply be called the Laplacian and denoted by \( \Delta \).

### 1.2.2 Multi-index Notations

Before continuing, define some new multiindex notations. Define \( \mathbb{N}_u = \{1, \ldots, u\} \). Then \( J \in \mathbb{N}_u^s \) is a multi-index \((j_1, \ldots, j_s)\) of integers between 1 and \( u \). In particular, define \( N_u = (1, \ldots, u) \in \mathbb{N}_u \). Let \( \sigma_J \) denote the permutation that puts the entries of \( J \) in increasing order maintaining the relative order of any repeated indices. Trivially, \( \sigma_{N_u} \) is the identity permutation. Additionally, define \( I_r = (i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_s) \), the multiindex in \( \mathbb{N}_u^{r-1} \) omitting the \( r \)th element.

For any two multiindices \( I \in \mathbb{N}_u^r, J \in \mathbb{N}_u^s \),

\[
I \xrightarrow{+} J := (i_1, \ldots, i_r, j_1, \ldots, j_s).
\]
If $K = I \hat{+} J$, then

$$K \hat{-} J := I.$$ 

For two multiindices $J \in \mathbb{N}_u^s$ and $K \in \mathbb{N}_u^t$ such that $r \leq s$, let $J \prec K$ denote the partial ordering given by every index in $J$ appearing in $K$ in the same order, with perhaps some extra terms separating them. For example,

$$(1, 2, 3) \prec (5, 1, 2, 4, 7, 3, 6)$$

Then define $\sigma_{K \succ J}$ to be the permutation of $K$ that maintains the relative order of repeated indices such that $\sigma_{K \succ J}(K) = I \hat{+} J$ and for some $I \in \mathbb{N}_u^{t-s}$ such that $I \prec K$. Extend the operation $\hat{-}$ to any case where $J \prec K$ by

$$K \hat{-} J := \sigma_{K \succ J}(K) \hat{-} J.$$ 

Notice that if $K \hat{-} J$ is defined, then $\sigma_{K \succ J}(K) = K$, so $K \hat{-} J = K \hat{+} J$.

For $I \in \mathbb{N}_u^r$, $J \in \mathbb{N}_u^s$, and $K \in \mathbb{N}_u^t$, $\hat{+}$ and $\hat{-}$ have the following properties when they are well-defined.

1. If $I \hat{+} J$ has no repeated indices, then for a permutation $\sigma$ such that $\sigma(I \hat{+} J) = J \hat{+} I$, $sgn(\sigma) = (-1)^{rs}$.

2. If $K \hat{+} J$ has no repeated indices and $I \succ (K \hat{+} J)$, then $I \hat{-}(K \hat{+} J) = (I \hat{-} J) \hat{-} K$ and $\sigma_{I \succ (K \hat{+} J)} = \sigma_{(I \hat{-} J) \hat{-} K} \sigma_{I \hat{-} J}$.

When $I \in \mathbb{N}_u^r$ is in strictly increasing order, then define $I^c := \mathbb{N}_u \hat{-} I$, where $\mathbb{N}_u$ is considered as a multiindex. It follows immediately that $I^c \in \mathbb{N}_u^{u-r}$.
Additionally, for \( I \in \mathbb{N}^s \) define the following notation.

\[
d z^I := dz^{i_1} \wedge \ldots \wedge dz^{i_s}
\]

\[
\frac{\partial}{\partial z^I} := \frac{\partial}{\partial z^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial z^{i_s}}
\]

Define \( d\bar{z}^I \) and \( \frac{\partial}{\partial \bar{z}^I} \) similarly.

Einstein summation notation is used liberally throughout, for both standard indices and multiindices.

### 1.2.3 Cohomology Groups and the Hodge Diamond

The cohomology group induced by the \( \overline{\partial} \) operator on \( A^{p,q}(E) \) is the Dolbeault cohomology group and is

\[
H^q(X, \Omega^p(E)) := \frac{\{ \phi \in \Gamma(X, A^{p,q}(E)) | \overline{\partial} \phi = 0 \}}{\{ \overline{\partial} \psi | \psi \in \Gamma(X, A^{p,q-1}(E)) \}}.
\]

The relevant Dolbeault cohomology group for the Hodge diamond is the case where \( E = \mathbb{C} \), in which case it reduces to

\[
H^q(X, \Omega^p) := \frac{\{ \phi \in \Gamma(X, A^{p,q}(X)) | \overline{\partial} \phi = 0 \}}{\{ \overline{\partial} \psi | \psi \in \Gamma(X, A^{p,q-1}(X)) \}}.
\]

A similar cohomology group can be defined using the standard exterior derivative \( d \). The cohomology groups \( H^{p,q}(X) \) are defined to be

\[
H^{p,q}(X) = \{ \phi \in A^{p,q}(X) : d\phi = 0 \}/dA^{k-1}(X) \cap A^{p,q}(X).
\]

That is, they’re the quotient group of the closed \( C^\infty \) \((p,q)\)-forms by the exact \( C^\infty \) \((p,q)\)-
forms.

The final of three groups under consideration is the group of $E$-valued harmonic $(p,q)$-forms:

$$\mathcal{H}^{p,q}(E) := \{ \phi \in \Gamma(X, \mathcal{A}^{p,q}(E)) | \Delta \phi = 0 \}.$$ 

In particular, when $E = \mathbb{C}$, this simplifies to

$$\mathcal{H}^{p,q}(X) := \{ \phi \in \Gamma(X, \mathcal{A}^{p,q}(X)) | \Delta \phi = 0 \},$$

or just $\mathcal{H}^{p,q}$ when the manifold is clear from context.

The Hodge and Dolbeault Theorems together give that the three groups

1. $H^q(X, \Omega^p)$,
2. $H^{p,q}(X)$, and
3. $\mathcal{H}^{p,q}$

are all isomorphic to each other and are finite dimensional. In particular, every element of the group $\mathcal{H}^{p,q}$ is a representative of a cohomology class in each of the other two cohomology groups, and is in fact the smallest element of that cohomology class with respect to the $L^2$ norm.

The Hodge numbers are

$$h^{p,q} := \dim H^q(X, \Omega^p) = \dim H^{p,q}(X) = \dim \mathcal{H}^{p,q}$$

These numbers can be arranged in the Hodge Diamond (Figure 1.1). Each row corresponds to a $k$ forms from $k = 0$ up to $k = 2n$. The Hodge Diamond can be simplified with a few helpful properties.
First consider the Serre Duality (which will be proven in Chapter 2).

**Theorem 1.1** (Serre Duality). Let $E$ be a Hermitian vector bundle over $X$ and let $E^*$ denote the dual space of $E$. Let $0 \leq p, q \leq n$. Then $H^q(X, \Omega^p(E))$ is isomorphic to $H^{n-q}(X, \Omega^{n-p}(E^*))$.

In particular, if $E = \mathbb{C}$, then $E^* = \mathbb{C}$ (action via multiplication), so $H^q(X, \Omega^p)$ is isomorphic to $H^{n-q}(X, \Omega^{n-p})$, and thus $h^{p,q} = h^{n-p,n-q}$. Applying this special case of the Serre Duality gives a reduced Hodge Diamond. That is, the Hodge Diamond is symmetric under rotation by $\pi$ radians.

Since $X$ is Kähler, then there’s an additional tool to simplify the Hodge Diamond and compute the values contained within: the Hodge Decomposition Theorem.

**Theorem 1.2** (Hodge Decomposition Theorem). Let $X$ be a compact Kähler manifold of dimension $n$. Then for each $k \in \{0, 1, \ldots, 2n\}$, the $k$th complex de-Rham cohomology group
Figure 1.2: The Hodge Diamond reduced by the Serre Duality

of $X$ can be written as a direct sum of cohomology groups:

$$H^k_{DR}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Additionally,

$$H^{q,p}(X) = H^{p,q}(\overline{X}).$$

The first half gives that the sum of each row of the Hodge Diamond is the corresponding Betti number $b^k := \dim H^k_{DR}(X, \mathbb{C})$. When considering a specific manifold, this is a helpful tool for computing the Hodge Diamond. It is the second half that is more immediately helpful for an abstract diamond. It gives the Hodge Diamond reflective symmetry about the vertical axis for all rows, not just the central row (Figure 1.3).
Additionally, since $X$ is connected and has no boundary, the maximum principle gives that holomorphic functions on $X$ must be constant. Thus $h^{0,0} = 1$. Since the Kähler form $\omega$ is harmonic, $\mathcal{H}^{1,1}(X)$ must be non-trivial, and thus $h^{1,1}$ is strictly positive. Additionally, since the Kähler form is $\bar{\partial}$-closed, but not $\bar{\partial}$-exact, then $\omega^k$ is also $\bar{\partial}$-closed, but not $\bar{\partial}$-exact for $k = 1, \ldots, n$, and thus represents a non-trivial element of $H^q(X, \Omega^p)$. Hence $h^{k,k}(X)$ is strictly positive for $k = 1, \ldots, n$.

### 1.2.4 Calabi-Yau Threefolds and Mirror Symmetry

A Calabi-Yau manifold is a Kähler manifold with several definitions, not all equivalent. With the assumption that $X$ is compact, the following are equivalent definitions ([4]) for $X$ being Calabi-Yau (with examples of sources taking that definition).
• The Ricci form, $-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \det g$, is zero. [9]

• The first Chern class of $X$ vanishes. [4]

• The holonomy group of $g$ is contained in $SU(n)$, the special unitary group of degree $n$ (the group of $n \times n$ unitary matrices of determinant 1). [45]

• The canonical bundle of $X$ is trivial. [46]

• $X$ admits a globally defined, nowhere-vanishing, holomorphic $n$-form. [20]

It is the fifth condition that is most relevant to the upcoming proofs, so that one will be taken as the definition of a Calabi-Yau manifold, with acknowledgment that it is not the most historically relevant definition nor necessarily the definition most commonly used.

This holomorphic $n$-form is in fact unique up to constant multiple, and thus $h^{n,0} = 1$. Additionally, it can be shown that $h^{k,0} = 0$ for $0 < k < n$.

Of particular interest is the case where $\dim X = 3$. In that case, the Hodge Diamond simplifies greatly to the one in Figure 1.4. That is, the Hodge diamond is entirely determined by $h^{1,1}$ and $h^{2,1}$.

In every mirror pair so far found, the Hodge diamonds of the Calabi-Yau threefolds are one-quarter rotation of each other. That is, $h^{1,1}$ of one is equal to $h^{2,1}$ of the other, and vice versa. The search for mirror pairs is thus in many ways a search for Calabi-Yau threefolds by these two numbers.

1.3 Expanding the Scope of Mirror Symmetry

A problem arises, when considering Calabi-Yau manifolds where $h^{2,1} = 0$: its mirror partner must have $h^{1,1} = 0$ and not be Kähler! This started the hunt for a larger set of manifolds
Figure 1.4: The Hodge Diamond of a Calabi-Yau Threefold

that could include all of the mirror pairs.

In [39] and [40], Schimmrigk, one of the key figures in the development of mirror symmetry, expanded the search to certain manifolds – of complex dimension 3 or often higher – that all had the following conditions on their Hodge numbers.

1. If neither $p = q$ nor $p + q = n$, then $h^{p,q} = 0$. That is, the only non-zero terms are on the vertical and horizontal axes.

2. $h^{n,0} = h^{0,n} = 0$. In particular, none of the manifolds are Calabi-Yau.

Working independently, Candelas, Derrick, and Parkes proposed a similar set of manifolds in [5]. The manifolds they considered instead had the following conditions.

1. The manifold is of dimension $2n + 1$ for $n > 1$. That is, the manifold is of odd dimension.

2. The central horizontal line, such that $p + q = 2n + 1$ satisfies the following conditions.

   (a) $h^{p,2n+1-p} = 0$ for $p > n + 2$. 

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That is, the middle row is of the form

$$0, \ldots, 0, 1, h^{n+1,n}, h^{n+1,n}, 1, 0, \ldots, 0.$$ 

Based on this work, Iliev and Manivel proposed the following set of manifolds in the search for mirror pairs. [20]

**Definition 1.1** (Manifold of Calabi-Yau Type). A manifold is said to be a $2n+1$-fold of Calabi-Yau type if $X$ is a smooth, compact, complex variety of complex dimension $2n+1$, $n \geq 1$, such that:

1. $h^{n+2, n-1}(X) = 1$ and $h^{n+r+1, n-r}(X) = 0$ for $r \geq 2$.

2. $h^{r,0}(X) = 0$ for $1 \leq r \leq 2n$.

3. For any generator $[\mu] \in H^{n+2, n-1}(X)$, the contraction

$$H^1(X, TX) \xrightarrow{\mu} H^n(X, \Omega^{n+1}_X)$$

is an isomorphism.

The contraction isomorphism corresponds to the isomorphism given by the Serre Duality in the Calabi-Yau threefold case. Additionally, notice that Calabi-Yau threefolds are in this class of manifolds for $n = 1$, so this includes all of the original desired manifolds.

Before considering new properties of manifolds of this type, we require some more preliminaries.
Chapter 2

Other Mathematical Preliminaries

2.1 Kähler Manifolds

We first consider useful properties on Kähler manifolds. This section is adapted from [29].

2.1.1 Holomorphic Normal Coordinates

When performing computations at a single point, Kähler manifolds allow for considerable simplification.

Theorem 2.1. Let \((X, \omega)\) be an \(n\)-dimensional Kähler manifold. For any fixed \(x_0 \in X\), there exist local holomorphic coordinates \((z^1, \ldots, z^n)\) such that

\[
\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} g_{i\overline{j}} dz^i \wedge d\overline{z}^j
\]
(as standard) with
\[ g_{i\overline{j}}(x_0) = \delta_{i\overline{j}} \text{ and } (dg_{i\overline{j}})(x_0) = 0. \]

These local holomorphic coordinates are called normal coordinates.

**Proof.** Satisfying the first condition is merely a matter of linear algebra. For any local holomorphic local coordinate system near \( x_0 \), apply the Gram-Schmidt orthogonalization progress to get a new holomorphic local coordinate system \( (w^1, \ldots, w^n) \) such that
\[ \omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n} h_{i\overline{j}} dw^i \wedge d\overline{w}^j \]
with
\[ h_{i\overline{j}}(x_0) = \delta_{i\overline{j}}. \]

It now remains to show that these coordinates can be adjusted to satisfy the second condition without losing the first. Using Taylor’s Theorem with \( x_0 \) (corresponding to \( w = 0 \)) as the base point,
\[ h_{i\overline{j}} = \sum_{m=0}^{\infty} \sum_{|K|+|L|=m} \frac{\partial^m h_{i\overline{j}}}{\partial w^K \partial \overline{w}^L} \Big|_{x_0} w^K \overline{w}^L = \delta_{i\overline{j}} + \sum_{m=1}^{\infty} \sum_{|K|+|L|=m} \frac{\partial^m h_{i\overline{j}}}{\partial w^K \partial \overline{w}^L} \Big|_{x_0} w^K \overline{w}^L. \]

The condition that \( (dh_{i\overline{j}})(x_0) = 0 \) is thus
\[ 0 = \sum_{m=1}^{\infty} \sum_{|K|+|L|=m} \frac{\partial^m h_{i\overline{j}}}{\partial w^K \partial \overline{w}^L} \Big|_{x_0} [w^K \overline{w}^L dw^k + w^K \overline{w}^L d\overline{w}^l]_{w=0}. \]

This condition is trivially satisfied for each term except when \( K = (k) \), \( L = \emptyset \) or when
$K = \emptyset, L = (l)$. Thus the restriction simplifies to

$$0 = \left. \frac{\partial h_{ij}}{\partial w^k} \right|_{x_0} dw^k(x_0) + \left. \frac{\partial h_{ij}}{\partial \bar{w}^l} \right|_{x_0} d\bar{w}^l(x_0),$$

and so removing the linear terms from the Taylor series is enough to produce the required coordinates. That is, the terms

$$\left. \frac{\partial h_{ij}}{\partial w^k} \right|_{x_0} w^k, \quad \left. \frac{\partial h_{ij}}{\partial \bar{w}^l} \right|_{x_0} \bar{w}^l$$

must be removed.

It is sufficient to find holomorphic local coordinates $(z^1, \ldots, z^n)$ such that

$$dz^k = \left( \delta_{ik} + \left. \frac{\partial h_{ik}}{\partial w^m} \right|_{x_0} w^m \right) dw^i,$$

for $1 \leq k \leq n$, because then

$$\delta_{kl} dz^k \wedge dz^l = \delta_{kl} \left( \delta_{ik} + \left. \frac{\partial h_{ik}}{\partial w^r} \right|_{x_0} w^r \right) dw^i \wedge \left( \delta_{lj} + \left. \frac{\partial h_{lj}}{\partial w^s} \right|_{x_0} w^s \right) d\bar{w}^j$$

$$= \left( \delta_{ij} + \left. \frac{\partial h_{ij}}{\partial w^r} \right|_{x_0} w^r + \delta_{bl} \left. \frac{\partial h_{bl}}{\partial \bar{w}^s} \right|_{x_0} \bar{w}^s \right) dw^i \wedge d\bar{w}^j.$$

Thus

$$\delta_{kl} dz^k \wedge dz^l - h_{ij} dw^i \wedge d\bar{w}^j = O(|w|^2) dw^i \wedge d\bar{w}^j.$$

Therefore $\delta_{kl} dz^k \wedge dz^l$ plus an appropriate quadratic adjustment is equal to $\omega$, so $(z^1, \ldots, z^n)$ is a system of holomorphic coordinates that satisfies the conditions. It remains to show that
such \((z^1, \ldots, z^n)\) exist that satisfy

\[
dz_k = \left( \delta_{ik} + \frac{\partial h_{ik}}{\partial w^m} \bigg|_{x_0} w^m \right) dw^i.
\]

In this case,

\[
z^k = w^k + \frac{1}{2} \frac{\partial h_{ik}}{\partial w^m} \bigg|_{x_0} w^m w^i
\]

satisfies the differential equation, but only because \(X\) is Kähler, and thus

\[
dz_k = dw^k + \frac{1}{2} \frac{\partial h_{ik}}{\partial w^m} \bigg|_{x_0} dw^m w^i + \frac{1}{2} \frac{\partial h_{ik}}{\partial w^m} \bigg|_{x_0} w^m dw^i
\]

\[
= dw^k + \frac{1}{2} \frac{\partial h_{ik}}{\partial w^m} \bigg|_{x_0} w^i dw^m + \frac{1}{2} \frac{\partial h_{ik}}{\partial w^m} \bigg|_{x_0} w^m dw^i
\]

\[
= \left( \delta_{ik} + \frac{\partial h_{ik}}{\partial w^m} \bigg|_{x_0} w^m \right) dw^i,
\]

as desired. Such \((z^1, \ldots, z^n)\) define holomorphic local coordinates if the neighborhood of \(x_0\) is shrunk as necessary, and therefore a correct choice of coordinates exists on a sufficiently small neighborhood.

\[
\square
\]

### 2.1.2 The Hodge \(\ast\)-operator

The Hodge \(\ast\)-operator is a linear operator on any finite-dimensional vector space \(V\) with ordered basis. It’s relevant here to vector bundles on Kähler manifolds, but it can easily be defined when the following conditions are met.

- The vector space \(V\) is an oriented, real, inner product space with orthonormal basis 
  \(\beta = \{e_1, \ldots, e_{2m}\}\);
- Extending the associated field to be \(\mathbb{C}\) gives a complex vector space \(W\) with Hermi-
tian inner product and orthonormal basis $\beta$. (That is, the basis for $W$ is conjugate-invariant.)

In both cases, it follows immediately that $\beta^* = \{e_1^*, \ldots, e_m^*\}$ is a basis for the dual space $V^*$ or $W^*$. The definition of $*$ and many of the properties that follow generalize to arbitrary vector spaces, but are here considered only for such a space $W$.

Define $*$ to be the linear operator

$$\ast : \bigwedge^k(W) \to \bigwedge^{m-k}(W)$$

such that

$$*(e_I) = \text{sgn}(\sigma_{I^c}) e_{I^c}$$

for $0 \leq k \leq m$ and $e_I := e_{i_1} \wedge \ldots e_{i_k}$ for $I = (i_1, \ldots, i_k)$ a strictly increasing multiindex.

**Proposition 2.1.** $*$ has the following properties.

1. The definition of $*$ is independent of choice of conjugate-invariant basis.

2. $**v = (-1)^k v$ for $v \in \bigwedge^k(W)$.

3. $\langle v, w \rangle *1 = v \wedge *w$ where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product on $W$ and $v, w \in \bigwedge^k(W)$.

**Proof.** 1. Every other conjugate-invariant orthonormal frame with the same orientation differs from the existing frame by a rotation. Every rotation in finite dimensions can be decomposed into a sequence of rotations that preserve all but two of the basis elements. It is thus sufficient to consider only the case where two vectors, $e_r$ and $e_s$, are rotating and all others are fixed.
Let $\beta' = \{e_1, \ldots, e_{r-1}, e'_r, e_{r+1}, \ldots, e_{s-1}, e'_s, e_{s+1}, \ldots, e_m\}$ be a different orthonormal basis for $V$ (and thus $W$) such that $(a_{ij})$ is the change of basis matrix from $\beta$ to $\beta'$. Note that when $i, j \neq r, s$, then $a_{ij} = \delta_{ij}$. Then for strictly increasing $I$,

$$e^*_I(f_I) = * \left( \sum_{J \in \mathbb{N}_m^k} a_{JI} e_J \right) = * \left( \sum_{J \in \mathbb{N}_m^k} a_{JI} \text{sgn}(\sigma_J) e_{\sigma_J(J)} \right)$$

$$= \sum_{J \in \mathbb{N}_m^k} a_{JI} \text{sgn}(\sigma_J) \text{sgn}(\sigma_J(J)^c) e_{\sigma_J(J)^c},$$

so for strictly increasing $K$,

$$e^*_K(*(f_I)) = \sum_{J \in \mathbb{N}_m^k} a_{JI} \text{sgn}(\sigma_J) \text{sgn}(\sigma_J(J)^c) \text{sgn}(\sigma_J(J)^c) \delta_{\sigma_J(J)^cK}$$

$$= \sum_{\tau \in S_k} a_{\tau(K^c)} \text{sgn}(\tau) \text{sgn}(\sigma_{K^c+K})$$

$$= \text{det}(a_{K^cI}) \text{sgn}(\sigma_{K^c+K}).$$

This will be 0 unless $K^c$ and $I$ are identical except for perhaps a replacement of $r$ with $s$ or vice versa. On the other hand,

$$e^*_K(\text{sgn}(\sigma_{I^c I^c}) f_{I^c}) = \text{sgn}(\sigma_{I^c I^c}) e^*_K(a_{LI} e_L)$$

$$= \text{sgn}(\sigma_{I^c I^c}) e^*_K(a_{LI} \text{sgn}(\sigma_L) e_{\sigma_L(L)})$$

$$= \sum_{\tau' \in S_{n-k}} \text{sgn}(\sigma_{I^c I^c}) a_{\tau'(K^c)} \text{sgn}(\tau')$$

$$= \text{sgn}(\sigma_{I^c I^c}) \text{det}(a_{KI^c}),$$

which is 0 except in the case where $K$ and $I^c$ are identical except perhaps for $r$ and $s$ (and thus the same holds for $I$ and $K^c$). It thus suffices to consider only such cases.

If $I$ includes bot $r$ and $s$, then $I = K^c$ and thus both $\text{det}(a_{KI^c}) = 1 = \text{det}(a_{K^cI})$ and
\[ \text{sgn} (\sigma_{I+I^c}) = \text{sgn} (\sigma_{K+c+K}). \] Hence

\[ e^*_K(*) = e^*_K(\text{sgn} (\sigma_{I+I^c}) f_{I^c}). \] If, on the other hand, \( I = K^c \), but they only share one of the two crucial indices (say \( r \)), then \( \det(a_{K^c}) = \cos \theta = \det(a_{K^c}) \) (where \( \theta \) is the angle of rotation) and

\[ \text{sgn} (\sigma_{I+I^c}) = \text{sgn} (\sigma_{K+c+K}). \] Hence

\[ e^*_K(*) = e^*_K(\text{sgn} (\sigma_{I+I^c}) f_{I^c}), \] as before. Finally, if \( I \) and \( K^c \) are different, then \( \det(a_{K^c}) = \sin \theta = -\det(a_{K^c}) \) (where \( \theta \) is the angle of rotation) and \( \text{sgn} (\sigma_{I+I^c}) = -\text{sgn} (\sigma_{K+c+K}) \) (since the two multiindices differ by a single transposition). Hence

\[ e^*_K(*) = e^*_K(\text{sgn} (\sigma_{I+I^c}) f_{I^c}) \] once more, and thus the equality holds in all cases. Since it holds for any choice of \( I \) and \( K \) of the correct length, it must hold for any vectors and dual vectors, and therefore

\[ (*) = \text{sgn} (\sigma_{I+I^c}) f_{I^c}. \] Thus the definition is not only independent of choice of basis, but is also intrinsic to the orientation.

2. Because \( * \) is linear, it’s sufficient to check for \( v = e_I \). In that case,

\[ * * (e_I) = * (\text{sgn} (\sigma_{I+I^c}) e_{I^c}) = \text{sgn} (\sigma_{I+c+I}) \text{sgn} (\sigma_{I+I^c}) e_I \]
\[ = \text{sgn} \left( \sigma_{I+c+I} \sigma_{I+I^c}^{-1} \right) e_I = (-1)^{k(2m-k)} e_I = (-1)^k e_I, \]
since the permutation $\sigma_{I^c} \cdot I$ is the one sending $I^c I$ to $I^c I$, and thus merely exchanges a block of $k$ indices and a block of $n - k$ indices. Since the identity holds on the basis, it holds on the entire space.

3. Consider the case $v = c_1 e_I$ and $w = c_2 e_I$ for $c_1, c_2 \in \mathbb{C}$. Since the inner product is sesquilinear, and $*$ is linear, it is sufficient to consider only multiples of the basis, and if $v$ and $w$ are not multiples the same basis vector, then orthogonality will make the inner product 0. Thus consider only

$$c_1 e_I \wedge * c_2 e_I = c_1 c_2 \operatorname{sgn}(\sigma_{I^c}) e_I \wedge e_{I^c} = c_1 c_2 e_{N_m} = c_1 c_2 * 1 = \langle c_1 e_I, c_2 e_I \rangle * 1$$

and the proposition is proven.

Let $E$ be a vector bundle on the manifold $X$. Let $\beta = \{e_1, \ldots, e_m\}$ be a local frame, with dual frame $\{e_1^*, \ldots, e_m^*\}$. Then if $s \in \Gamma(X, \mathcal{A}^{p,q}(E))$ such that $s = v_I \otimes e_I$ (for $v_I$ a locally-defined $(p,q)$-form on $X$), then define

$$*s = (*v_I) \otimes e_I^*.$$

To extend the properties of $*$ on a vector space to $*$ on sections, first define a new operator $\wedge_E$. If $s^{(1)} = v_I \otimes e_I \in \Gamma(X, \mathcal{A}^{p,q}(E))$ and $s^{(2)} = w_J \otimes e_J^* \in \Gamma(X, \mathcal{A}^{p,q}(E^*))$, then

$$s^{(1)} \wedge_E s^{(2)} = v_I \wedge w_I.$$

**Proposition 2.2.** Let $s^{(j)} = v_i^{(j)} \otimes e_i \in \Gamma(X, \mathcal{A}^{p,q}(E))$ for $j = 1, 2$. Then $*$ has the following properties.
1. \( \ast \ast s^{(1)} = (-1)^{p+q}s^{(1)} \).

2. \( \langle s^{(1)}, s^{(2)} \rangle_1 = s^{(1)} \wedge_E \ast s^{(2)} \) where \( \langle \cdot, \cdot \rangle \) is the Hermitian inner product on \( \mathcal{A}^{p,q}(X) \).

**Proof.** Both statements follow from the properties of \( \ast \) on \( \mathcal{A}^{p,q}(X) \). First,

\[
\ast \ast s^{(1)} = \ast \left( \ast v^{(1)}_i \wedge_E e_i^* \right) = \ast \ast v^{(1)}_i \wedge_E e_i^{**} = (-1)^{p+q}v^{(1)}_i \wedge e_i = (-1)^{p+q}s^{(1)}.
\]

For the second statement,

\[
\langle s^{(1)}, s^{(2)} \rangle_1 \ast 1 = \langle v^{(1)}_i, v^{(2)}_i \rangle_1 \ast 1
\]

and

\[
s^{(1)} \wedge_E \ast s^{(2)} = v^{(1)}_i \wedge *v^{(2)}_i,
\]

so the statement follow immediately from the original property on the forms \( v^{(1)}_i \) and \( v^{(2)}_i \). \( \square \)

Let \( \partial^* \) and \( \bar{\partial}^* \) denote the dual operators of \( \partial \) and \( \bar{\partial} \), respectively, with respect to the global inner product

\[
(\phi, \psi) = p! \cdot q! \int_X \langle \phi, \psi \rangle dV_g.
\]

These dual operators can be written in terms of the original operators and \( \ast \).

**Lemma 2.1.** Let \( X \) be a Kähler manifold. Then \( \bar{\partial}^* = -\ast \partial^* \) and \( \partial^* = -\ast \bar{\partial}^* \).

**Proof.** Let \( \phi \in \Gamma(X, \mathcal{A}^{p,q}(E)) \), \( \psi \in \Gamma(X, \mathcal{A}^{p,q+1}(E)) \). Then for some appropriate constant \( C \),

\[
(\phi, -\ast \partial \ast \psi) = C \int_X \langle \phi, -\ast \partial \ast \psi \rangle \ast 1 = \int_X -C(-1)^{(n-p+1)+(n-q-1)} \phi \wedge_E \bar{\partial} \ast \psi.
\]
Thus \(- \ast \partial \ast = \overline{\partial}\). Taking the conjugate of this immediately gives \(- \ast \overline{\partial} \ast = \partial \ast \).

This leads to the most important theorem relating to the \(*\)-operator: the Serre Duality, here stated slightly differently from the original statement in Chapter 1.

**Theorem 2.2** (Serre Duality). Let \(E\) be a Hermitian vector bundle over \(X\) and let \(0 \leq p, q \leq n\). Then \(\tilde{*} := \ast\overline{\ast}\) induces an isomorphism between \(H^q(X, \Omega^p(E))\) and \(H^{n-q}(X, \Omega^{n-p}(E^*))\).

Portions of the proof of this theorem will be relevant later.

**Proof.** First, prove that \(\tilde{*}\) is an isomorphism on \(\Gamma(X, \mathcal{A}^{p,q}(E))\). Let \(s^{(1)} = v_I \otimes e_I \in \Gamma(X, \mathcal{A}^{p,q}(E))\) and \(s^{(2)} = w_J \otimes e_j^* \in \Gamma(X, \mathcal{A}^{n-q,n-p}(E^*))\).

If \(0 = \tilde{*} s^{(1)}\), then

\[
0 = (-1)^{p+q} \tilde{*} 0 = (-1)^{p+q} \tilde{*} s^{(1)} = (-1)^{p+q} \ast \overline{s^{(1)}} = (-1)^{p+q} \ast \overline{s^{(1)}} = s^{(1)}.
\]

Hence \(s^{(1)} = 0\), and therefore \(*\) is injective. Similarly,

\[
s^{(2)} = (-1)^{(n-p)+(n-q)} \ast \overline{s^{(2)}} = (-1)^{p+q} \tilde{*} s^{(2)} = \tilde{*} (-1)^{p+q} \ast \overline{s^{(2)}} = \tilde{*} (-1)^{p+q} \ast \overline{s^{(2)}} = 0,
\]

and thus every element of \(\Gamma(X, \mathcal{A}^{n-q,n-p}(E^*))\) is in the image of \(\tilde{*}\). \(\tilde{*}\) is thus surjective, and therefore an isomorphism. It remains to show that this induces an isomorphism on the cohomology groups.

Consider the action of \(\tilde{*}\) on \(\mathcal{H}^{p,q}(E)\), the space of harmonic \((p,q)\)-forms. Let \(\phi \in \mathcal{H}^{p,q}(E)\). Then

\[
\overline{\partial} \tilde{*} \phi = (-1)^{(n-p)+(n-q-1)} \ast \overline{\partial} \ast \overline{\phi} = (-1)^{p+q} \ast \overline{\partial} \ast \overline{\phi} = (-1)^{p+q} \ast \overline{\partial} \ast \overline{\phi} = 0,
\]
since harmonic forms are $\partial^*$-closed. Similarly,

$$\bar{\partial}^* \phi = -\ast \partial \ast \bar{\phi} = (-1)^{p+q+1} \ast (\bar{\partial} \bar{\phi}) = 0,$$

since harmonic forms are $\partial$-closed. Thus $\ast$ is both $\partial$-closed and $\partial^*$-closed, and is therefore harmonic. Hence

$$\ast : \mathcal{H}^{p,q}(E) \to \mathcal{H}^{n-p,n-q}(E^*)$$

is well-defined. By similar argument,

$$\ast : \mathcal{H}^{n-p,n-q}(E^*) \to \mathcal{H}^{p,q}(E)$$

is also well-defined. Since both spaces are finite-dimensional by the Hodge Theorem, and there are injective maps between them, they must have the same dimension, and thus the injective maps are bijective.

Also by the Hodge Theorem, every element of both $H^q(X, \Omega^p(E))$ and $H^{n-q}(X, \Omega^{n-p}(E^*))$ has a harmonic representative, and thus the isomorphism between the harmonic representatives induces an isomorphism on the equivalence classes, as desired.

\[\square\]

### 2.2 Deformations of Complex Manifolds

#### 2.2.1 Introduction

This section is based on the lectures of Kunihiko Kodaira, as presented in his book with James Morrow ([31]).

**Definition 2.1.** The triple $(\mathcal{X}, \pi, B)$ is called a family of compact, complex manifolds if:
1. $\mathcal{X}$ and $B$ are complex manifolds;

2. The map $\pi : \mathcal{X} \to B$ is holomorphic;

3. For every $t \in B$, $X_t = \pi^{-1}(t)$ is a connected, compact, complex manifold; and

4. $\pi_*$, the differentiation of $\pi$, is of full rank at every $x \in \mathcal{X}$.

Additionally, $\mathcal{X}$ is called the total space, $B$ is quite naturally called the parameter space, and $X_t$ – the actual “manifolds” of the family of manifolds – are called the fibers.

**Definition 2.2.** Let $X$ and $Y$ be compact, complex manifolds. Then $Y$ is said to be a deformation of $X$ if there exists a family of compact, complex manifolds such that both $X$ and $Y$ are fibers.

Of particular relevance is local deformations: deformations that are ”close” to a manifold $X$ in the sense that their base point in the parameter space is close to the base point of $X$. When only considering local deformations, we can assume without loss of generality that $B$ is an open neighborhood of $\mathbb{C}^m$ at the origin, where $m$ is the complex dimension of $B$, that $X = X_0$, the fiber at the origin, and that $\dim X = n$.

By convention, $t = (t^1, \ldots, t^m)$ will denote the local coordinates of $B$ and $z = (z^1, \ldots, z^n)$ will denote the local coordinates of the fiber $X$.

Under this notation, the infinitesimal deformation of $X$ is

$$\frac{dX_t}{dt} \bigg|_{t=0}.$$

It is far more useful to consider infinitesimal deformations in the context of cohomology groups, in particular $H^1(X, \Theta)$, where $\Theta$ is the sheaf of germs of holomorphic $(1,0)$-vector fields on $X$. 

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To begin, consider the simple case of \( m = 1 \), and \( B = B(0; \epsilon) \), the open ball of radius \( \epsilon \) around the origin of \( \mathbb{C} \). Let \( \cup_{\alpha} U_{\alpha} \) be an atlas of \( \pi^{-1}(B) \) such that

\[
(z_{\alpha}^{1}(x), \ldots, z_{\alpha}^{n}(x), t_{\alpha}(x))
\]

is a holomorphic coordinate system on \( U_{\alpha} \). Then on \( U_{\alpha} \cup U_{\beta} \neq \emptyset \), there exist transition functions

\[
z_{\alpha}^{i}(x) = f_{\alpha\beta}^{i}(z_{\beta}, t_{\beta}),
\]

for \( i = 1, \ldots, n \). The complex structure of \( X_{t} \) is determined by the collection of such transition functions for fixed \( t \). In that sense, the variation of complex structure is determined by the derivatives of those functions. Let

\[
\theta_{\alpha\beta} = \sum_{i=1}^{n} \left. \frac{\partial f_{\alpha\beta}^{i}(z_{\beta}, t_{\beta})}{\partial t_{\beta}} \right|_{t=0} \cdot \frac{\partial}{\partial z_{\alpha}^{i}}.
\]

**Lemma 2.2.** Let \( \theta_{\alpha\beta} \) be defined as above. Then on \( X \cap U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \),

\[
\theta_{\alpha\gamma} = \theta_{\alpha\beta} + \theta_{\beta\gamma}.
\]

This follows immediately from the chain rule.

**Lemma 2.3.** \((\theta_{\alpha\beta})\) defines a cocycle in the cochain group \( C^{1}(\{U_{\alpha} \cap X\}, \Theta) \).

**Proof.** By the preceding lemma,

\[
\theta_{\alpha\gamma} = \theta_{\alpha\beta} + \theta_{\beta\gamma}.
\]
Setting $\gamma = \alpha$ gives

$$0 = \theta_{\alpha\alpha} = \theta_{\alpha\beta} + \theta_{\beta\alpha},$$

so $\theta_{\alpha\beta} = -\theta_{\beta\alpha}$. It then immediately follows that

$$0 = \theta_{\alpha\beta} + \theta_{\beta\gamma} - \theta_{\alpha\gamma} = \theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha},$$

so $\theta_{\alpha\beta}$ satisfies the cocycle conditions.

$\theta_{\alpha\beta}$ thus represents a cohomology class, but we must first show that this choice of cohomology class $(\theta_{\alpha\beta}) \in \check{H}^1(X, \Theta)$ is well-defined with respect to the cover $U_\alpha$ and the choice of local coordinates.

Let $\{V_\beta\}$ be a different open cover with transition functions

$$w^i_\alpha(x) = g^i_{\alpha\beta}(w_\beta, t_\beta).$$

This defines

$$\hat{\theta}_{\beta\gamma} = \sum_{i=1}^n \left. \frac{\partial g^i_{\beta\gamma}(w_\beta, t_\gamma)}{\partial t_\gamma} \right|_{t=0} \cdot \frac{\partial}{\partial w^i_\beta}.$$

The cover $\{U_\alpha \cap V_\beta\}$ is a refinement of both over covers. Reindex this cover and call it $\{W_k\}$. The transition functions are then also naturally restricted to this new cover, and thus so are the cocycles. Let $h_k$ be the function mapping the first coordinates to the second coordinates on $W_k$. That is,

$$w^i_i = h^i_k(z_k, t_k)$$
for each $i = 1, \ldots, n$. Thus whenever $W_k \cap W_l \neq \emptyset$,

$$h^i_k(f_{kl}(z_l, t_l), t_k) = g^i_{kl}(h_l(z_l, t_l), t_k).$$

Taking the derivative with respect to $t$ and using the chain rule gives

$$\frac{\partial h^i_k}{\partial t_k} + \frac{\partial h^i_k}{\partial z^j_k} \cdot \frac{\partial f_{kl}}{\partial t_l} = \frac{\partial g^i_{kl}}{\partial t_k} + \frac{\partial g^i_{kl}}{\partial w^l_k} \cdot \frac{\partial h^j_l}{\partial t_l}. $$

Multiplying everything by $\frac{\partial}{\partial w^i_l}$ then gives

$$\frac{\partial h^i_k}{\partial t_k} \cdot \frac{\partial}{\partial w^i_l} + \frac{\partial h^i_k}{\partial z^j_k} \cdot \frac{\partial f_{kl}}{\partial t_l} \cdot \frac{\partial}{\partial w^i_l} = \frac{\partial g^i_{kl}}{\partial t_k} \cdot \frac{\partial}{\partial w^i_k} + \frac{\partial g^i_{kl}}{\partial w^l_k} \cdot \frac{\partial h^j_l}{\partial t_l} \cdot \frac{\partial}{\partial w^i_l}.$$

which by chain rule reduces to

$$\frac{\partial h^i_k}{\partial t_k} \cdot \frac{\partial}{\partial w^i_l} + \frac{\partial f_{kl}}{\partial t_l} \cdot \frac{\partial}{\partial z^j_k} = \frac{\partial g^i_{kl}}{\partial t_k} \cdot \frac{\partial}{\partial w^i_k} + \frac{\partial}{\partial t_l} \cdot \frac{\partial}{\partial w^l_k}.$$

Rearranging and evaluating at $t = 0$ gives

$$\theta^i_{kl} - \hat{\theta}_{kl} = \frac{\partial h^i_k}{\partial t_k} \bigg|_{t=0} \cdot \frac{\partial}{\partial w^i_l} - \frac{\partial h^i_k}{\partial t_k} \bigg|_{t=0} \cdot \frac{\partial}{\partial w^l_k}. $$

Since the first term on the right is defined on $W_k$ and the second one is defined on $W_l$, this is a coboundary on $W_k \cap W_l$. Hence $\theta_{kl}$ and $\hat{\theta}_{kl}$ are both representatives of the same cohomology class, and therefore $(\theta_{\alpha\beta}) \in \hat{H}^1(X, \Theta)$ is well-defined.

**Definition 2.3.** The infinitesimal deformation

$$\left( \frac{dX_t}{dt} \right) \bigg|_{t=0}$$

is the cohomology class $(\theta_{\alpha\beta}) \in \hat{H}^1(X, \Theta).$
In the case where \( m > 1 \), where \( t = (t^1, \ldots, t^m) \), then this just gives

\[
\left. \left( \frac{\partial X_i}{\partial t^i} \right) \right|_{t=0}
\]

as the infinitesimal deformation in the direction \( \frac{\partial}{\partial t^i} \). This map

\[
\frac{\partial}{\partial t^i} \mapsto \left. \left( \frac{\partial X_i}{\partial t^i} \right) \right|_{t=0}
\]

is called the Kodaira-Spencer map. Due to the isomorphism between the Čech and Dolbeault cohomologies, the equivalent map from the tangent space to the Dolbeault cohomology group is also called the Kodaira-Spencer map.

### 2.2.2 The Lie superbracket

Now that there is some framework for discussing deformations, a bit more notation is required.

**Definition 2.4.** Let \( X \) be a smooth, complex manifold, and let \( \nu \in \mathcal{A}^{0,p}(T^{(1,0)}(X)) \) and \( \psi \in \mathcal{A}^{0,q}(T^{(1,0)}(X)) \), locally defined by

\[
\nu = \nu^i J \frac{\partial}{\partial z^i} d\bar{z}^J,
\]

\[
\psi = \psi^k L \frac{\partial}{\partial z^k} d\bar{z}^L
\]

Then the Lie superbracket is defined as

\[
[v, \psi] := \left( \nu^i J \frac{\partial \psi^k L}{\partial z^i} \frac{\partial}{\partial z^k} - \psi^k L \frac{\partial \nu^i J}{\partial z^k} \frac{\partial}{\partial z^i} \right) d\bar{z}^J \wedge d\bar{z}^L.
\]

The Lie superbracket is trivially linear with respect to addition and multiplication by con-
Proposition 2.3. Let \( v \in A^{0,p}(T^{(1,0)}(X)) \), \( \psi \in A^{0,q}(T^{(1,0)}(X)) \), and \( \chi \in A^{0,r}(T^{(1,0)}(X)) \), the Lie superbracket has the following properties.

1. \([v, \psi] = -(1)^{pq}[\psi, v]\). In particular, when \( p = q = 1 \), \([v, \psi] = [\psi, v]\).

2. \((-1)^{pr}[v, [\psi, \chi]] + (-1)^{qp}[\psi, [\chi, v]] + (-1)^{rq}[\chi, [v, \psi]] = 0\). In particular, \([v, [v, v]] = 0\); and when \( p = q = r = 1 \), \([v, [\psi, \chi]] + [\psi, [\chi, v]] + [\chi, [v, \psi]] = 0\).

3. For \( f \in C(X) \), \([fv, \psi] = f[v, \psi] - (-1)^{pq}\psi(f) \wedge v\) and \([v, f\psi] = f[v, \psi] + v(f) \wedge \psi\). In particular, when \( p = q = 1 \), \([fv, \psi] = f[v, \psi] + \psi(f) \wedge v\).

4. \(\overline{\partial}[v, \psi] = [\overline{\partial}v, \psi] + (-1)^{p}[v, \overline{\partial}\psi]\). In particular, \([v, v] \) is \(\overline{\partial}\)-closed.

The first two properties justify calling this operation a Lie superbracket.

Proof. Proving this proposition is mostly a matter of tedious computation.

\[
[v, \psi] = \left( \psi^L_J \frac{\partial v^k_L}{\partial z^i} \frac{\partial}{\partial z^k} - v^k_L \frac{\partial \psi^L_J}{\partial z^i} \frac{\partial}{\partial z^k} \right) dz^J \wedge dz^L
\]

\[= -(1)^{pq} \left( v^k_L \frac{\partial \psi^L_J}{\partial z^k} \frac{\partial}{\partial z^i} - \psi^L_J \frac{\partial v^k_L}{\partial z^k} \frac{\partial}{\partial z^i} \right) dz^L \wedge dz^J.
\]

This proves the first statement. For the second statement,

\[
[v, [\psi, \chi]] = \left( v^m_N \frac{\partial \psi^i_J}{\partial z^m} \frac{\partial \chi^k_L}{\partial z^i} - v^m_N \frac{\partial \chi^i_L}{\partial z^m} \frac{\partial \psi^k_J}{\partial z^i} - \psi^m_N \frac{\partial \chi^i_L}{\partial z^m} \frac{\partial \psi^k_J}{\partial z^i} + \chi^m_L \frac{\partial \psi^i_J}{\partial z^m} \frac{\partial \chi^k_L}{\partial z^i} \right) \frac{\partial}{\partial z^k} dz^N \wedge dz^J \wedge dz^L
\]
and

\[
[\psi, [\chi, \nu]] = \left( \psi^m_j \frac{\partial v^i_k}{\partial z^m} \frac{\partial \nu^j_l}{\partial z^i} - \psi^m_j \frac{\partial \nu^i_k}{\partial z^m} \frac{\partial \psi^j_l}{\partial z^i} + \nu^m_j \frac{\partial \chi^i_l}{\partial z^m} \frac{\partial \nu^j_k}{\partial z^i} \right. \\
+ \psi^m_j \frac{\partial^2 v^i_k}{\partial z^i \partial z^m} - \psi^i_j \nu^m_k \frac{\partial^2 \psi^j_l}{\partial z^i \partial z^m} \right) \frac{\partial}{\partial z^k} d\bar{z}^l \wedge d\bar{z}^m \wedge d\bar{z}^N.
\]

Thus,

\[
[v, [\psi, \chi]] + (-1)^{pq+r} [\psi, [\chi, \nu]] = \left( \chi^m_L \frac{\partial \psi^i_k}{\partial z^m} \frac{\partial \nu^j_l}{\partial z^i} - \chi^m_L \frac{\partial \nu^i_k}{\partial z^m} \frac{\partial \psi^j_l}{\partial z^i} + \nu^m_L \frac{\partial \chi^i_l}{\partial z^m} \frac{\partial \nu^j_k}{\partial z^i} \right. \\
- \psi^m_j \frac{\partial \nu^i_k}{\partial z^m} \frac{\partial \chi^j_l}{\partial z^i} + \psi^m_j \frac{\partial^2 \nu^i_k}{\partial z^i \partial z^m} - \psi^i_j \nu^m_k \frac{\partial^2 \psi^j_l}{\partial z^i \partial z^m} \right) \frac{\partial}{\partial z^k} d\bar{z}^N \wedge d\bar{z}^l \wedge d\bar{z}^N,
\]

which is just \(-(-1)^{r(p+q)} [\chi, [v, \psi]]\). Therefore

\[
(-1)^{pr} [v, [\psi, \chi]] + (-1)^{qr} [\psi, [\chi, v]] + (-1)^{rq} [\chi, [v, \psi]]
\]

\[
= (-1)^{pr} \left( [v, [\psi, \chi]] + (-1)^{pq+r} [\psi, [\chi, v]] + (-1)^{r(p+q)} [\chi, [v, \psi]] \right) = 0.
\]

For the third statement,

\[
[f \nu, \psi] = \left( f \nu^i_j \frac{\partial \psi^k}{\partial z^i} \frac{\partial \nu^l}{\partial z^k} - \psi^k \frac{\partial (f \nu^i_j)}{\partial z^k} \frac{\partial \nu^l}{\partial z^i} \right) d\bar{z}^l \wedge d\bar{z}^N
\]

\[
= \left( f \nu^i_j \frac{\partial \psi^k}{\partial z^i} \frac{\partial \nu^l}{\partial z^k} - \psi^k \frac{\partial (f \nu^i_j)}{\partial z^k} \frac{\partial \nu^l}{\partial z^i} - \psi^k \nu^i_j \frac{\partial f}{\partial z^i} \frac{\partial \nu^l}{\partial z^k} \right) d\bar{z}^l \wedge d\bar{z}^N
\]

\[
= f [v, \psi] - (-1)^{pq} \psi(f) \wedge \nu.
\]

It follows immediately from the first statement that

\[
[v, f \psi] = -(-1)^{pq} [f \psi, \nu] = -(-1)^{pq} f [\psi, \nu] + \nu(f) \wedge \psi = f [v, \psi] + \nu(f) \wedge \psi.
\]
Finally,

\[
\overline{\partial}[v, \psi] = \left( \frac{\partial}{\partial \bar{z}^j} \left( \nu^i \frac{\partial \psi^k_L}{\partial z^i} \right) \frac{\partial}{\partial z^k} - \frac{\partial}{\partial \bar{z}^i} \left( \psi^k_L \frac{\partial \nu^i}{\partial z^k} \right) \frac{\partial}{\partial z^k} \right) d\bar{z}^j \wedge d\bar{z}^k \wedge dz^L
\]

\[
= \left( \frac{\partial \nu^i}{\partial \bar{z}^j} \frac{\partial \psi^k_L}{\partial z^j} \frac{\partial}{\partial z^k} - \psi^k_L \frac{\partial^2 \nu^i}{\partial \bar{z}^j \partial z^k} \frac{\partial}{\partial z^k} \right) d\bar{z}^j \wedge d\bar{z}^k \wedge dz^L
\]

\[
+ (-1)^p \left( \frac{\partial^2 \psi^k_L}{\partial \bar{z}^j \partial z^k} \frac{\partial}{\partial z^i} - \frac{\partial \nu^i}{\partial \bar{z}^j} \frac{\partial \psi^k_L}{\partial z^k} \frac{\partial}{\partial z^i} \right) d\bar{z}^j \wedge d\bar{z}^k \wedge dz^L
\]

\[
=[\overline{\partial}v, \psi] + (-1)^p[v, \overline{\partial}\psi]
\]

For the final special case,

\[
\overline{\partial}[v, v] = [\overline{\partial}v, v] + (-1)^p[v, \overline{\partial}v] = [\overline{\partial}v, v] - (-1)^{p^2 + p}[\overline{\partial}v, v] = 0.
\]

\[
\quad \square
\]

2.2.3 Deformation Theorems: Existence

Proving that infinitesimal deformations exist comes down to solving a particular differential equation.

**Theorem 2.1.** Let \( \phi(t) \) be a family of \( C^1 \mathcal{T}^{(1,0)}(X) \)-valued \((0,1)\)-forms that is holomorphic with respect to \( t \). Let \( \overline{\partial} \) be the d-bar operator on \( X \). Let \( \theta \) be a \( \overline{\partial} \)-closed \( T^{(1,0)}(X) \)-valued \((0,1)\)-form. Let \( \epsilon > 0 \). If for \(|t| < \epsilon\),

\[
\overline{\partial}\phi(t) - \frac{1}{2} [\phi(t), \phi(t)] = 0, \quad \frac{\partial \phi(t)}{\partial t} = 0, \quad \phi(0) = 0, \quad \phi'(0) = \theta
\]

then there exists a compact, complex family \((\mathcal{X}, \pi, B)\) such that

1. \( B \) is an open neighborhood of \( \mathbb{C} \) at the origin;
2. $X = \pi^{-1}(0)$;

3. The infinitesimal deformation of the family $(\mathfrak{X}, \pi, B)$ is $[\theta] \in H^1(X, \Theta)$, where $\Theta$ is the sheaf of germs of holomorphic $(1,0)$-vector fields on $X$.

If this differential equation is solvable for every $[\theta] \in H^1(X, \Theta)$, then the deformations of $X$ are said to be unobstructed. There are two main theorems for unobstructed deformations: the Kodaira-Nirenberg-Spencer Theorem and the Tian-Todorov Theorem.

**Theorem 2.3** (Kodaira-Nirenberg-Spencer [31]). Let $X$ be a compact, complex manifold. Assume that $H^2(X, \Theta) = 0$. Then for every $[\theta] \in H^1(X, \Theta)$, there is a family of compact, complex manifolds such that the infinitesimal deformation of the family is $[\theta]$.

Since $H^2(X, \Theta) = 0$ is a sufficient condition for unobstructed deformations, this group is called the obstruction group.

Even though most manifolds do not have a trivial obstruction group, the Kodaira-Nirenberg-Spencer Theorem is enough to prove that certain classes of manifolds have unobstructed deformations.

**Definition 2.5** (Fano Manifold). A complex, projective manifold $X$ of complex dimension $n$ is called a Fano manifold if its anticanonical bundle $K_X^{-1} = \bigwedge^n T^{(1,0)}(X)$ is ample.

That definition is not particularly useful in this situation, so instead use an equivalent one provided by the Kodaira Embedding Theorem[22].

**Definition 2.6** (Fano Manifold, Version 2). A complex, projective manifold $X$ of complex dimension $n$ is called a Fano manifold if its anticanonical bundle $K_X^{-1} = \bigwedge^n T^{(1,0)}(X)$ is positive. That is, $X$ is Fano if there exists a smooth hermitian metric on $K_X^{-1}$ such that the corresponding first Chern class is positive.
With this definition, it’s now possible to prove:

**Theorem 2.4.** *Fano manifolds have unobstructed deformations.*

Proving this requires one more theorem.

**Theorem 2.5** (Kodaira-Nakano Vanishing Theorem[33]). *Let* $X$ *be an* $n$-*dimensional compact complex manifold, and let* $L$ *be a positive holomorphic line bundle over* $X$. *Then*

$$H^q(X, \Omega^p \otimes L) = 0$$

*whenever* $p + q \geq n + 1$.

We are now ready to prove Theorem 2.4.

*Proof of Theorem 2.4.* Assume $X$ is Fano. By twice applying the Serre Duality (Theorem 2.2),

$$H^2(X, \Theta) = H^{n-2}(X, \Omega^n \otimes \Omega^1) = H^2(X, \Omega^{n-1}(K_X^{-1})).$$

However, since $X$ is Fano, then by the Kodaira-Nakano Vanishing Theorem

$$H^2(X, \Omega^{n-1}(K_X^{-1})) = 0$$

because $2 + (n - 1) = n + 1 \geq n + 1$. Hence the obstruction group is trivial, and by the Kodaira-Nirenburg-Spence Theorem, the infinitesimal deformations are unobstructed.

The Calabi-Yau case is covered by the other major existence theorem:

**Theorem 2.6** (Tian [43], Todorov [45]). *Let* $X$ *be a Calabi-Yau manifold. Then for each* $[\theta] \in H^1(X, \Theta)$, *there is a family of compact, complex manifolds such that the infinitesimal deformation of the family is* $[\theta]$. 

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2.2.4 Hölder Norms and the Green’s Operator

Following the standard notation of Gilbarg and Trudinger[11], let \( \mathcal{U} = \{ U_\zeta, x_\zeta \} \) be an atlas on any 2n-dimensional manifold (most importantly, including n-dimensional complex manifolds). For any multi-index \( I \in \mathbb{Z}_{\geq 0}^{2n} \), define

\[
D^I_\zeta := \left( \frac{\partial}{\partial x_1^\zeta} \right)^{i_1} \cdots \left( \frac{\partial}{\partial x_{2n}^\zeta} \right)^{i_{2n}}
\]

Then for \( \phi \in \mathcal{A}^{p,q} \left( \wedge^u T(1,0)(X) \right) \) where

\[
\phi = \phi^I_{\zeta,K,J} \frac{\partial}{\partial z^K} dz^J \wedge dz^L,
\]

\( k \in \mathbb{Z}_{\geq 0} \), and \( 0 \leq \alpha \leq 1 \), the Hölder norm is

\[
\| \phi \|_{k,\alpha} := \max_{\zeta} \left\{ \sum_{|L| \leq k} \left[ \sup_{z \in U_\zeta} \left| D^L_\zeta \phi^I_{\zeta,K,J}(z) \right| + \sup_{y,z \in U_\zeta} \left| \frac{D^L_\zeta \phi^I_{\zeta,K,J}(y) - D^L_\zeta \phi^I_{\zeta,K,J}(z)}{|y - z|^\alpha} \right| \right] \right\}.
\]

If \( \| \phi \|_{k,\alpha} \) is finite, we say \( \phi \in \mathcal{C}^{k,\alpha} \). Additionally, the special case of \( \| \cdot \|_{0,0} \) is just double the supremum norm.

The Hölder norm satisfies all standard properties of a norm, particularly the Triangle Inequality, and it has a few additional properties.

**Proposition 2.4.** The Hölder norm satisfies

- \( \| \overline{\partial} \phi \|_{k,\alpha} \leq C \| \phi \|_{k+1,\alpha} \)
- \( \| \partial \phi \|_{k,\alpha} \leq C \| \phi \|_{k+1,\alpha} \)
- \( \| \partial^* \phi \|_{k,\alpha} \leq C \| \phi \|_{k+1,\alpha} \)
- \( \| \overline{\partial}^* \phi \|_{k,\alpha} \leq C \| \phi \|_{k+1,\alpha} \)
where $C$ is different between each line and is independent of $\phi$ and $\nu$, but may depend on $k$, $\alpha$, $n$, and $q$.

Hölder norms provide a particularly potent tool when combined with Laplacians and Green’s operators. Let $X$ be a compact, complex manifold with Hermitian metric $g$. Let $\overline{\partial}^{\ast}$ be the dual operator of $\partial$ on $T^{(1,0)}(X)$ with respect to the metric $g$. Then the Hodge Laplacian on $\mathcal{A}^{p,q}(T^{(1,0)}(X))$ is

$$\Delta = \overline{\partial}^{\ast} \partial + \overline{\partial} \partial^{\ast}.$$ 

Recall that $\mathcal{H}^{p,q}$ (or just $\mathcal{H}$ when $p$ and $q$ are clear from context) is the set of harmonic elements of $\mathcal{A}^{p,q}(T^{(1,0)}(X))$. Define $H$ to be the orthogonal projection onto $\mathcal{H}$.

**Proposition 2.5.** $\mathcal{H}^{p,q}$ has the following properties.

- $\mathcal{H}^{p,q} = \ker \overline{\partial} \cap \ker \overline{\partial}^{\ast}$;
- $\mathcal{H}^{p,q} \perp \text{Im} \overline{\partial}$; and
- $\mathcal{H}^{p,q} \perp \text{Im} \partial^{\ast}$.

**Proof.** If $\phi \in \ker \overline{\partial} \cap \ker \overline{\partial}^{\ast}$, then

$$\Delta \phi = \overline{\partial} \partial^{\ast} \phi + \overline{\partial}^{\ast} \partial \phi = 0,$$

so $\ker \overline{\partial} \cap \ker \overline{\partial}^{\ast} \subseteq \mathcal{H}^{p,q}$. Conversely, if $\phi \in \mathcal{H}^{p,q}$, then

$$0 = (\Delta \phi, \phi) = (\overline{\partial} \partial^{\ast} \phi, \phi) + (\overline{\partial}^{\ast} \partial \phi, \phi) = \| \partial \phi \|^2 + \| \overline{\partial} \phi \|^2.$$
Since both $\|\partial^* \phi\|^2$ and $\|\partial \phi\|^2$ are non-negative, then the must both be zero, and therefore $\mathcal{H}^{p,q} \subseteq \ker \partial \cap \ker \partial^*$. Hence the first statement is proven.

The second and third statements follow immediately from the first statement, $\ker \partial \perp \text{Im} \partial^*$, and $\ker \partial^* \perp \text{Im} \partial$.

The Green’s operator for the Hodge Laplacian is the operator

$$G : (\mathcal{H}^{p,q})^\perp \to (\mathcal{H}^{p,q})^\perp$$

such that $I = H + \Delta G$ once $G$ is extended linearly to the entire space by $G(\mathcal{H}^{p,q}) = \{0\}$.

**Proposition 2.6.** The Green’s operator for the Hodge Laplacian has the following properties.

- $G\partial = \partial G$;
- $G\partial^* = \partial^* G$; and
- $G\Delta = \Delta G$.

**Proof.** For any $\phi \in \mathcal{A}^{p,q}(\mathbf{T}^{(1,0)}(X))$, by Proposition 2.5,

$$0 = (\partial I - I\partial)\phi = (\partial(H + \Delta G) - (H + \Delta G)\partial)\phi = \Delta(\partial G - G\partial)\phi.$$

Therefore $(\partial G - G\partial)\phi \in \mathcal{H}^{p,q}$. Since both $\partial G\phi$ and $G\partial\phi$ have no harmonic component, they must be equal. Hence the first statement is proven.

The proof of the second statement is analogous to the proof of the first statement, and the third statement follows immediately from the first two.

The interaction of the Green’s operator with the Hőlder norms is based on Schauder Estimates.
Theorem 2.7 (Schauder Estimate for the Hodge Laplacian). Let \( k \) be a non-negative integer and \( \alpha \in (0, 1) \). Let \( \phi \in \mathcal{A}^{p,q} \left( T^{(1,0)}(X) \right) \). Then for \( C \) independent of \( \phi \),

\[
\| \phi \|_{k+2,\alpha} \leq C \left( \| \Delta \phi \|_{k,\alpha} + \| \phi \|_{0,0} \right).
\]

This is a special case of the Schauder estimates for elliptic operators with bounded coefficients as found in [11] and produces an estimate for the norm of the Green’s operator of a function.

Lemma 2.4. Let \( k \) be a non-negative integer and \( \alpha \in (0, 1) \). Let \( \phi \in \mathcal{A}^{(0,1)} \left( T^{(1,0)}(X) \right) \). Then for \( C \) independent of \( \phi \),

\[
\| G\phi \|_{k+2,\alpha} \leq C \| \phi \|_{k,\alpha}.
\]

Notice that this trivially holds for components of \( \phi \) that are in \( \mathcal{H} \). This also immediately provides a better estimate for \( \| \phi \|_{k+2,\alpha} \) when \( \phi \in \mathcal{H}^\perp \):

Corollary 2.1. Let \( k \) be a non-negative integer and \( \alpha \in (0, 1) \). Let \( \phi \in (\mathcal{H}^{p,q})^\perp \). Then for \( C \) independent of \( \phi \),

\[
\| \phi \|_{k+2,\alpha} \leq C \| \Delta \phi \|_{k,\alpha}.
\]

Proof. For \( \phi \in \mathcal{H}^{p,q} \),

\[
\| \phi \|_{k+2,\alpha} = \| (I - H)\phi \|_{k+2,\alpha} = \| \Delta G\phi \|_{k+2,\alpha} = \| G\Delta \phi \|_{k+2,\alpha} \leq C \| \Delta \phi \|_{k,\alpha}.
\]

The Hölder norms have a final useful property.
Theorem 2.8 ([14]). For non-negative \( p, q, u, \) and \( k \), and \( \alpha \in (0,1) \), \( \mathcal{A}^{p,q}(\bigwedge^u T^{(1,0)}(X)) \) is a Banach space with respect to \( \| \cdot \|_{k,\alpha} \).

2.2.5 Deformation Theorems: Completeness and Stability

Before proving sufficient conditions for the existence of infinitesimal deformations of manifolds of Calabi-Yau type, consider two final theorems of deformations.

Theorem 2.9 (Kuranishi’s Completeness Theorem[23][24]). Let \( X \) be a compact complex manifold. Let \( m = \dim H^1(M, \Omega) \), \( \{ \beta_k \}_{k=1}^m \) be a basis of \( \mathcal{H} \), \( B = B(0; \epsilon) \subset \mathbb{C}^m \), and \( \eta(t) = \sum_{k=1}^m \eta_k t_k \). Let \( \phi(t) \) be a solution of

\[
\phi(t) = \eta(t) + \frac{1}{2} \bar{\partial}G[\phi(t), \phi(t)],
\]

and let

\[
B' = \{ t \in B | H[\phi(t), \phi(t)] = 0 \}.
\]

Let \( \psi \in \mathcal{A}^{(0,1)}(T^{(1,0)}(X)) \) satisfy

\[
\bar{\partial}\psi - \frac{1}{2} [\psi, \psi].
\]

For \( \chi \in T^{(1,0)}(X) \), let \( f_\chi : X \to X \) be the function that translates a point \( z \) 1 unit along the geodesic in the \( \chi(z) \) direction. Then

1. For each \( t \in B' \), \( \phi(t) \) determines a complex structure \( X_t \) on \( X \);

2. \( \psi \) defines a complex structure \( X_\psi \) on \( X \); and

3. If \( \| \psi \|_{k,0} \) is sufficiently small, there is a unique \( \chi \in (\mathcal{H}^{0,0})^\perp \) such that \( \psi \circ f_\chi = \phi(t) \) for
some $t \in B'$, and thus $X_\psi$ is biholomorphically equivalent to $X_t$.

Kuranishi’s Completeness Theorem characterizes complex structures “close” to a known complex structure, though it is not enough to determine all possible complex structures of a manifold.

The only major question remaining is when deformations of Kähler manifolds are Kähler. Hironaka proved that non-Kähler deformations exist in [18], but there is still a positive result for sufficiently small deformations.

**Theorem 2.10 ([22]).** Let $(\mathcal{X}, \pi, B)$ is a family of compact complex manifolds such that $0 \in B$ and $X_0$ is Kähler. Then $X_t$ is also Kähler for sufficiently small $t$.

### 2.3 Hodge Structures

*This section is adapted from [14].*

#### 2.3.1 Motivation

Recall that if $X$ is a compact Kähler manifold of dimension $n$, then the set of $C^\infty$ $k$-forms on $X$, denoted by $\mathcal{A}^n(X)$, can be decomposed:

$$\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X),$$

and the cohomology groups $H^{p,q}(X)$ are defined to be

$$H^{p,q}(X) = \{ \phi \in \mathcal{A}^{p,q}(X) : d\phi = 0 \}/d\mathcal{A}^{k-1}(X) \cap \mathcal{A}^{p,q}(X).$$
That is, it’s the quotient group of the closed $C^\infty (p, q)$-forms by the exact $C^\infty (p, q)$-forms.

**Theorem 2.11** (Hodge Decomposition Theorem). Let $X$ be a compact Kähler manifold of dimension $n$. Then for each $k \in \{0, 1, \ldots, 2n\}$, the $k$th complex de-Rham cohomology group of $X$ can be written as a direct sum of cohomology groups:

$$H_{DR}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

These decompositions can be extended to filtrations. Define

$$F^pA^k(X) = A^{k,0}(X) \oplus \cdots \oplus A^{p,k-p}(X).$$

Then

$$A^k(X) = F^0A^k(X) \supset \cdots \supset F^kA^k(X) = A^{k,0}(X)$$

is a decreasing filtration on $A^k(X)$ that extends to a decreasing filtration on $H_{DR}^k(X, \mathbb{C})$ via the groups

$$F^pH_{DR}^k(X, \mathbb{C}) = H^{k,0}(X) \oplus \cdots \oplus H^{p,k-p}(X)$$

$$= \{ \phi \in F^pA^k(X) : d\phi = 0 \}/dA^{k-1}(X) \cap F^pA^k(X).$$

This idea can be generalized into a *Hodge structure of weight $k$*.

**2.3.2 Hodge Structures**

The Hodge structure of weight $k$ can be considered in two forms: one based on the *Hodge components* and the other based on the *Hodge filtration*. 

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1. **By Hodge components.** Let $H_Z$ be a finitely generated Abelian group such that the complexification $H = H_Z \otimes \mathbb{C}$ decomposes into groups $H^{p,q}$ (called Hodge components):

$$H = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{p,q} = \overline{H^{q,p}}$. Then the Hodge structure of weight $k$ is the pairing of $H_Z$ and $\{H^{p,q} : p + q = k\}$, denoted $\{H_Z, H^{p,q}\}$.

2. **By Hodge filtration.** Let $H_Z$ be a finitely generated Abelian group such that the complexification $H = H_Z \otimes \mathbb{C}$ has a decreasing filtration $F^p$ (called a Hodge filtration):

$$H = F^0 \supset \cdots \supset F^k$$

such that

$$H \cong F^p \oplus \overline{F^{k-p+1}}$$

Then the Hodge structure of weight $k$ is the pairing of $H_Z$ and $\{F^p\}_{p=0,\ldots,k}$, denoted $\{H_Z, F^p\}$.

These two definitions are equivalent via the relations

$$F^p = H^{k,0} \oplus \cdots \oplus H^{p,k-p}$$

and

$$H^{p,q} = F^p \cap \overline{F^q},$$

so the Hodge structure of weight $k$ can be denoted by either $\{H_Z, H^{p,q}\}$ or $\{H_Z, F^p\}$.
2.3.3 Polarized Hodge Structures

First define the *Weil operator* $C : H \to H$ by

$$
C|_{H^{p,q}} = (\sqrt{-1})^{p-q}
$$

A Hodge structure $\{H^p, H^{p,q}\}$ of weight $k$ is called polarized if there exists a bilinear form $Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$, called the Hodge bilinear form or the polarization, that is skew-symmetric for odd $k$ and symmetric for even $k$, and which satisfies the *Hodge-Riemann relations* on Hodge components:

1. $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p' = k - p$ and $q' = k - q$,

2. $Q(C\phi, \phi) > 0$ for any non-zero $\phi \in H^{p,q}$.

Equivalently, in terms of a Hodge filtration $\{F^p\}$, the Hodge-Riemann relations are

1. $Q(F^p, F^{k-p+1}) = 0$ for $p \in \{0, 1, \ldots, k\}$,

2. $Q(C\phi, \phi) > 0$ for any non-zero $\phi \in H$.

Such a $Q$ is called the polarization of the Hodge structure. A polarized Hodge structure of weight $k$ is represented by a triple $\{H^p, H^{p,q}, Q\}$ or $\{H^p, F^p, Q\}$.

2.3.4 Primitive Cohomology Groups

Let $X$ now be an algebraic variety, and let $\omega$ by the first Chern class of a positive line bundle on $X$. Then $(X, \omega)$ is called a polarized algebraic variety. On such a variety, define the Lefschetz operator $L : H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C})$ by $L([\phi]) = [\phi \wedge \omega]$. 

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Theorem 2.12 (Hard Lefschetz Theorem). Let \((X, \omega)\) be a polarized algebraic variety of dimension \(n\). Then for any non-negative integer \(k < n\),

\[
L^{n-k} : H^k(X, \mathbb{C}) \to H^{2n-k}(X, \mathbb{C})
\]

is an isomorphism.

The kernel of \(L^{n-k+1}\) in \(H^k(X, \mathbb{C})\) is called the primitive cohomology group and denoted \(P^k(X, \mathbb{C})\).

Theorem 2.13 (Lefschetz Decomposition Theorem). On a polarized algebraic variety \((X, \omega)\) of dimension \(n\),

\[
H^k(X, \mathbb{C}) \cong \bigoplus_{l=0}^{\lfloor \frac{k}{2} \rfloor} L^l P^{k-2l}(X, \mathbb{C})
\]

is a decomposition of the cohomology group.

Thus the complex cohomology groups are determined by the primitive cohomology groups. We can now properly identify a Hodge structure based on the primitive cohomology groups.

\[
H_Z = P^k(X, \mathbb{C}) \cap H^k(X, \mathbb{Z})
\]

and

\[
H^{p,q} = P^k(X, \mathbb{C}) \cap H^{p,q}(X)
\]

define a Hodge structure \(\{H_Z, H^{p,q}\}\) of weight \(k\). On this structure define a bilinear form \(Q : H_Z \times H_Z \to \mathbb{Z}\) by

\[
Q(\phi, \psi) = (-1)^{\frac{k(k-1)}{2}} \int_X \phi \wedge \psi \wedge \omega^{n-k}
\]
This $Q$ satisfies the Hodge-Riemann relations, so we have a polarized Hodge structure.

**Definition 2.7.** [15] [16] [17] Let $(\mathcal{X}, \pi, B)$ be a deformation space. Then a polarized variation of Hodge structure of weight $n$ is Hodge filtration on $\mathcal{X}$ that restricts to a polarized Hodge filtration of weight $n$ on each fiber $X_i$ and which satisfies the Griffiths Transversality:

Let $\nabla$ be the integrable holomorphic connection on $H_{\mathbb{Z}}$. Then

\[ \nabla(F^n) \subset F^{n-1} \otimes \Omega^1. \]

### 2.4 BCOV Torsion

#### 2.4.1 Definition

**Definition 2.8 (BCOV Torsion).** The BCOV torsion (as defined by Bershadsky- Ceccotti-Ooguri-Vafa in [1], [2]) is

\[
T = \prod_{1 \leq p,q \leq n} (\det \Delta'_{p,q})^{(-1)^{p+q}pq}
\]

where $\Delta'_{p,q}$ is the non-singular part of the $\overline{\partial}$-Laplace operator on $(p,q)$ forms with respect to the metric on the fiber, and the determinant is taken via zeta function regularization.

That is, if $\{\lambda_n\}$ is the increasing (though not necessarily strictly increasing) sequence of (positive) eigenvalues of $\Delta'_{p,q}$, then the zeta function is

\[
\zeta_{\Delta'_{p,q}}(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}
\]

and well-defined whenever $\text{Re}(s) > 1$. It can be extended to be regular at $s = 0$ as in [35],
so that the determinant of $\Delta'_{p,q}$ is defined to be

$$\det \Delta'_{p,q} = \exp(-\zeta'_{\Delta'_{p,q}}(0)).$$

An explicit computation of the BCOV torsion is highly non-trivial. Using a formula analogous to the one to be proven in Chapter 4, in [10] Fang, Lu, and Yoshikawa were able to compute it explicitly in for some Calabi-Yau threefolds.

### 2.4.2 Preparations for Proofs About the BCOV Torsion

Let $X$ be a polarized manifold of Calabi-Yau type of dimension $2n+1$. Let $\mathfrak{X}$ be the universal family polarized by the moduli space $\mathcal{M}$. Then there exists a proper, surjective holomorphic map $\pi : \mathfrak{X} \to \mathcal{M}$ defining the universal deformation family. From this universal family, define a local deformation space $\mathfrak{X}'$ of $X$, parameterized by $\mathcal{M}'$ with projection $\pi' : \mathfrak{X}' \to \mathcal{M}'$ such that for some interior point $O \in \mathcal{M}'$, $\pi'^{-1}(O) \cong X$. Let $X_t$ denote $\pi'^{-1}(t)$. In particular, $X_O \cong X$.

Assume without loss of generality that

- $\mathcal{M}'$ is an open neighborhood of the origin in $\mathbb{C}^m$
- If $\omega_t$ is the polarized Kähler form on $X_t$, then $\int_{X_t} \omega_t^n = 1$.

On the moduli space there exist the relative Hodge bundles $R^q\pi_*\Omega^p_{\mathfrak{X}'/\mathcal{M}'} \to \mathcal{M}'$ in place of the cohomology groups. Following the methods of Fang-Lu in [9], define the following holomorphic coefficient vector bundle.

$$E = \bigoplus_{p=1}^{n} (-1)^p \Omega^p_{\mathfrak{X}'/\mathcal{M}'}.$$
Definition 2.9. [9] For each $t \in \mathcal{M}$ and $X = X_t$, let $\omega_{PH}^k$ denote Hermitian form of the pullback of the natural Hermitian metric on the bundle $\bigoplus_{p+q=k} \text{Hom}(PR^q\pi_*\Omega^p_{\mathcal{X}'/\mathcal{M}'}) \to PR^k\pi_*(\mathbb{C})$ to $T\mathcal{M}'$ for $k \leq 2n + 1$. Following the Lefschetz Decomposition, define

$$\omega_{H^k} = \omega_{PH^k} + \omega_{PH^{k-2}} + \cdots.$$ 

Both $\omega_{H^k}$ and $\omega_{PH^k}$ are called generalized Hodge metrics.

Proposition 2.7. Abusing notation slightly, let $c_1(E)$ denote the Ricci form of the vector bundle $E$. Then

$$\omega_{H^i} = \sum_{p=0}^i p c_1(R^{i-p}\pi_*\Omega^p_{\mathcal{X}'/\mathcal{M}'})$$

for $0 \leq i \leq 2n + 1$. [9]

By [32], there exists a corresponding determinant line bundle over $\mathcal{M}$:

$$\lambda = \bigwedge_{0 \leq p,q \leq n} \det(H^{p,q}(X, E, \overline{\partial}))(-1)^{p+q}$$

where $H^{p,q}(X, E, \overline{\partial}) = R^q\pi_*\Omega^p_{\mathcal{X}'/\mathcal{M}'}$ are holomorphic vector bundles over $\mathcal{M}'$. The bundle $\lambda$ naturally admits the $L^2$ metric. Define the Quillen metric[37] on $\lambda$ via the $L^2$ metric and the BCOV Torsion:

$$\| \cdot \|_Q^2 = \| \cdot \|_{L^2}^2 T.$$
Chapter 3

Deformations of Manifolds of
Calabi-Yau Type

All of the examples of manifolds of Calabi-Yau type considered in [20] are Fano, and thus, by Theorem 2.4, they have unobstructed deformations. This just raises a question: what about manifolds of Calabi-Yau type that aren’t Fano?

Recall from Chapter 2 that finding infinitesimal deformations of a complex manifold is equivalent to solving a certain differential equation on that manifold. In particular, for a manifold $X$ with local coordinates $z = (z^1, \ldots, z^n)$ and a parameter space $B$ with local coordinates $t$, we had the following theorem.

**Theorem 2.1.** Let $\phi(t)$ be a family of $C^4$ $T^{(1,0)}(X)$-valued $(0,1)$-forms that is holomorphic with respect to $t$. Let $\overline{\partial}$ be the $d$-bar operator on $X$. Let $\theta$ be a $\overline{\partial}$-closed $T^{(1,0)}(X)$-valued $(0,1)$-form. Let $\epsilon > 0$. If for $|t| < \epsilon$,

$$\overline{\partial} \phi(t) - \frac{1}{2} [\phi(t), \phi(t)] = 0, \quad \frac{\partial \phi(t)}{\partial t} = 0, \quad \phi(0) = 0, \quad \phi'(0) = \theta$$

48
then there exists a compact, complex family \((X, \pi, B)\) such that

1. \(B\) is an open neighborhood of \(\mathbb{C}\) at the origin;

2. \(X = \pi^{-1}(0)\);

3. The infinitesimal deformation of the family \((X, \pi, B)\) is \([\theta] \in H^1(X, \Theta)\), where \(\Theta\) is the sheaf of germs of holomorphic \((1, 0)\)-vector fields on \(X\).

One method of solving the differential equation, as used by Tian [43], is to consider a formal power series solution.

**Lemma 3.1.** Solving

\[
\overline{\partial}\phi(t) - \frac{1}{2} [\phi(t), \phi(t)] = 0, \quad \frac{\partial \phi(t)}{\partial t} = 0, \quad \phi(0) = 0, \quad \phi'(0) = \theta
\]

is equivalent to showing that there exists a formal power series \(\phi(t) = \sum_{i=1}^{\infty} t^i \phi_i\) over \(\mathcal{A}^{0,1}(T^{(1,0)}(X))\) such that

\[
\left[ \sum_{r+s=i} [\phi_r, \phi_s] \right] = 0 \in H^1(X, \Theta)
\]

for all \(i \geq 2\) and \(\phi_1 = \theta\).

**Proof.** This is a simple matter of plugging in the formal power series \(\phi(t) = \sum_{i=0}^{\infty} t^i \phi_i\) (notice the starting index), since \(\frac{\partial \phi(t)}{\partial t} = 0\). First off, the condition \(\phi(0) = 0\) is equivalent to \(\phi_0 = 0\). Second, \(\phi'(0) = \theta\) becomes \(\phi_1 = \theta\). We now approach the differential equation itself. Because the Lie superbracket is linear with respect to addition and multiplication by constant (including \(t\) in this case) and \(\overline{\partial}\phi_1 = \overline{\partial}\theta = 0\) by assumption, \(\overline{\partial}\phi(t) - \frac{1}{2} [\phi(t), \phi(t)] = 0\)
becomes

\[ 0 = \overline{\partial} \sum_{i=2}^{\infty} t^i \phi_i - \frac{1}{2} \left[ \sum_{i=1}^{\infty} t^i \phi_i, \sum_{i=1}^{\infty} t^i \phi_i \right] \]
\[ = \sum_{i=1}^{\infty} t^i \left( \overline{\partial} \phi_i - \frac{1}{2} \sum_{r+s=i} [\phi_r, \phi_s] \right). \]

Solving the system is thus a matter of finding each \( \phi_i \) such that \( \overline{\partial} \phi_i = \frac{1}{2} \sum_{r+s=i} [\phi_r, \phi_s] \) for \( i \geq 2 \). Since by Proposition 2.3,

\[ \overline{\partial} \sum_{r+s=i} [\phi_r, \phi_s] = \sum_{r+s=i} [\overline{\partial} \phi_r, \phi_s] - \sum_{r+s=i} [\phi_r, \overline{\partial} \phi_s] \]
\[ = \sum_{r+s=i} [\overline{\partial} \phi_r, \phi_s] - \sum_{r+s=i} [\overline{\partial} \phi_s, \phi_r] \]
\[ = 0, \]

then \( \left[ \sum_{r+s=i} [\phi_r, \phi_s] \right] \in H^1(X, \Theta) \). Finding a solution to the differential equation is then just a matter of showing it’s the 0 element – that is, that it’s \( \overline{\partial} \)-exact.

\[ \square \]

Because the definition of manifolds of Calabi-Yau type is based on that of Calabi-Yau manifolds, the proof of the Tian-Todorov Theorem (Theorem 2.6) is of great interest. At the core of Tian’s proof in [43] is the isomorphism

\[ A^{0,1} (T^{(1,0)}(X)) \to A^{n+1,n}(X) \]

induced by contraction with the (unique up to constant multiple) nowhere-zero holomorphic \((n,0)\)-form that must exist for all Calabi-Yau manifolds.

The main idea of the proof is to apply this isomorphism to the differential equation from Theorem 2.1 to get a new differential equation on \((n - 1, 1)\)-forms, find a solution to that equation, and use the isomorphism to produce a solution in \( A^{0,1} (T^{(1,0)}(X)) \).
The main complication of replicating this for manifolds of Calabi-Yau type is that on such manifolds, the corresponding contraction map need not be an isomorphism; it merely induces one on the corresponding cohomology groups. To this end, we will consider only manifolds of Calabi-Yau type for which the contraction map is sufficiently “close” to an isomorphism.

### 3.1 The Contraction Map

Recall that $X$ is said to be a $2n + 1$-fold of Calabi-Yau type if $X$ is a smooth, compact, complex variety of complex dimension $2n + 1$, $n \geq 1$, such that:

1. $h^{n+2,n-1}(X) = 1$ and $h^{n+r+1,n-r}(X) = 0$ for $r \geq 2$;
2. $h^{r,0}(X) = 0$ for $1 \leq r \leq 2n$; and
3. For any generator $[\mu] \in H^{n+2,n-1}(X)$, the contraction

$$H^1(X, TX) \xrightarrow{\mu} H^n(X, \Omega_X^{n+1})$$

is an isomorphism.

It is important to consider the properties of the contraction map that defines this isomorphism.

#### 3.1.1 Defining the Contraction Map

Let $X$ be a Kähler manifold of Calabi-Yau type, fix a non-trivial harmonic $(n + 2, n - 1)$-form $\mu$. By the Hodge Theorem and $h^{n+2,n-1}(X) = 1$, this choice is unique up to constant multiple and $\bar{\partial}$-exact component. Let this form be given in local coordinates by $\mu = \mu_{1,\overline{j}} \, dz^I \wedge d\overline{z}^J$,.
where $\mu_{I,J}$ is skew-symmetric with respect to both $I$ and $J$. For $u \leq n + 2$, define

$$A : \mathcal{A}^{p,q} \left( \bigwedge^u T^{(1,0)}(X) \right) \to \mathcal{A}^{n+2+p-u,n+1+q}(X)$$

by

$$A(\phi) = \iota_\phi(\mu).$$

**Proposition 3.1.** $A$ induces the isomorphism $H^1(X, TX) \xrightarrow{\mu} H^n(X, \Omega^n_X)$ independent of choice of representative of $[\mu] \in H^{n+2,n-1}(X)$.

This proposition will be proven after a few other properties of $A$.

It is immediate from the definition of manifold of Calabi-Yau type that there exists such a isomorphism via contraction, but it does not immediately follow that $A$ directly produces such an isomorphism for every choice of $\mu$.

For an example where not every choice of representative works equally well, refer back to the proof of the Serre Duality in Chapter 2. In that case, the operator $\tilde{*}$ didn’t even take $\overline{\partial}$-closed forms to $\partial$-closed forms! It was only by restricting the operator to the harmonic forms that produced a well-defined map (and thus induced the isomorphism of the duality).

In this case, we have no such restriction on $\mu$. While each $\mu$ produces a different $A$, each such $A$ induces the same homomorphism (and thus isomorphism, by definition of manifold of Calabi-Yau type). This allows the choice of $\mu$ to be any representative of the cohomology class. In particular, it will be convenient to choose $\mu$ to be harmonic.

Let $p, q, u$ all be non-negative integers, $u \leq i$, and define

$$a^{(i,j)}_{(u,p,q)} := (-1)^{(p+q)(i-u)+qj} \binom{i}{u}.$$
Proposition 3.2. Let \( \phi \in \mathcal{A}^{p,q}(\bigwedge^u T^{(1,0)}(X)) \). Then \( A \) has the following properties.

1. If \( \phi \) is given in local coordinates by \( \phi^{I'}_{K',I'} \frac{\partial}{\partial x^I} \, dz^{K'} \wedge dz^{L'} \), then \( A(\phi) \) is given in local coordinates by \( a_{(u,p,q)}^{(n+2,n-1)} \phi^{I'}_{K',I'} \mu^I_{I} \, dz^K \wedge dz^{L} \).

2. \( A \) is linear over \( \mathcal{A}^{0,q} \left( \bigwedge^u T^{(1,0)}(X) \right) \). That is, if \( f \in C(X) \) and both \( \phi \) and \( \psi \) are elements of \( \mathcal{A}^{0,q} \left( \bigwedge^u T^{(1,0)}(X) \right) \), then \( A(f \phi) = f A(\phi) \) and \( A(\phi + \psi) = A(\phi) + A(\psi) \).

3. \( A(\overline{\partial} \phi) = \overline{\partial} A(\phi) \).

4. If \( \alpha \in \mathcal{A}^{0,r}(X) \), then \( A(\phi \wedge \alpha) = (-1)^{r(2n+1-u)} A(\phi) \wedge \alpha \) and \( A(\alpha \wedge \phi) = \alpha \wedge A(\phi) \).

Proof. First compute \( A \) explicitly.

\[
A(\phi) = t_\phi(\mu) = \text{sgn} \left( \sigma_{>K''} \right) \phi^{I''-K'' \mu I, \overline{J}} \frac{\partial}{\partial x^I} \, dz^{K'} \wedge dz^{L'} \wedge dz^{K''} \wedge dz^J
\]

\[
= (-1)^{q(n+2-u)+q(n-1)+p(n+2-u)} \binom{n+2}{u} \phi^{I'}_{K',I'} \mu^I_{I} \, dz^{K''+K'} \wedge dz^{L''+L'}
\]

\[
= a_{(u,p,q)}^{(n+2,n-1)} \phi^{I'}_{K',I'} \mu^I_{I} \, dz^{K} \wedge dz^{L}
\]

The second statement is trivial. Since

\[
\overline{\partial} A(\phi) = \overline{\partial} \left( a_{(u,p,q)}^{(n+2,n-1)} \phi^{I'}_{K',I'} \mu^I_{I} \, dz^{K} \wedge dz^{L} \right)
\]

\[
= a_{(u,p,q)}^{(n+2,n-1)} \left[ \frac{\partial \phi^{I'}_{K',I'}}{\partial z^I} \mu^I_{I} + \phi^{I'}_{K',I'} \frac{\partial \mu^I_{I}}{\partial z^I} \right] \, dz^{K} \wedge dz^{L}
\]

\[
= (-1)^{p+q} a_{(u,p,q+1)}^{(n+2,n-1)} \frac{\partial \phi^{I'}_{K',I'}}{\partial z^I} \mu^I_{I} \, dz^{K} \wedge dz^{L+1(1)} + (-1)^{n+2+p+q-u} t_\phi(\overline{\partial} \mu),
\]

\[53\]
and because

\[ A(\bar{\partial} \phi) = A \left( \frac{\partial \phi_{K', \overline{U}}}{\partial \overline{z}^l} \frac{\partial}{\partial z^l} d\overline{z}^l \wedge dK' \wedge d\overline{z}^L \right) \]

\[ = (-1)^{p+q} A \left( \frac{\partial \phi_{K', \overline{U}}}{\partial \overline{z}^l} \frac{\partial}{\partial z^l} dz^{K'} \wedge d\overline{z}^{L'(l)} \right) \]

\[ = (-1)^{p+q} a_{(u p q+1)}^{(n+2, \overline{p}-1)} \frac{\partial \phi_{K', \overline{U}}}{\partial \overline{z}^l} \mu_{I+K'-\overline{K}', \overline{L}-(L'+(l))} \alpha dK \wedge d\overline{z}^{L} \]

it immediately follows that because \( \bar{\partial} \mu = 0 \), then \( \bar{\partial} A(\phi) = A(\bar{\partial} \phi) \). Hence \( A \) and \( \bar{\partial} \) commute.

For the fourth statement, let \( \alpha = \alpha_{\overline{\mathcal{U}}} d\overline{z}^{\overline{p}} \in A^{0,r}(X) \). Then

\[ A(\phi \wedge \alpha) = (-1)^{r(2n+1-u)} a_{(u p, \overline{q} + 1)}^{(n+2, \overline{p}-1)} \frac{\partial \phi}{\partial \overline{z}^l} \mu_{\overline{K}'+K'-\overline{K}', \overline{L}-(L'+(l))} \alpha_{\overline{\mathcal{U}}} dK \wedge d\overline{z}^{L} \]

Therefore

\[ A(\alpha \wedge \phi) = (-1)^{r} A(\phi \wedge \alpha) = (-1)^{r+q+r(2n+1-u)} A(\phi) \wedge \alpha = \alpha \wedge A(\phi), \]

finishing the final part of the proposition. \( \square \)

We now have enough to prove Proposition 3.1.

**Proof of Proposition 3.1.** Let \( \phi \in A^{0,1}(T^{(1,0)}(X)) \) and \( \mu \in A^{n+2,n-1}(X) \) be \( \bar{\partial} \)-closed. Let \( \psi \in T^{(1,0)}(X) \) and \( \eta \in A^{n+1,n-1}(X) \). Then \( A(\bar{\partial} \psi) = \bar{\partial} A(\psi) \), so \( A \) preserves \( \bar{\partial} \)-exactness. Additionally,

\[ \bar{\partial} A(\phi) = A(\bar{\partial} \phi) = A(0) = 0, \]
so $A$ preserves $\bar{\partial}$-closedness. Hence $A$ induces an isomorphism simply by restriction of domain. To prove that choice of representative doesn’t change the cohomology class, consider

$$
\iota_\phi (\mu + \bar{\partial} \eta) = \iota_\phi (\mu) + \iota_\phi (\bar{\partial} \eta) = \iota_\phi (\mu) + C \bar{\partial} \iota_\phi (\mu) + C' \iota_\phi (\eta) = A(\phi) + C \bar{\partial} \iota_\phi (\mu),
$$

where $C$ and $C'$ are appropriate non-zero constants. Additionally for any holomorphic $f \in C^\infty(X),$

$$
\iota_\phi (f \mu) = f \iota_\phi (\mu) = f A(\phi)
$$

Therefore any other choice of $\mu$ does not change the cohomology class of $A(\phi)$. \hfill \Box

Hence that we know the choice of $A$ as the map that induces the isomorphism is justified.

### 3.1.2 Further Properties of $A$

Now consider the relationship between $A$ and the Lie superbracket. First define the operation $\bar{\wedge}$ to be the composition of $\wedge$ and the map

$$
\mathcal{A}^{p,q} \left( \bigwedge^u T^{(1,0)}(X) \otimes \bigwedge^{u'} T^{(1,0)}(X) \right) \to \mathcal{A}^{p',q'} \left( \bigwedge^{u+u'} T^{(1,0)}(X) \right)
$$

that naturally identifies tensor products with wedge products. Then for the vector-valued forms $\phi \in \mathcal{A}^{p,q} \left( \bigwedge^u T^{(1,0)}(X) \right)$ and $\psi \in \mathcal{A}^{p',q'} \left( \bigwedge^{u'} T^{(1,0)}(X) \right),$

$$
\phi \bar{\wedge} \psi = (-1)^{(p+q)(p'+q')} + uu' \psi \bar{\wedge} \phi.
$$

In particular, when $p = p' = 0$ and $q = q' = u = u' = 1$, the conditions in which the notation will most commonly be used, $\phi \bar{\wedge} \psi = \psi \bar{\wedge} \phi$ and $A(\phi \bar{\wedge} \psi) = -\iota_\phi A(\psi)$. It is now convenient to
prove a lemma inspired by a similar lemma for Calabi-Yau manifolds used by Tian[43].

**Lemma 3.2.** Let \( \phi, \psi \in \mathcal{A}^{0,1}(T^{(1,0)}(X)) \). Then

\[
A[\phi, \psi] = -\partial A(\phi \wedge \psi) + \iota_\phi(\partial A(\psi)) + \iota_\psi(\partial A(\phi)).
\]

**Proof.** Following the method of Tian, define a function \( F(\phi, \psi) = A[\phi, \psi] + \partial A(\phi \wedge \psi) - \iota_\phi(\partial A(\psi)) - \iota_\psi(\partial A(\phi)) \) and prove that \( F \) is a tensor. This requires knowledge of four facts about \( F \):

1. If \( \theta \in \mathcal{A}^{0,1}(T^{(1,0)}(X)) \), then \( F(\phi, \psi + \theta) = F(\phi, \psi) + F(\phi, \theta) \). (This is trivial.)
2. If \( \theta \in \mathcal{A}^{0,1}(T^{(1,0)}(X)) \), then \( F(\phi + \theta, \psi) = F(\phi, \psi) + F(\theta, \psi) \). (This is also trivial.)
3. \( F(f \phi, \psi) = f F(\phi, \psi) \)
4. \( F(\phi, f \psi) = f F(\phi, \psi) \)

To prove the third statement, consider \( F(f \phi, \psi) - f F(\phi, \psi) \).

\[
F(f \phi, \psi) - f F(\phi, \psi) = A(\psi(f) \wedge \phi) - \partial f \wedge \iota_\phi A(\psi) - \iota_\psi(\partial f \wedge A(\phi)) = A(\psi(f) \wedge \phi) - \partial f \wedge \iota_\phi A(\psi) - \psi(f) \wedge A(\phi) + \partial f \wedge \iota_\psi (A(\phi)) = 0,
\]

by the fourth statement of Proposition 3.2. To prove the fourth statement, it’s sufficient to prove that \( F(\psi, \phi) = F(\phi, \psi) \).

\[
F(\psi, \phi) = A[\psi, \phi] + \partial A(\psi \wedge \phi) - \iota_\psi(\partial A(\phi)) - \iota_\phi(\partial A(\psi)) = F(\phi, \psi),
\]
using the symmetry of both the Lie superbracket and \( \tilde{\wedge} \) for forms of this type.

Since \( F \) is a tensor, it is entirely defined by action on the constant forms. To compute \( F \) on such forms, we require one additional formula: \( \partial A(\phi) \) when \( \phi \) is constant. Consider the constant form \( \phi = \frac{\partial}{\partial z}dz^j \in A^{0,q}(\Lambda^uT^{(1,0)}(X)) \). Then since \( \mu \) is harmonic on a Kähler manifold, \( \partial \mu = 0 \) and thus

\[
\partial A(\phi) = \partial \left( a^{(n+2, n-1)}_{(u,0,q)} \mu_{I+K,L} dz^K \wedge d\bar{z}^L \wedge d\bar{z}^j \right)
\]

\[
= \frac{a^{(n+2, n-1)}_{(u,0,q)}(-1)^{r-1}}{n + 3 - u} \partial_{K'} \mu_{I+K',\bar{L}} d\bar{z}^{K'} \wedge d\bar{z}^L \wedge d\bar{z}^j
\]

\[
= \frac{a^{(n+2, n-1)}_{(u,0,q)}(-1)^{u+s-1}}{n + 3 - u} \partial_{K'} \mu_{I+K',\bar{L}} d\bar{z}^{K'} \wedge d\bar{z}^L \wedge d\bar{z}^j.
\]

Hence for \( \phi = \frac{\partial}{\partial z}d\bar{z}^j \) and \( \psi = \frac{\partial}{\partial z^i}d\bar{z}^j \),

\[
F(\phi, \psi) = a^{(n+2, n-1)}_{(u,2,0)} \left( \partial_{K'} \mu_{I+K',\bar{L}} d\bar{z}^{K'} \wedge d\bar{z}^L \wedge d\bar{z}^j \right)
\]

\[
+ \frac{a^{(n+2, n-1)}_{(1,0,q)}}{n+2} \left[ \iota_\phi \left( \partial_{K'} \mu_{I+K',\bar{L}} d\bar{z}^{K'} \wedge d\bar{z}^L \wedge d\bar{z}^j \right) + \iota_\psi \left( \partial_{K'} \mu_{I+K',\bar{L}} d\bar{z}^{K'} \wedge d\bar{z}^L \wedge d\bar{z}^j \right) \right]
\]

\[
= \left( \frac{-n+2}{n+2} + \frac{(n+2)(n+1)}{n+1} \right) \left( \partial_{K'} \mu_{I+K',\bar{L}} \right) d\bar{z}^{K'} \wedge d\bar{z}^L \wedge d\bar{z}^j \wedge d\bar{z}^l
\]

\[
= 0.
\]

Thus \( F \) is zero on constant forms, and therefore always 0. \( \square \)

This immediate corollary is essential to constructing a formal power series solution to the differential equation in Theorem 2.1.

**Corollary 3.1.** If for \( \phi, \psi \in A^{0,1}(T^{(1,0)}(X)) \), \( A(\phi) \) and \( A(\psi) \) are both \( \partial \)-closed, then \( A[\phi, \psi] \) is \( \partial A \)-exact.

**Proof.** Given the assumption,

\[
A[\phi, \psi] = -\partial A(\phi \tilde{\wedge} \psi) + \iota_\phi(\partial A(\psi)) + \iota_\psi(\partial A(\phi)) = -\partial A(\phi \tilde{\wedge} \psi).
\]
3.2 Existence of Deformations

In [20], Iliev and Manivel considered several manifolds of Calabi-Yau type with unobstructed deformations. However, all such manifolds were Fano, which by Theorem 2.4 have unobstructed deformations. What are other conditions on these manifolds that can guarantee such deformations exist?

3.2.1 The First Proof

**Theorem 3.1 (Lu-Reale).** Let $X$ be a Kähler manifold of Calabi-Yau type with contraction map $A$ that satisfies the following conditions.

1. Every cohomology class of $H^{n+1,n}(X)$ has its harmonic representative in the subspace $A\left(A^{0,1}(T^{(1,0)}(X))\right)$. (While a representative must normally exist in $\text{Im}(A)$ for each cohomology class for manifold of Calabi-Yau type, it need not be purely harmonic.)

2. If $\psi \in A^{0,2}(T^{(1,0)}(X))$ is a non-zero harmonic function, then $A(\psi) \notin \ker H$. That is, $A(\psi)$ has a non-trivial harmonic component.

3. If $\phi \in A^{0,1}(T^{(1,0)}(X))$ and $HA(\phi) \neq 0$, then $H\phi \neq 0$.

4. $\text{Im} \bar{\partial} \cap A^{n+1,n}(X) \subset A\left(A^{0,1}(T^{(1,0)}(X)) \cap \ker \partial\right)$.

Then the infinitesimal deformations of the complex structure of $X$ are unobstructed.

**Proof.** Recall from Lemma 3.1 that it is sufficient to find $\phi_i \ (i \geq 2)$ such that $\mathcal{D}\phi_i = \frac{1}{2} \sum_{r+s=i} [\phi_r, \phi_s]$ for $\phi_1 = \theta$. 

\[ A[\phi, \psi] = \partial A(-\phi \wedge \psi). \]
By the first assumption, without loss of generality choose \( \theta \) to be one of the representatives such that \( A(\theta) \) is harmonic. Since \( X \) is Kähler, the \( \overline{\partial} \) Laplacian and \( \partial \) Laplacian are equal, and thus \( A(\theta) \) is \( \partial \)-closed.

Now assume as inductive hypothesis that for \( j \leq i \)

1. \( \overline{\partial} \phi_r = \frac{1}{2} \sum_{r+s=j} [\phi_r, \phi_s] \),

2. \( A(\phi_j) \) is \( \partial \)-closed.

Recall from Lemma 3.1 that

\[
\overline{\partial} \phi_{i+1} = \frac{1}{2} \sum_{p+q=i+1} [\phi_p, \phi_q]
\]

is a necessary and sufficient condition for the solvability of the system. Then by Corollary 3.1,

\[
A\left(\frac{1}{2} \sum_{r+s=i+1} [\phi_r, \phi_s]\right) = \frac{1}{2} \sum_{r+s=i+1} A([\phi_r, \phi_s]) = -\frac{1}{2} \sum_{r+s=i+1} \partial A(\phi_r \overline{\partial} \phi_s),
\]

so \( A\left(\frac{1}{2} \sum_{r+s=i+1} [\phi_r, \phi_s]\right) \) is \( \partial \)-exact. Additionally, by Lemma 3.1, it is \( \overline{\partial} \)-closed. It thus has Hodge decomposition \( \alpha_{i+1} + \overline{\partial} \beta_{i+1} \). Without loss of generality, assume \( \beta_{i+1} \in \text{Im} \overline{\partial}^* \).

Therefore, by Proposition 3.2,

\[
A\left(\frac{1}{2} \sum_{r+s=i+1} [\phi_r, \phi_s]\right) = A(\alpha_{i+1}) + \overline{\partial} A(\beta_{i+1}),
\]

and thus since \( \ker \Delta \) is orthogonal to both \( \text{Im} \partial \) and \( \text{Im} \overline{\partial} \),

\[
0 = -H \left(\frac{1}{2} \sum_{r+s=i+1} \partial A(\phi_r \overline{\partial} \phi_s)\right) = HA(\alpha_{i+1}) + H\overline{\partial} A(\beta_{i+1}) = HA(\alpha_{i+1}).
\]
Since \( H A(\alpha_{i+1}) = 0 \), but \( \alpha_{i+1} \) is harmonic, then \( \alpha_{i+1} = 0 \) by the second assumption. Hence

\[
\frac{1}{2} \sum_{r+s=i+1} [\phi_r, \phi_s] = \overline{\partial} \beta_{i+1}.
\]

Thus \( \beta_{i+1} \) is a reasonable choice for \( \phi_{i+1} \) to solve the differential equation, but it doesn’t necessarily satisfy the conditions to bring induction forward. The goal is now to find a \( \overline{\partial} \)-closed form to add to \( \beta_{i+1} \) so that the image under \( A \) is \( \partial \)-closed.

\[
\overline{\partial} A(\beta_{i+1}) = A \left( \frac{1}{2} \sum_{r+s=i+1} [\phi_r, \phi_s] \right) = -\partial \left( \frac{1}{2} \sum_{r+s=i+1} A(\phi_r \tilde{\omega} \phi_s) \right)
\]

is both \( \overline{\partial} \)-closed and \( \partial \)-exact. Therefore by the \( \partial \overline{\partial} \)-Lemma, there exists \( \eta_{i+1} \in A^{n,n}(X) \) such that \( \overline{\partial} A(\beta_{i+1}) = \partial \overline{\partial} \eta_{i+1} \). Therefore \( A(\beta_{i+1}) + \partial \eta_{i+1} \) is \( \overline{\partial} \)-closed. \( H \beta_{i+1} = 0 \), so by the third assumption, \( H (A(\beta_{i+1}) + \partial \eta_{i+1}) = 0 \). By the complex Hodge Theorem, \( A(\beta_{i+1}) + \partial \eta_{i+1} \) is thus \( \overline{\partial} \)-exact.

Applying the complex Hodge Theorem again, there exist \( \beta'_{i+1}, \beta''_{i+1} \) such that

\[
A(\beta_{i+1}) = \overline{\partial} \beta'_{i+1} + \overline{\partial} \beta''_{i+1}
\]

and since \( \eta_{i+1} \) can be chosen without loss of generality to have no \( \overline{\partial} \)-closed component, there exists \( \eta''_{i+1} \) such that

\[
\eta_{i+1} = \overline{\partial} \eta''_{i+1}.
\]

Since

\[
A(\beta_{i+1}) + \partial \eta_{i+1} = \overline{\partial} \beta'_{i+1} + \overline{\partial} \beta''_{i+1} - \overline{\partial} \partial \eta''_{i+1}
\]
is $\bar{\partial}$-exact, then $\bar{\partial} \beta''_{i+1} - \partial^* \partial \eta''_{i+1} = 0$, and thus

$$A(\beta_{i+1}) + \partial \eta_{i+1} = \bar{\partial} \beta'_{i+1}.$$ 

By the final assumption, there exists $\bar{\partial}$-closed $\gamma_{i+1} \in A^{0,1}(T^{(1,0)}(X))$ such that

$$A(\gamma_{i+1}) = \bar{\partial} \beta'_{i+1} = A(\beta_{i+1}) + \partial \eta_{i+1}.$$ 

Now consider $\beta_{i+1} - \gamma_{i+1}$.

$$\bar{\partial}(\beta_{i+1} - \gamma_{i+1}) = \bar{\partial} \beta_{i+1} = \frac{1}{2} \sum_{r+s=i+1} [\phi_r, \phi_s],$$

so $\beta_{i+1} - \gamma_{i+1}$ is still a solution. Additionally,

$$\partial A(\beta_{i+1} - \gamma_{i+1}) = \partial (-\partial \eta_{i+1}) = 0,$$

Setting $\phi_{i+1} = \beta_{i+1} - \gamma_{i+1}$ therefore satisfies the inductive hypotheses. Hence the formal power series solution $\phi(t) = \sum_{i=0}^\infty t^i \phi_i$ exists.

Since a solution exists for sufficiently small $t$, then by Theorem 2.1, for any element $\theta \in H^1(X, \Theta)$, there exists a complex family $(\mathfrak{X}, \pi, B)$ such that

1. $B$ is an open neighborhood of $\mathbb{C}$ at the origin;

2. $X = \pi^{-1}(0)$;

3. The infinitesimal deformation of the family $(\mathfrak{X}, \pi, B)$ is $\vartheta \in H^1(X, \Theta)$. 

□
3.2.2 The Second Proof

By imposing stricter conditions, it’s possible to provide a shorter proof. This requires a few extra assumptions that \(A\) is “sufficiently close” to being an isomorphism. Consider the case where

1. \(\partial A (A^0.q (\bigwedge^u T^{(1,0)}(X))) \subseteq A (A^0.q (\bigwedge^{u-1} T^{(1,0)}(X)))\);
2. \(A\) is injective on \(A^0.q (\bigwedge^{u-1} T^{(1,0)}(X)) \cap A^{-1}(\text{Im}(\partial A))\);

for particular \((u, q)\). For convenience, call these the \(\tilde{\partial}\) Conditions for \((u, q)\).

Let \(A\) satisfy the \(\tilde{\partial}\) Conditions for \((u, q)\). Then on \(A^0.q (\bigwedge^u T^{(1,0)}(X))\) define a new operator

\[
\tilde{\partial} := A^{-1} \partial A,
\]

where \(A^{-1}\) is understood to be the inverse restricted to \(\text{Im}(\partial A)\).

**Proposition 3.3.** Let \(A\) satisfy the \(\tilde{\partial}\) Conditions for \((u, q)\). Then

- \(\tilde{\partial}^2 = 0\) when \(A\) satisfies the \(\tilde{\partial}\) Conditions for \((u - 1, q)\); and
- \(\tilde{\partial} \tilde{\partial} = -\tilde{\partial} \tilde{\partial}\) when \(A\) satisfies the \(\tilde{\partial}\) Conditions for \((u, q + 1)\).

**Proof.** For the first statement,

\[
\tilde{\partial}^2 = (A^{-1} \partial A)^2 = A^{-1} \partial^2 A = 0.
\]
For the second statement,

\[ \tilde{\partial} \tilde{\partial} = A^{-1} \partial A \tilde{\partial} = A^{-1} \partial \tilde{\partial} A = -A^{-1} \tilde{\partial} A = -\tilde{\partial} A^{-1} \partial = -\tilde{\partial} \tilde{\partial}. \]

\[ \square \]

**Lemma 3.3** (\( \tilde{\partial} \tilde{\partial} \) - Lemma). Let \( A \) satisfy the \( \tilde{\partial} \) Conditions for \((2,2)\) and let the induced map

\[ A^{0,1} \left( \bigwedge^2 T^{(1,0)}(X) \right) \rightarrow A^{n,n}(X) / \ker \tilde{\partial} \]

be surjective. Then if \( \phi \in \ker \tilde{\partial} \cap A^{0,2} \left( \bigwedge^2 T^{(1,0)}(X) \right) \), there exists \( \psi \in A^{0,1} \left( \bigwedge^2 T^{(1,0)}(X) \right) \) such that \( \tilde{\partial} \phi = \tilde{\partial} \tilde{\partial} \psi \).

**Proof.** Since \( \tilde{\partial} \tilde{\partial} \phi = 0 \), then \( \tilde{\partial} \partial A \phi = 0 \). By the \( \tilde{\partial} \tilde{\partial} \) Lemma, then there exists \( \beta \in A^{n,n}(X) \) such that \( \partial \tilde{\partial} \beta = \partial A \phi \). Without loss of generality, choose \( \beta \in \text{Im} A \), as allowed by the assumption. Then for some \( \psi \in A^{0,1} \left( \bigwedge^2 T^{(1,0)}(X) \right) \) such that \( \beta = A(\psi) \), and therefore

\[ \tilde{\partial} \phi = A^{-1} \partial \tilde{\partial} \beta = A^{-1} \partial \tilde{\partial} A \psi = \tilde{\partial} \tilde{\partial} \psi. \]

\[ \square \]

**Corollary 3.2** (Corollary to Lemma 3.2). Let \( A \) satisfy the \( \tilde{\partial} \) Conditions for \((2,2)\) and \((1,1)\). For \( \phi, \psi \in A^{0,1} \left( T^{(1,0)}(X) \right) \)

\[ [\phi, \psi] = -\tilde{\partial}(\phi \tilde{\partial} \psi) + A^{-1} \left[ \iota_\phi A(\tilde{\partial} \psi) + \iota_\psi A(\tilde{\partial} \phi) \right]. \]

**Proof.** By Lemma 3.2,

\[ A[\phi, \psi] = -\partial A(\phi \tilde{\partial} \psi) + \iota_\phi (\partial A(\psi)) + \iota_\psi (\partial A(\phi)). \]
By assumption, $A^{-1}$ exists when restricted to the subspaces $\partial A \left( A^{0.2} \left( \bigwedge^2 T^{(1,0)}(X) \right) \right)$ and $\partial A \left( A^{0.1} \left( T^{(1,0)}(X) \right) \right)$, so

$$[\phi, \psi] = -\bar{\partial}(\phi \check{\partial} \psi) + A^{-1} \left[ \iota_\phi (\partial A(\psi)) + \iota_\psi (\partial A(\phi)) \right].$$

Then since $A$ is invertible on $\partial A \left( A^{0.1} \left( T^{(1,0)}(X) \right) \right)$,

$$[\phi, \psi] = -\bar{\partial}(\phi \check{\partial} \psi) + A^{-1} \left[ \iota_\phi A(\check{\partial} \psi) + \iota_\psi A(\check{\partial} \phi) \right].$$

This has a natural corollary just like Lemma 3.2 did.

**Corollary 3.3** (Corollary to Corollary 3.2). Let $A$ satisfy the $\check{\partial}$ Conditions for $(2,2)$ and $(1,1)$. For $\phi, \psi \in A^{0.1} \left( T^{(1,0)}(X) \right) \cap \ker \bar{\partial}$,

$$[\phi, \psi] = -\bar{\partial}(\phi \check{\partial} \psi).$$

It is important to note here that in the case where $n = 1$ and $X$ is a Calabi-Yau threefold, $A$ is an isomorphism and all of these conditions are trivially satisfied.

**Theorem 3.2** (Lu-Reale). Let $X$ be a Kähler manifold of Calabi-Yau type with contraction map $A$ that satisfies the $\check{\partial}$ Conditions for $(2,2)$, $(1,1)$, and $(2,1)$. Additionally, let the induced map

$$A^{0.1} \left( \bigwedge^2 T^{(1,0)}(X) \right) \to A^{n,n} (X) / \ker \bar{\partial}$$

be surjective, and assume that for every $\alpha \in \ker \bar{\partial} \cap A^{0.1} \left( T^{(1,0)}(X) \right)$ there exists $\beta \in T^{(1,0)}(X)$ such that $\bar{\partial}(\alpha + \bar{\partial} \beta) \in \ker \bar{\partial}^*$. Then the infinitesimal deformations of the complex structure of $X$ are unobstructed.
Proof. Once again, recall from Lemma 3.1 that it is sufficient to prove there exists a formal power series $\phi(t) = \sum_{i=1}^{\infty} t^i \phi_i$ in $A^{0,1}(T^{(1,0)}(X))$ such that

$$\left[ \sum_{r+s=i} [\phi_r, \phi_s] \right] = 0 \in H^1(X, \Theta)$$

for all $i \geq 2$ and $\phi_1 = \theta$.

Without loss of generality choose $\theta$ such that $\tilde{\partial} \theta$ is both $\bar{\partial}$-closed (since any choice of $\theta$ is) and $\bar{\partial}^*$-closed (since by assumption, there is a choice of $\theta$ that makes this true). That is, choose $\theta$ such that $\bar{\partial} \theta$ is harmonic. Thus, since $X$ is of Calabi-Yau type, if $\tilde{\partial} \theta$ is non-zero then $A \tilde{\partial} \theta$ has non-zero harmonic component. However,

$$A \tilde{\partial} \theta = \partial A \theta,$$

which has no harmonic component. Hence $A \tilde{\partial} \theta$ must be trivial. Since $X$ is Kähler, the $\bar{\partial}$ Laplacian and $\partial$ Laplacian are equal, and thus choosing $\theta$ such that $A(\theta)$ is harmonic is sufficient, if possible.

Since $\phi'(0) = \theta$, set $\phi_1 = \theta$. Now assume as inductive hypothesis that for $r \leq i$

1. $\bar{\partial} \phi_i = \frac{1}{2} \sum_{r+s=i} [\phi_r, \phi_s]$,

2. $\phi_r$ is $\bar{\partial}$-closed.

Applying Corollary 3.2

$$\bar{\partial} \phi_{i+1} = \frac{1}{2} \sum_{r+s=i+1} [\phi_r, \phi_s] = -\frac{1}{2} \sum_{r+s=i+1} \left( \bar{\partial}(\phi_r \bar{\partial} \phi_s) - A^{-1} \left[ t_{\phi_r} A(\bar{\partial} \phi_s) + t_{\phi_s} A(\bar{\partial} \phi_r) \right] \right)$$

as a necessary and sufficient condition for the solvability of the system. By inductive hy-
pothesis and Corollary 3.3, this reduces to solving

$$\bar{\partial} \phi_{i+1} = -\frac{1}{2} \sum_{r+s=i+1} \bar{\partial} (\phi_r \bar{\partial} \phi_s),$$

so $\frac{1}{2} \sum_{r+s=i+1} [\phi_r, \phi_s]$ is $\bar{\partial}$-exact. By Lemma 3.1, $\frac{1}{2} \sum_{r+s=i+1} [\phi_r, \phi_s]$ is $\bar{\partial}$-closed. Thus

$$\bar{\partial} \left( -\frac{1}{2} \sum_{r+s=i+1} \bar{\partial} (\phi_r \bar{\partial} \phi_s) \right) = 0.$$

By the $\bar{\partial}\bar{\partial}$ Lemma, there exists $\psi_{i+1} \in A^{0,1} (\wedge^2 T^{(1,0)}(X))$ such that

$$-\frac{1}{2} \sum_{r+s=i+1} \bar{\partial} (\phi_r \bar{\partial} \phi_s) = \bar{\partial}\bar{\partial} \psi_{i+1}.$$

This $\psi_{i+1}$ can be chosen up to a $\bar{\partial}$-exact element. Setting $\phi_{i+1} = -\bar{\partial} \psi_{i+1}$ thus gives a $\bar{\partial}$-closed solution to the differential equation. Hence the formal power series solution $\phi(t) = \sum_{i=0} t^i \phi_i$ exists.

Since a solution exists for sufficiently small $t$, then by Theorem 2.1, for any element $\theta \in H^1(X, \Theta)$, there exists a complex family $(\mathfrak{X}, \pi, B)$ such that

1. $B$ is an open neighborhood of $\mathbb{C}$ at the origin;

2. $X = \pi^{-1}(0)$;

3. The infinitesimal deformation of the family $(\mathfrak{X}, \pi, B)$ is $\vartheta \in H^1(X, \Theta)$.

Of particular note is that all assumptions of the proof are satisfied when the contraction map is an isomorphism. In that case, it is functionally similar to Tian’s proof for Calabi-Yau manifolds in [43].
### 3.3 Convergence of Solutions

Now that it is known that solutions exist under certain conditions, the next question is whether or not those solutions converge with respect to particular norms. A natural norm to consider is the one induced by the $L^2$ inner product, as that is what was used to define the operators $\mathcal{D}$ and $\Delta$, but it is possible to do better than that. When considering solutions to differential equations, the Hölder norms are preferred.\[14\] In order to use them in a satisfactory manner, we first require information about how the interact with the new operators defined earlier in this chapter.

**Proposition 3.4.** $A$ is a bounded operator with respect to $\|\cdot\|_{k,\alpha}$.

**Proof.** Let $\phi \in A^{p,q}(\bigwedge u T^{(1,0)}(X))$. Since $\|\mu\|_{k,\alpha}$ is a finite, non-negative constant, then for some $C > 0$ independent of $\phi$ possibly varying from line to line,

$$
\|A(\phi)\|_{k,\alpha} = \left| a^{(n+2,\alpha-1)}_{(u,p,q)} \right| \max_{\zeta} \left\{ \sum_{|J| \leq k} \left[ \sup_{\zeta \in U_{\zeta}} \sum_{J' + J'' = J} \left| D_{\zeta}^{J' \phi_{\zeta,K,T}}(\phi_{\zeta,K'}(z)) \right| \right] \right. 
$$

$$
+ \sup_{y,z \in U_{\zeta}} \sum_{J' + J'' = J} \left| D_{\zeta}^{J' \phi_{\zeta,K,T}}(\mu_{\zeta,K,T}(z)) \right| \left| D_{\zeta}^{J' \mu_{\zeta,K,T}}(z) \right|
$$

$$
\leq C \max_{\zeta} \left\{ \sum_{|J| \leq k} \left[ \sup_{\zeta \in U_{\zeta}} \sum_{J' + J'' = J} \left| D_{\zeta}^{J' \phi_{\zeta,K,T}}(\phi_{\zeta,K'}(z)) \right| \right] \right. 
$$

$$
+ \sup_{y,z \in U_{\zeta}} \sum_{J' + J'' = J} \left| D_{\zeta}^{J' \phi_{\zeta,K,T}}(\mu_{\zeta,K,T}(z)) \right| \left| D_{\zeta}^{J' \mu_{\zeta,K,T}}(z) \right|
$$

$$
\left. \sup_{y,z \in U_{\zeta}} \sum_{J' + J'' = J} \left| D_{\zeta}^{J' \phi_{\zeta,K,T}}(\mu_{\zeta,K,T}(z)) \right| \left| D_{\zeta}^{J' \mu_{\zeta,K,T}}(z) \right| \right\} \frac{|y-z|^\alpha}{|y-z|^\alpha}
$$

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\[
\leq C \max_{\zeta} \left\{ \sum_{|J| \leq k} \left[ \sup_{z \in U_{\zeta}} \sum_{\substack{j'' + j'' = J}} \left( |D_{\zeta}^{j} \phi_{\xi, K''}(z)| + |D_{\zeta}^{j''} \phi_{\xi, K''}(z)| \right) \right] \right. \\
+ \left. \sup_{y, z \in U_{\zeta}} \sum_{\substack{j'' + j'' = J}} \left| \frac{|D_{\zeta}^{j} \phi_{\xi, K''}(y) - D_{\zeta}^{j''} \phi_{\xi, K''}(z)|}{|y - z|^\alpha} \right| \right\} \\
\leq C \|\phi\|_{k, \alpha}.
\]

Since

\[
\|A(\phi)\|_{k, \alpha} \leq C \|\phi\|_{k, \alpha}
\]

for \(C\) independent of \(\phi\), \(A\) is a bounded operator. \(\Box\)

Consider now the subspace decomposition \(A^{0,1} \left( T^{(1,0)}(X) \right) = \ker A \oplus W\) for any appropriate \(W\). Such a decomposition always exists because \(A\) is linear. In particular, if \(A\) is an isomorphism, then \(W = A^{0,1} \left( T^{(1,0)}(X) \right)\). Whether or not \(A\) is an isomorphism,

\[A_{|W} : W \to \text{Im} \ A\]

is an isomorphism. It’s thus possible to discuss the boundedness of

\[(A_{|W})^{-1} : \text{Im} \ A \to W.\]

**Corollary 3.4.** \((A_{|W})^{-1}\) is a bounded operator with respect to \(\|\cdot\|_{k, \alpha}\).

**Proof.** Since \(A\) is bounded and \(A^{0,q} \left( \Lambda^{n} T^{(1,0)}(X) \right)\) is a Banach space with respect to \(\|\cdot\|_{k, \alpha}\) by Theorem 2.8, the Bounded Inverse Theorem gives that \((A_{|W})^{-1}\) is also a bounded operator on \(A^{n+2-u, n-1+q}(X) \cap \text{Im} \ A\) with respect to \(\|\cdot\|_{k, \alpha}\). \(\Box\)
We also require the boundedness of $\tilde{\Lambda}$.

**Proposition 3.5.** Let $\phi \in \mathcal{A}^{p,q}(\bigwedge^n T^{(1,0)}(X))$ and $\psi \in \mathcal{A}^{p',q'}(\bigwedge^{n'} T^{(1,0)}(X))$. Then

$$\|\phi \tilde{\Lambda} \psi\|_{k,\alpha} \leq 2 \|\phi\|_{k,\alpha} \|\psi\|_{k,\alpha}.$$ 

**Proof.**

$$\|\phi \tilde{\Lambda} \phi'\|_{k,\alpha} = \max_{\zeta} \left\{ \sum_{|J| \leq k} \left( \sup_{z \in U_{\zeta}, |I'\cup J' = 0} \left| D_{\zeta}^I \left( \phi_{\zeta,K,L}(z) \psi_{\zeta,K',L'}(z) \right) \right| \right) + \sup_{y,z \in U_{\zeta}} \sum_{|J| = J'} \left| D_{\zeta}^I \phi_{\zeta,K,L}(y) \right| \left| D_{\zeta}^{J'} \psi_{\zeta,K',L'}(z) \right| \right\}.$$ 

$$\leq \max_{\zeta} \left\{ \sum_{|J| \leq k} \left( \sup_{z \in U_{\zeta}} \sum_{|J'\cup J'' = J'} \left| D_{\zeta}^{J''} \psi_{\zeta,K',L'}(z) \right| \right) \right\} \cdot \max_{\zeta} \left\{ \sum_{|J| \leq k} \left( \sup_{y,z \in U_{\zeta}} \left| D_{\zeta}^I \phi_{\zeta,K,L}(y) - D_{\zeta}^{J'} \phi_{\zeta,K,L}(z) \right| \right) \right\}.$$ 

$$\leq 2 \|\phi\|_{k,\alpha} \|\psi\|_{k,\alpha}.$$
The necessary statements of boundedness are now in place to show the convergence of the solutions.

**Theorem 3.3.** The formal power series solutions of Theorem 3.1 and Theorem 3.2 converge with respect to the $\mathcal{C}^{k,\alpha}$ norm.

This will be proven in three pieces.

**Lemma 3.4.** If

$$\|\phi_i\|_{k,\alpha} \leq C \sum_{r+s=i} \|\phi_r\|_{k,\alpha} \|\phi_s\|_{k,\alpha}$$

for some positive constant $C$ (possibly depending on $k$ and $\alpha$), then the formal power series solution in Lemma 3.1 converges for sufficiently small $t$.

**Proof.** Define $c_i = \sum_{j=1}^i \|\phi_j\|_{k,\alpha} |t|^j$ for $i \geq 1$. Thus for $i \geq 2$

$$c_i \leq C \left( \sum_{r+s=i} \|\phi_r\|_{k,\alpha} \|\phi_s\|_{k,\alpha} \right) |t|^i = C \sum_{r+s \leq i} \|\phi_r\|_{k,\alpha} |t|^r \|\phi_s\|_{k,\alpha} |t|^s \leq C \sum_{r,s=1}^{i-1} \|\phi_r\|_{k,\alpha} |t|^r \|\phi_s\|_{k,\alpha} |t|^s = C c_{i-1}^2.$$

Therefore $c_i \leq C' c_{i-1}^2$ for a positive constant $C'$.

Since $c_1 = \|\theta\|_{k,\alpha} |t|$, consider $|t| \leq \frac{1}{C'' \|\theta\|_{k,\alpha}}$. Then $c_1 \leq \frac{1}{C''}$. Assume, by induction, that $c_{i-1} \leq \frac{1}{C''}$. Then,

$$c_i \leq C' c_{i-1}^2 \leq \frac{1}{C''}.$$
so \( c_i \leq \frac{1}{c^i} \) for all \( i \geq 1 \) when \(|t|\) is sufficiently small. Because \((c_i)\) is a bounded, increasing sequence, it converges. Since

\[
c_i \geq \left\| \sum_{j=1}^{i} \beta_j t^j \right\|_{k,\alpha},
\]

then \( \sum_{j=1}^{\infty} \phi_j t^j \) converges in \( C^{k,\alpha} \) norm.

It is now sufficient to show that the solutions for each theorem are so bounded.

**Lemma 3.5.** There exists a positive constant \( C \) (possibly depending on \( k \) and \( \alpha \)) such that

\[
\|\phi_i\|_{k,\alpha} \leq C \sum_{r+s=i} \|\phi_r\|_{k,\alpha} \|\phi_s\|_{k,\alpha}
\]

for \( \sum_{i=1}^{\infty} \phi_i t^i \) the formal power series solution from Theorem 3.1.

**Proof.** Recall first that the terms of the power series are \( \phi_i = \beta_i - \gamma_i \) where \( \gamma_i \) is \( \overline{\partial} \)-closed and \( A(\phi_i) = -\partial \eta_i \) for \( \eta_i \) chosen without loss of generality to be \( \overline{\partial} \)-exact.

Consider the decomposition \( A_0^0 (T^{(1,0)}(X)) = \ker A \oplus W \) as in Corollary 3.4. Without loss of generality, choose \( W \) such that \( W' \subset W \), where \( W' \) is (not necessarily uniquely) defined by \( \ker \overline{\partial} = W' \oplus (\ker \overline{\partial} \cap \ker A) \). Then without loss of generality, each \( \gamma_i \) can be chosen to be an element of \( W' \), and thus an element of \( W \).

In what follows, Proposition 2.4, Corollary 2.1, Proposition 3.5, and other basic properties of Hölder norms will be used without full citation. Consider now \( i \geq 2, k \geq 2, 1 > \alpha > 0 \). By Corollary 3.4, for positive constants \( C \) independent of each function (but not necessarily independent of \( n, k, \) or \( \alpha \)) and possibly changing from term to term,

\[
\|\gamma_i\|_{k,\alpha} = \| (A|_W)^{-1} (A(\gamma_i)) \|_{k,\alpha} \leq C \| A(\gamma_i) \|_{k,\alpha} \leq C \left( \| A(\beta_i) \|_{k,\alpha} + \| \partial \eta_i \|_{k,\alpha} \right),
\]
and (by prior assumption that without loss of generality, $\eta_i$ is $\partial^*$-exact)

$$\|\partial \eta_i\|_{k,\alpha} \leq C \| \Delta \partial \eta_i \|_{k-2,\alpha} \leq C \left( \| \partial^* \partial \partial \eta_i \|_{k-2,\alpha} + \| \partial \partial^* \partial \eta_i \|_{k-2,\alpha} \right)$$

$$= C \| \partial^* \partial \partial \eta_i \|_{k-2,\alpha} = C \| \partial^* \partial A(\beta_i) \|_{k-2,\alpha} \leq C \| A(\beta_i) \|_{k,\alpha},$$

then

$$\|\gamma_i\|_{k,\alpha} \leq C \|A(\beta_i)\|_{k,\alpha} \leq C \|\beta_i\|_{k,\alpha}.$$ 

Therefore

$$\|\phi_i\|_{k,\alpha} \leq \|\beta_i\|_{k,\alpha} + \|\gamma_i\|_{k,\alpha} \leq C \|\beta_i\|_{k,\alpha} \leq C \| \Delta \beta_i \|_{k-2,\alpha} = C \| \partial^* \partial \beta_i \|_{k-2,\alpha}$$

$$= C \| \partial^* \sum_{r+s=i} [\phi_r, \phi_s] \|_{k-2,\alpha} \leq C \| \sum_{r+s=i} [\phi_r, \phi_s] \|_{k-1,\alpha} \leq C \sum_{r+s=i} \| \phi_r \|_{k,\alpha} \| \phi_s \|_{k,\alpha}. $$

$\square$
Lemma 3.6. There exists a positive constant $C$ (possibly depending on $k$ and $\alpha$) such that

$$\|\phi_i\|_{k,\alpha} \leq C \sum_{r+s=i} \|\phi_r\|_{k,\alpha} \|\phi_s\|_{k,\alpha}$$

for $\sum_{i=1}^{\infty} \phi_i t^i$ the formal power series solution from Theorem 3.2.

Proof. Recall that the series was constructed from $\phi_{i+1} = -\tilde{\partial} \psi_{i+1}$. The first step of this task is to show that $\psi_{i+1}$ can be chosen such that $\tilde{\partial} \phi_{i+1}$ is $\overline{\partial}^*$-exact. As with the initial selection of $\theta$, assume that $\phi_{i+1}$ has a non-trivial harmonic component. Then $A(\phi_{i+1})$ also has a harmonic component because $X$ is of Calabi-Yau type. However

$$A(\phi_{i+1}) = \partial A(\psi_{i+1}),$$

which has no harmonic component. Hence the harmonic component of $\phi_{i+1}$ is trivial. Next, by assumption on $A$, it’s possible to choose $\psi$ such that $\tilde{\partial} \psi$ is $\overline{\partial}^*$-closed. Since it has no harmonic component, it must also be $\overline{\partial}^*$-exact. With appropriate choice of $\psi$ now made, it’s time to consider norms.

Once again, Proposition 2.4, Corollary 2.1, Proposition 3.5, and other basic properties of H"older norms will be used without full citation. Consider now $i \geq 2, k \geq 2, 1 > \alpha > 0$. By Proposition 3.4 and Corollary 3.4, for positive constants $C$ independent of each function (but not necessarily independent of $n, k$, or $\alpha$) and possibly changing from term to term,

$$\|\phi_i\|_{k,\alpha} \leq C \|\Delta \phi_i\|_{k-2,\alpha} = C \|\overline{\partial^*} \phi_i\|_{k-2,\alpha} = C \|\overline{\partial^*} \sum_{r+s=i} \tilde{\partial}(\phi_r \tilde{\partial} \phi_s)\|_{k-2,\alpha} \leq C \|\sum_{r+s=i} \tilde{\partial}(\phi_r \tilde{\partial} \phi_s)\|_{k-1,\alpha} \leq C \sum_{r+s=i} \|\phi_r \tilde{\partial} \phi_s\|_{k,\alpha} \leq C \sum_{r+s=i} \|\phi_r\|_{k,\alpha} \|\phi_s\|_{k,\alpha}.$$
It’s now possible to prove Theorem 3.3.

*Proof of Theorem 3.3.* By Lemma 3.5 and Lemma 3.6,

\[ \|\phi_i\|_{k,\alpha} \leq C \sum_{r+s=i} \|\phi_r\|_{k,\alpha} \|\phi_s\|_{k,\alpha} \]

for both power series solutions. Then by Lemma 3.4, the corresponding power series solutions must converge in the $C^{k,\alpha}$ norm. \qed
Chapter 4

Kähler Metrics on the Moduli Space

4.1 Modulus Metric

4.1.1 Defining the Modulus Metric

Let \( \mathcal{M} \) be the \( m \)-dimensional moduli space of manifolds of Calabi-Yau type of dimension \( 2n + 1 \) such that \( P^{2n+1}(X, \mathbb{C}) = H^{2n+1}(X, \mathbb{C}) \) (as is trivially true for the \( n = 1 \) Calabi-Yau three-fold case). Let \( z = (z^1, \ldots, z^{2n+1}) \) be the holomorphic local coordinates on the fibers and let \( t = (t^1, \ldots, t^m) \) be the holomorphic local coordinates on the moduli space. Define the sesquilinear form

\[
(\phi, \psi) = \sqrt{-1}^{2n+1} \int_X \phi \wedge \overline{\psi}
\]

on the primitive cohomology group \( P^{2n+1}(X, \mathbb{C}) \). This satisfies the Hodge-Riemann relations:

1. \( (\phi, \psi) = 0 \) unless \( \phi \) and \( \psi \) are both elements of \( H^{p,q} \); and

2. \( (-1)^q(\phi, \overline{\phi}) > 0 \) for any non-zero \( \phi \in H^{p,q} \).
for $p + q = 2n + 1$. $(-1)^{n-1}(\cdot, \cdot)$ is an inner product on $H^{n+2,n-1}$ and $H^{n,n+1}$, and $(-1)^{n}(\cdot, \cdot)$ is an inner product on $H^{n+1,n}$ and $H^{n-1,n+2}$. Since the manifolds $X$ are of Calabi-Yau type, $h^{n+2,n-1} = 1$ is the first non-zero Hodge number in the middle row.

Let $\mu$ be a non-zero section of $H^{n+2,n-1}$. (Such a section must exist since $H^{n+2,n-1} = H^{n+2,n-1}(X)$.) Define:

$$W = (-1)^{n-1}(\mu, \overline{\mu}).$$

This is just the square of the norm of $\mu$ with respect to the norm induced by the inner product on $H^{n+2,n-1}$. Note that the choice of $\mu$ (and thus the value of $W$) is unique up to non-zero constant multiple. As will become apparent, the specific choice of $\mu$ will not matter for any crucial results.

For the remainder of this section, let $\partial_i := \frac{\partial}{\partial t_i}$. The derivatives $\partial_i \mu$ have components in $H^{n+3,n-2}$, $H^{n+2,n-1}$, and $H^{n+1,n}$. The first group either doesn’t exist at all (if $n = 1$) or has dimension 0 (if $n > 1$), so only the latter two components are non-trivial. It is advantageous to identify the $H^{n+1,n}$ component. To that end, define

$$D_i \mu = \partial_i \mu - (\partial_i W) W^{-1} \mu.$$

**Proposition 4.1.** $D_i \mu$ is thus the projection of $\partial_i \mu$ into $H^{n+1,n}$.

**Proof.** $D_i \mu$ differs from $\partial_i \mu$ by a term in $H^{n+2,n-1}$, so the $H^{n+1,n}$ component is unchanged. However, since

$$(D_i \mu, \overline{\mu}) = (\partial_i \mu, \overline{\mu}) - (\partial_i W) W^{-1} (\mu, \overline{\mu}) = 0,$$

the $H^{n+2,n-1}$ component of $D_i \mu$ is trivial, and thus only the $H^{n+1,n}$ component remains.
Hence $D_i \mu$ is the $H^{n+1,n}$ component of $\partial_i \mu$. 

$\{ D_i \mu \mid i = 1, \ldots, m \}$ has a useful property.

**Proposition 4.2.** $\{ D_i \mu \mid i = 1, \ldots, m \}$ is a basis of $H^{n+1,n} = H^{n+1,n}(X)$.

**Proof.** The dimension $m$ of the moduli space is equal to the dimension of the universal deformation space. Recall that this later dimension is equal to that of $H^1(X, T X)$, which is equal to the dimension of $h^{n+1,n}$ via the isomorphism induced by $A$. Hence $m = h^{n+1,n}$ and the set is the correct size to be a basis.

It remains to show that the set is linearly independent.

$$c_1 D_t \mu + \cdots + c_m D_t^m \mu = D_{c_1 t^1 + \cdots c_m t^m} \mu,$$

so this condition is equivalent to saying that for some appropriate local coordinates $t'$, obtained through linear transformation of $t$, that $\frac{\partial}{\partial t'} \mu \in H^{n+2,n-1}$. However, this $t'$ corresponds to a non-trivial deformation, and thus a non-trivial element of $H^{n+1,n}$. Therefore it cannot have only an $H^{n+2,n-1}$ component, so the set is linearly independent. As a maximal linearly independent set, it is a basis.

It will be useful later to consider alternate choices of $\mu$. In particular, sections of the form $f(t) \mu$ can be used to rescale $\mu$ to simplify computations. Since

$$\partial_i (f(t) \mu) = (\partial_i f(t)) \mu + f(t) (\partial_i \mu),$$

then $\partial_i (f(t) \mu) - f(t) (\partial_i \mu) \in H^{n+2,n-1}$. Hence

$$D_i (f(t) \mu) = f(t) D_i \mu.$$
That is, the operator $D_i$ acts independent of rescaling by function of $t$ - far more useful than $\partial_i$, which just acts independent of rescaling by constant!

With the properties of $D_i\mu$ well in-hand, define a new tensor

$$g_{i\overline{j}} := -\partial_i \overline{\partial}_j \log W.$$ 

In the $n = 1$ case, where $X$ is Calabi-Yau, this is the Weil-Petersson metric.[43] Its properties for $n > 1$ are to be determined.

**Lemma 4.1** (Lu-Reale). $g_{i\overline{j}}$ defines a metric on the moduli space, and

$$g_{i\overline{j}} = (-1)^n \frac{(D_i\mu, \overline{D_j}\mu)}{W}.$$ 

**Proof.**

$$g_{i\overline{j}} = -\partial_i \overline{\partial}_j \log W = -\partial_i \overline{\partial}_j W = -\frac{\partial_i \overline{\partial}_j W}{W} + \frac{(\overline{\partial}_j W) (\partial_i W)}{W^2} = (-1)^{n-1} W^{-1} (\partial_i \mu - (\partial_i W) W^{-1} \mu, \overline{\partial}_j \mu) = (-1)^n W^{-1} (D_i\mu, \overline{D_j}\mu)$$

This is a positive definite matrix because $W$ is positive and $(-1)^n (c_i D_i\mu, \overline{c_j D_j}\mu)$ is positive for any non-zero constant vector $(c_i)$. 

**Corollary 4.1.** $g_{i\overline{j}}$ is independent of rescaling of $\mu$ by $f(t)$. 

**Proof.** Since $D_i$ acts independent of $f(t)$, all instances of $f(t)$ factor out and cancel.

**Definition 4.1.** Call the metric defined by $g_{i\overline{j}}$ the modulus metric and denote its Hermitian form by $\omega_M$. 

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4.1.2 Properties of the Modulus Metric

Let \( \{\zeta_1, \ldots, \zeta_m\} \) be an orthonormal basis of \( H^{n+1,n} \) such that the representatives of each equivalence class are harmonic. Define \( \sigma_{i\bar{\beta}} \) by \( D_i \Omega_{\alpha} = \sigma_{i\bar{\beta}} \delta_{\chi\bar{\beta}} \zeta_{\chi} \).

**Proposition 4.3.** \( \sigma_{i\bar{\beta}} \) has the following properties.

1. \( \sigma_{i\bar{\beta}} = (-1)^n \left( D_i \Omega_{\alpha}, \zeta_{\bar{\beta}} \right) \).
2. \( g_{i\bar{j}} = W^{-1} \sigma_{i\bar{\beta}} \delta_{\alpha\bar{\beta}} \sigma_{j\bar{\alpha}} \).

**Proof.**

\[
(D_i \mu, \zeta_{\bar{\beta}}) = (\sigma_{i\bar{\xi}} \delta_{\chi\bar{\xi}} \zeta_{\chi}, \zeta_{\bar{\beta}}) = (-1)^n \sigma_{i\bar{\beta}}
\]

Thus \( \sigma_{i\bar{\beta}} = (-1)^n \left( D_i \mu, \zeta_{\bar{\beta}} \right) \).

Similarly,

\[
g_{i\bar{j}} = (-1)^n W^{-1} \left( \sigma_{i\bar{\xi}} \delta_{\chi\bar{\xi}} \zeta_{\chi}, D_j \mu \right) = (-1)^n W^{-1} \sigma_{i\bar{\xi}} \delta_{\chi\bar{\xi}} \left( \zeta_{\chi}, D_j \mu \right) = W^{-1} \sigma_{i\bar{\xi}} \delta_{\chi\bar{\xi}} \sigma_{j\bar{\alpha}}.
\]

Depending on the situation, it is more convenient to think of \( H^{n+1,n} \) being spanned either by the natural choice of basis \( \{D_i \mu\} \) or by the orthonormal basis \( \{\zeta_i\} \). Results will be presented in both forms when both are useful.

Finding the curvature of \( g_{i\bar{j}} \) requires taking derivatives of \( D_i \mu \), so consider those next.

**Proposition 4.4.** \( D_i \mu \) has the following properties.

1. \( \overline{\partial_j} D_i \mu = g_{i\bar{j}} \mu \).
2. \( \partial_k D_{i\mu} = \partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu - (\partial_i W) W^{-1} D_k \mu. \)

**Proof.** These are both simple computations.

\[
\overline{\partial_j} D_{i\mu} = \overline{\partial_j} \partial_i \mu - (\overline{\partial_j} \partial_i W) W^{-1} \mu - (\partial_i W) (\overline{\partial_j} W^{-1}) \mu - (\overline{\partial_j} \partial_i W) W^{-1} (\overline{\partial_j} \mu)
\]

\[
= -(\overline{\partial_j} \partial_i W) W^{-1} \mu + (\partial_i W) W^{-1} (\overline{\partial_j} W) W^{-1} \mu
\]

\( \overline{\partial_j} D_{i\mu} \in H^{n+2,n-1} \) is expected, since \( \overline{\partial_j} \partial_i \mu = 0 \), and \( \overline{\partial_j} (\partial_i - D_i) \mu \in H^{n+2,n-1} \).

\[
\partial_k D_{i\mu} = \partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu - (\partial_i W) (\partial_k W^{-1}) \mu - (\partial_i W) W^{-1} (\partial_k \mu)
\]

\[
= \partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu - (\partial_i W) (\partial_k W^{-1}) \mu - (\partial_i W) W^{-1} (D_k \mu)
\]

\[
- (\partial_i W) W^{-1} (\partial_k W) W^{-1} \mu
\]

\[
= \partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu - (\partial_i W) W^{-1} D_k \mu
\]

Once again, it is advantageous to know a particular component of a derivative – in this case, the \( H^{n,n+1} \) component of \( \partial_i D_{i\mu} \). Define \( D_k D_{i\mu} = \partial_k D_{i\mu} - (-1)^n (\partial_k D_{i\mu}, \overline{\zeta_\beta}) \delta_{\alpha \beta} \zeta_\alpha \).

**Proposition 4.5.** \( D_k D_{i\mu} \) has the following properties.

1. \( D_k D_{i\mu} = \partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu - (-1)^n (\partial_k \partial_i \mu, \overline{\zeta_\beta}) \delta_{\alpha \beta} \zeta_\alpha. \)

2. \( D_k D_{i\mu} = \partial_k D_{i\mu} + (\partial_i W) W^{-1} D_k \mu - (-1)^n (\partial_k \partial_i \mu, \overline{\zeta_\beta}) \delta_{\alpha \beta} \zeta_\alpha. \)

3. \( D_k D_{i\mu} = \partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu - (\partial_i W) W^{-1} D_k \mu - (\partial_k W) W^{-1} D_i \mu - \Gamma^r_{ki} D_r \mu. \)

4. \( D_k D_{i\mu} \) is the projection of both \( \partial_k D_{i\mu} \) and \( \partial_k \partial_i \mu \) into \( H^{n,n+1} \).
Proof. By Proposition 4.4,

\[ D_k D_i \mu = \partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu - (\partial_i W) W^{-1} D_k \mu + (-1)^{n-1} (\partial_k \partial_i \mu, \overline{\zeta_\beta}) \delta_{\alpha \beta} \zeta_\alpha \]

\[ + (-1)^n (\partial_i W) W^{-1} (D_k \mu, \overline{\zeta_\beta}) \delta_{\alpha \beta} \zeta_\alpha \]

\[ = \partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu + (-1)^{n-1} (\partial_k \partial_i \mu, \overline{\zeta_\beta}) \delta_{\alpha \beta} \zeta_\alpha \]

This proves the first part. The second part follows immediately from the this. For the third statement, start by rewriting the definition of \( D_k D_i \mu \) in terms of the basis \( \{D_i \mu\} \):

\[ D_k D_i \mu = \partial_k D_i \mu - (-1)^n (\partial_k D_i \mu, \overline{D_s \mu}) W^{-1} g^{r \bar{s}} D_r \mu \]

\[ = \partial_k D_i \mu - (-1)^n \partial_k (D_i \mu, \overline{D_s \mu}) W^{-1} g^{r \bar{s}} D_r \mu \]

\[ = \partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu - (\partial_i W) W^{-1} (D_k \mu) - (\partial_k W) W^{-1} D_i \mu \]

\[ - (\partial_k g_{i \bar{s}}) g^{r \bar{s}} D_r \mu. \]

Since \( D_k D_i \mu \) differs from \( \partial_k \partial_i \mu \) and \( \partial_k D_i \mu \) by terms in \( H^{n+2,n-1} \) and \( H^{n+1,n} \), the fourth statement is equivalent to three conditions:

1. \( D_k D_i \mu \) has no \((n + 2, n - 1)\)-component

2. \( D_k D_i \mu \) has no \((n + 1, n)\)-component

3. \( D_k D_i \mu = D_i D_k \mu \).

Proving each in turn,

\[ (D_k D_i \mu, \overline{\pi}) = \partial_k \partial_i W - (\partial_k \partial_i W) W^{-1} W + (-1)^{n-1} (\partial_k \partial_i \mu, \overline{\zeta_\beta}) \delta_{\alpha \beta} (\zeta_\alpha, \pi) = 0. \]
Thus $D_k D_i \mu$ has no $(n + 2, n - 1)$-component. Similarly, for any $\beta$,

\[
(D_k D_i \mu, \zeta_{\beta}) = (\partial_k \partial_i \mu, \zeta_{\beta}) - (\partial_k \partial_i W) W^{-1} (\mu, \zeta_{\beta}) + (-1)^{n-1} (\partial_k \partial_i \mu, \zeta_\epsilon) \delta_{\alpha \tau} (\zeta_\alpha, \zeta_\beta) \\
= (\partial_k \partial_i \mu, \zeta_{\beta}) + (-1)^{n-1} (\partial_k \partial_i \mu, \zeta_\epsilon) \delta_{\alpha \tau} (-1) \delta_{\alpha \beta} \\
= 0
\]

via the Hodge-Riemann relations. Hence $D_k D_i \mu$ has no $(n + 1, n)$-component.

Since $\partial_k \partial_i \mu - (\partial_k \partial_i W) W^{-1} \mu - (-1)^n (\partial_k \partial_i \mu, \zeta_{\beta}) \delta_{\chi \beta} \zeta_\chi$ is symmetric with respect to $k$ and $i$, then $D_k D_i \mu = D_i D_k \mu$. Combining all of these facts, $D_k D_i \mu$ is the projection of both $\partial_k D_i \mu$ and $\partial_k \partial_i \mu$ into $H^{n, n+1}$.

Just as with $D_i$, since $D_k D_i$ is a projection differential operator, it acts independently of $f(t)$ multiple of $\mu$, making it ideal notation for later conclusions.

Almost all of the pieces are in place to compute the Riemannian curvature tensor. Before that, define some space-saving notation:

\[
B_{ik\alpha} = \delta_{\alpha \eta} W^{-1} \left[ \sigma_{\eta \gamma} W^{-1} (\partial_k W) + \sigma_{k \eta} W^{-1} (\partial_i W) - (-1)^n (\partial_k \partial_i \mu, \zeta_\eta) \right].
\]

It immediately follows that

\[
B_{kia} = B_{ika}.
\]

Additionally define

\[
W''_{ijkl} := (-1)^{n-1} W^{-1} (D_k D_i \mu, \overline{D_l D_j \mu}).
\]

**Proposition 4.6.** $W''_{ijkl}$ has the following properties:
1. $W''_{kjl} = W''_{ijkl}$

2. $W''_{dlj} = W''_{ijkl}$

3. $W''_{jlk} = W''_{ijkl}$

4. $W''_{ijkl}$ is real and non-negative.

5. $W''_{ijkl}$ is independent of $f(t)$ multiple of $\mu$.

Proof. These properties follow immediately from Proposition 4.5, $(-1)^{n-1}(\cdot, \cdot)$ being an inner product on $H_{n,n+1}$, and $f(t)$-independence of $D_k D_l$.

Now equipped with the notations $D_k D_l \mu$, $B_{i\alpha \alpha}$ and $W''_{ijkl}$, we are ready to start the computations for the curvature tensor.

**Proposition 4.7.** $g_{ij}$ has the following properties.

1. $\partial_k g_{ij} = - (\partial_k W)^{-1} g_{ij} - (\partial_i W)^{-1} g_{kj} + (-1)^n W^{-1} \left( \partial_k \partial_i \mu, D_j \mu \right)$

2. $\partial_k g_{ij} = - \overline{g_{ij}} B_{i\alpha \alpha}$

3. $\partial_k \overline{g}_{ij} = g_{ij} g_{kl} + g_{il} g_{kj} - W''_{ijkl} + (-1)^{n-1} W B_{i\alpha \alpha} \delta_{\alpha \beta} \overline{D_{j\beta}}$

Proof. The first statement is a matter of simple computation:

\[
\partial_k g_{ij} = - (-1)^n \left( \partial_k W^{-1} \right) \left( D_i \mu, \overline{D_j \mu} \right) + (-1)^n W^{-1} \left( \partial_k D_i \mu, \overline{D_j \mu} \right) \\
+ (-1)^n W^{-1} \left( D_i \mu, \overline{\partial_k D_j \mu} \right) \\
= - (-1)^n W^{-2} \left( \partial_k W \right) \left( D_i \mu, \overline{D_j \mu} \right) + (-1)^n W^{-1} \left( \partial_k D_i \mu, \overline{D_j \mu} \right) \\
- (-1)^n W^{-2} \left( \partial_i W \right) \left( D_k \mu, \overline{D_j \mu} \right) \\
= - \left( \partial_k W \right) W^{-1} g_{ij} - \left( \partial_i W \right) W^{-1} g_{kj} + (-1)^n W^{-1} \left( \partial_k \partial_i \mu, \overline{D_j \mu} \right).
\]
The second statement follows immediately from this, Proposition 4.3, and the definition of $B_{ik\alpha}$. From here,

\[
\overline{\partial}_{l} \left[ -W^{-1} (\partial_k W) g_{ij} \right] = g_{kl} g_{ij} - W^{-1} (\partial_k W) (\overline{\partial}_{i} g_{\bar{j}}) = g_{ij} g_{kl} + W^{-1} (\partial_k W) \sigma_{\bar{j}l} B_{j\bar{i}\beta}.
\]

Similarly,

\[
\overline{\partial}_{l} \left[ -W^{-1} (\partial_i W) g_{k\bar{j}} \right] = g_{k\bar{j}} g_{il} + W^{-1} (\partial_i W) \sigma_{\bar{k}l} B_{j\bar{i}\beta}.
\]

Finally,

\[
\overline{\partial}_{l} \left[ (-1)^n W^{-1} (\partial_k \partial_i \mu, \overline{D}_j \overline{\mu}) \right] = (-1)^n (\overline{\partial}_{l} W^{-1}) (\partial_k \partial_i \mu, \overline{D}_j \overline{\mu}) + (-1)^n W^{-1} (\partial_k \partial_i \mu, \overline{\partial}_l D_j \overline{\mu}).
\]

Application of Proposition 4.4 simplifies this to

\[
\overline{\partial}_{l} \left[ (-1)^n W^{-1} (\partial_k \partial_i \mu, \overline{D}_j \overline{\mu}) \right] = (-1)^n W^{-2} (\overline{\partial}_{l} W) (\partial_k \partial_i \mu, \overline{D}_j \overline{\mu}) + (-1)^n W^{-2} (\partial_k \partial_i \mu, D_l D_j \overline{\mu}) + W^{-1} (\partial_k \partial_i \mu, \overline{\zeta}_\tau (\zeta_\chi, \overline{\partial}_j \overline{\mu})).
\]

Since only the $H^{n,n+1}$ component of $\partial_k \partial_i \mu$ affects $(\partial_k \partial_i \mu, \overline{D}_j \overline{\mu})$, this term reduces to $(D_k D_i \mu, \overline{D}_l \overline{D}_j \overline{\mu})$. Thus by the definition of $B_{ik\alpha}$ and $W''_{ij\overline{k}\overline{l}}$,

\[
\overline{\partial}_{l} \left[ (-1)^n W^{-1} (\partial_k \partial_i \mu, \overline{D}_j \overline{\mu}) \right] = (-1)^n (\partial_k \partial_i \mu, \overline{\zeta}_\beta) B_{j\beta} - W''_{ij\overline{k}\overline{l}}
\]
Combining these,

$$
\partial_k \partial_l g_{ij} = g_{ij} g_{kl} + g_{ik} g_{jl} - W'_{ijkl}
$$

$$
\quad + \left[ (\partial_k W) W^{-1} \sigma_{ij} + (\partial_i W) W^{-1} \sigma_{kj} + (-1)^{n-1} (\partial_k \partial_i \mu, \zeta_{\beta}) \right] \overline{B_{jl\beta}}
$$

$$
\quad = g_{ij} g_{kl} + g_{ik} g_{lj} - W'_{ijkl} + (-1)^{n-1} W B_{ik\alpha} \delta_{\alpha\beta} \overline{B_{jl\beta}}.
$$

Before reaching the primary result of this section, this proposition has one important consequence:

**Corollary 4.2.** \((g_{ij})\) defines a Kähler metric.

**Proof.** A necessary and sufficient condition for \(g_{ij}\) to define a Kähler metric is

$$
\partial_k g_{ij} = \partial_i g_{kj}.
$$

This follows immediately from the previous proposition:

$$
\partial_k g_{ij} = -\overline{\sigma_{j\alpha}} B_{ika} = -\overline{\sigma_{j\alpha}} B_{kia} = \partial_i g_{kj}.
$$

The way is now clear to find the curvature tensor.

**Theorem 4.1** (Lu-Reale). Let \(R_{ijkl}\) be the curvature tensor for the metric defined by \(g_{ij}\).

Then

$$
R_{ijkl} = g_{ij} g_{kl} + g_{ik} g_{lj} - W'_{ijkl}.
$$
Proof.

\[ R_{ijkl} = \partial_k \overline{\partial_l} g_{ij} - g^{ir} (\partial_k g_{ir}) (\overline{\partial_l} g_{ij}) \]

From the previous proposition, this means

\[ R_{ijkl} = g_{ij} g_{kl} + g_{il} g_{kj} - W'_{ijkl} + (-1)^{n-1} W B_{ika} \delta_{\alpha \beta} B_{jkl} - g^{ir} [-\sigma_{s\alpha} B_{ika}] [-\sigma_{r\beta} B_{jkl}] . \]

Since for any \( u \),

\[ (\sigma_{s\alpha} g^{ir} D_{r\mu}, \overline{D}_{u\mu}) = -W \sigma_{s\alpha} g^{ir} g_{r\mu} = -W \sigma_{u\alpha}, \]

then

\[ (-1)^{n-1} W \delta_{\alpha \tau} - \sigma_{s\alpha} g^{ir} \sigma_{r\tau} = (-1)^{n-1} W \delta_{\alpha \tau} + W (\zeta_{\alpha}, \overline{\zeta}_{\tau}) = (-1)^{n-1} W \delta_{\alpha \tau} + (-1)^n W \delta_{\alpha \tau} = 0, \]

which gives the desired conclusion. \( \square \)

In the case where \( n = 1 \), and thus \( X \) is Calabi-Yau, this agrees with the result of Strominger [42] (cf. [41], [34]), albeit via the alternate Hodge-theoretic methods of Wang [46] as refined by Lu and Sun [27].

It is unknown whether or not the volume form induced by this metric will produce a rational volume for the moduli space, as was found by Lu and Sun in [28].
4.2 Partial Hodge Metric

4.2.1 Defining the Partial Hodge Metric

Definition 4.2. The partial Hodge metric $\omega_\kappa$ is defined to be

$$\omega_\kappa = \kappa \omega_\mathcal{M} + \text{Ric} (\omega_\mathcal{M})$$

for $\kappa > m + 1$ a real number.

Since $\omega_\mathcal{M}$ is Kähler, it follows immediately that $\omega_\kappa$ is as well. We desire the curvature properties of $\omega_\kappa$. With this new notation, consider the partial Hodge metric in local coordinates.

Proposition 4.8. In local coordinates, the partial Hodge metric is given by

$$\left[ (\kappa - m - 1)g_{i\overline{j}} + g^{kl}W''_{i\overline{j}kl} \right] dz^i \otimes dz^j.$$

Proof. Let $h_{i\overline{j}}$ denote the coefficients of the partial Hodge metric tensor. Then by Theorem 4.1,

$$h_{i\overline{j}} = \kappa g_{i\overline{j}} - g^{kl} \left( g_{i\overline{j}kl} + g_{i\overline{k}l}g_{j\overline{l}} - W''_{i\overline{j}kl} \right)$$

$$= (\kappa - m - 1) g_{i\overline{j}} + g^{kl}W''_{i\overline{j}kl}.$$

□

Corollary 4.3. The partial Hodge metric is independent of choice of $\mu$.

Proof. By Corollary 4.1 and Proposition 4.6, all terms are independent of rescaling. □

Once again, we desire the curvature tensor, denoted $\tilde{R}_{i\overline{j}kl}$ to distinguish it from the curvature...
tensor for the modulus metric. Computing the derivatives of the second term in a convenient way won’t be so easy. For that, we need the derivatives of $D_k D_i \mu$.

**Proposition 4.9.** $\overline{\partial}_j D_k D_i \mu \in H^{n+1,n}$. In particular, $\overline{\partial}_j D_k D_i \mu = W''_{ijkl} g^{rs} D_r D_s \mu$. Additionally, $\partial_r D_k D_i \mu \in H^{n,n+1} \oplus H^{n-1,n+2}$. In particular,

$$\partial_r D_k D_i \mu = \partial_r \partial_k \partial_i \mu - (\partial_r \partial_k \partial_i \mu) W^{-1} \mu - (-1)^n W^{-1} \left( \partial_r \partial_k \partial_i \mu, \overline{D}_j \mu \right) g^{us} D_u \mu$$

$$- (-1)^n W^{-1} \left( \partial_k \partial_i \mu, \overline{D}_v \mu \right) g^{uv} D_v D_u \mu.$$

**Proof.** Direct computation provides the first result.

$$\overline{\partial}_j D_k D_i \mu = \overline{\partial}_j \partial_k \partial_i \mu - (\overline{\partial}_j \partial_k \partial_i W) W^{-1} \mu - (\partial_k \partial_i W) (\overline{\partial}_j W^{-1}) \mu - (\partial_k \partial_i W) W^{-1} \overline{\partial}_j \mu$$

$$- \overline{\partial}_j [(\partial_k W) W^{-1}] D_k \mu - (\partial_i W) W^{-1} \overline{\partial}_j D_k \mu - \overline{\partial}_j [(\partial_k W) W^{-1}] D_i \mu - (\partial_k W) W^{-1} \overline{\partial}_j D_i \mu$$

$$- (\overline{\partial}_j \partial_k g_{i\overline{\sigma}}) g^{\overline{r} \overline{s}} D_r \mu - (\partial_k g_{i\overline{\sigma}}) (\overline{\partial}_j g^{\overline{r} \overline{s}}) D_r \mu - (\partial_k g_{i\overline{\sigma}}) g^{\overline{r} \overline{s}} \overline{\partial}_j D_r \mu.$$

Using the appropriate formulas for the curvature tensor, this simplifies to

$$= - (\partial_k \partial_i \mu, \overline{D}_j \mu) W^{-1} \mu - R_{i\overline{sk}j} g^{\overline{r} \overline{s}} D_r \mu - (\partial_k g_{i\overline{\sigma}}) \mu$$

$$+ g_{i\overline{\sigma}} D_k \mu - (\partial_i W) W^{-1} g_{k\overline{\sigma}} \mu + g_{k\overline{\sigma}} D_i \mu - (\partial_k W) W^{-1} g_{i\overline{\sigma}} \mu$$

and then further to

$$= g_{i\overline{\sigma}} g_{k \overline{\rho}} g^{\overline{r} \overline{\tau}} D_r \mu + g_{k \overline{\sigma}} g_{i \overline{\rho}} g^{\overline{r} \overline{\tau}} D_r \mu - R_{i\overline{sk}j} g^{\overline{r} \overline{\tau}} D_r \mu.$$

The result then follows immediately from the definition of the tensor $W''_{ijkl}$. For the second result,

$$(\partial_r D_k D_i \mu, \overline{\mu}) = \partial_r (D_k D_i \mu, \overline{\mu}) + (D_k D_i \mu, \partial_r \overline{\mu}) = 0,$$
so there is no $H^{n+2,n-1}$ component. Similarly,

$$(\partial_r D_k D_i \mu, \overrightarrow{D_j} \mu) = \partial_r (D_k D_i \mu, \overrightarrow{D_j} \mu) + (D_k D_i \mu, \partial_r \overrightarrow{D_j} \mu) = 0,$$

so there is no $H^{n+1,n}$ component. Using Proposition 4.4

$$\partial_r D_k D_i \mu = \partial_r \partial_k \partial_i \mu - (\partial_r \partial_k \partial_i W) W^{-1} \mu - (\partial_k \partial_i W) W^{-1} D_r \mu$$

$$- (\partial_r \partial_i W) W^{-1} D_k \mu - (\partial_i W) W^{-1} D_r D_k \mu - (\partial_i W) W^{-1} \Gamma^u_{rk} D_u \mu$$

$$- (\partial_k \partial_i W) W^{-1} D_r \mu - (\partial_i W) W^{-1} D_r D_i \mu - (\partial_i W) W^{-1} \Gamma^u_{ri} D_u \mu$$

$$- (\partial_r \partial_k g_{ij}) g^{u\nu} D_u \mu - \Gamma^u_{ki} D_r D_u \mu - (\partial_r W) W^{-1} \Gamma^u_{ki} D_u \mu.$$

This is rather unwieldy, but direct computation of the Christoffel symbols and derivatives of $g_{ij}$ via Propositions 4.4 and 4.7, it simplifies to

$$\partial_r \partial_k \partial_i \mu - (\partial_r \partial_k \partial_i W) W^{-1} \mu - (-1)^n W^{-1} (\partial_r \partial_k \partial_i \mu, \overrightarrow{D_j} \mu) g^{u\nu} D_u \mu$$

$$- (-1)^n W^{-1} (\partial_k \partial_i \mu, \overrightarrow{D_j} \mu) g^{u\nu} D_r D_u \mu.$$

Notice that in the second case, the first three terms exactly match the projection of $\partial_r \partial_k \partial_i \mu$ into $H^{n,n+1} \oplus H^{n-1,n+2}$. It is only the final term, on the second line, that produces any difference from this projection. In order to make this formula more manageable, define $P_{ikr}$ and $D_r D_k D_i \mu$ to be the $H^{n,n+1}$ and $H^{n-1,n+2}$ components, respectively, of $\partial_r \partial_k \partial_i \mu$. The formula then simplifies to

**Proposition 4.10.**

$$\partial_r D_k D_i \mu = P_{ikr} + D_r D_k D_i \mu - (-1)^n W^{-1} (\partial_k \partial_i \mu, \overrightarrow{D_r} \mu) g^{u\nu} D_r D_u \mu.$$
Notice that since $X$ is Kähler and each $\zeta_\alpha$ is harmonic, then $\overline{\zeta_\alpha}$ is also harmonic, and thus \{$\overline{\zeta_1}, \ldots, \overline{\zeta_m}$\} is an orthonormal basis for $H^{n,n+1}$. It immediately follows that

$$P_{ikr} = (-1)^{n-1}(\partial_r \partial_k \partial_i \mu, \zeta_\alpha) \delta_{\alpha\beta} \overline{\zeta_\beta}.$$ 

Similarly, since $\overline{\mu}$ is harmonic, it forms a basis of $H^{n-1,n+2}$, and thus

$$D_r D_k D_i \mu = \frac{(-1)^n (\partial_r \partial_k \partial_i \mu, \overline{\mu})}{(-1)^n (\overline{\mu}, \overline{\mu})} \overline{\mu} = -(-1)^n (\partial_r \partial_k \partial_i \mu, \mu) W^{-1} \overline{\mu}.$$ 

From the definitions, it immediately follows that both new terms are symmetric with respect to all three indices. Additionally, $D_r D_k D_i \mu$ is also the projection of $\partial_r D_k D_i \mu$ into $H^{n-1,n+2}$, since $\partial_r \partial_k \partial_i \mu$ and $\partial_r D_k D_i \mu$ differ only by terms in $H^{n,n+1}$. As a final concession to readability, define

$$U_{ikr} = P_{ikr} - \Gamma^u_{ri} D_k D_u \mu - \Gamma^u_{rk} D_i D_u \mu - \Gamma^u_{ki} D_r D_u \mu - \frac{\partial_r W}{W} D_k D_i \mu - \frac{\partial_k W}{W} D_r D_i \mu - \frac{\partial_i W}{W} D_r D_k \mu.$$ 

The use of this notation will become apparent during the computation of the Riemann curvature tensor. For now, notice that $U_{ikr}$ is symmetric with respect to all of its indices, just like $P_{ikr}$, and additionally it’s an element of $H^{n,n+1}$.

In order to use these notations in a useful manner, we’ll need to know their derivatives. For these computation, we require a simple, well-known property that’s true for every Kähler metric:

**Proposition 4.11.** $\overline{\partial_s} \Gamma^k_{ri} = R_{i\overline{j}r\overline{k}} g^{\overline{k} \overline{j}}$.

**Proof.** Since $g_{\overline{j}k} g^{\overline{k} \overline{j}} = \delta_{ik}$, the product rule gives

$$\overline{\partial_s} g^{\overline{j} \overline{k}} = \left(\overline{\partial_s} g^{\overline{u} \overline{j}}\right) g_{u \overline{l}} g^{\overline{k} \overline{l}} = -g^{\overline{u} \overline{j}} (\overline{\partial_s} g_{u \overline{l}}) g^{\overline{k} \overline{l}}.$$ 

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Using this fact with a little reindexing gives the result:

$$\overline{\partial}_s \Gamma_{kri} = (\overline{\partial}_s \partial_r g_{ij}) g^{k\overline{\partial}^j} + (\partial_r g_{ij}) (\overline{\partial}_s g^{k\overline{\partial}^j}) = (\overline{\partial}_s \partial_r g_{ij}) g^{k\overline{\partial}^j} - (\partial_r g_{i\overline{\partial}}) g^{i\overline{\partial}} (\overline{\partial}_s g_{j\overline{\partial}}) g^{j\overline{\partial}} = R_{j\overline{\partial}r\overline{\partial}} g^{k\overline{\partial}}.$$

With that property handy, the computation of derivatives can begin. While these are not particularly illustrative by any means, being able to call upon these facts later will be great help.

**Proposition 4.12.** $D_r D_k D_i \mu$, $P_{ikr}$, and $U_{ikr}$ have the following properties:

1. $\overline{\partial}_j D_r D_k D_i \mu = (-1)^n (D_r D_k D_i \mu, \mu) W^{-1} \overline{D_j} \mu$

2. $\overline{\partial}_j P_{ikr} = (-1)^{n-1} W^{-1} (P_{ikr}, D_s \overline{D_j} \mu) g^{u\overline{\partial}} D_u \mu - (-1)^n (D_r D_k D_i \mu, \mu) W^{-1} \overline{D_j} \mu$

3. $\overline{\partial}_j U_{ikr} = (-1)^{n-1} W^{-1} (U_{ikr}, D_s \overline{D_j} \mu) g^{u\overline{\partial}} D_u \mu - (-1)^{n-1} (D_k D_i \mu, D_r \mu) W^{-1} \overline{D_j} \mu + W''_{j\overline{\partial}r} g^{i\overline{\partial}} D_i D_u \mu + W''_{k\overline{\partial}r} g^{i\overline{\partial}} D_i D_u \mu + W''_{j\overline{\partial}k} g^{i\overline{\partial}} D_i D_u \mu - g_{r\overline{\partial}} D_i D_u \mu - g_{k\overline{\partial}} D_i D_u \mu - g_{j\overline{\partial}} D_i D_u \mu.$

**Proof.** Since $\overline{\partial}_j \mu = 0$,

$$\overline{\partial}_j D_r D_k D_i \mu = - (\partial_r \partial_k \partial_i \mu) (\overline{\partial}_j W) W^{-2} \overline{\mu} + (-1)^n (\partial_r \partial_k \partial_i \mu) W^{-1} \overline{D_j} \mu = (-1)^n (D_r D_k D_i \mu, \mu) W^{-1} \overline{D_j} \mu$$

Since

$$P_{ikr} = \partial_r D_k D_i \mu - D_r D_k D_i \mu + (-1)^n W^{-1} (\partial_r \partial_i \mu, \overline{D_j} \mu) g^{u\overline{\partial}} D_r D_u \mu,$$

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then, again since $\overline{\partial}_{j\mu}$,

$$
\overline{\partial}_{j} P_{ikr} = \overline{\partial}_{j} \partial_{r} D_{k} D_{i\mu} - \overline{\partial}_{j} D_{r} D_{k} D_{i\mu} - (-1)^n \left( \overline{\partial}_{j} W \right) W^{-2} \left( \partial_{k} \partial_{l\mu}, D_{v\mu} \right) g^{uv} D_{r} D_{u\mu} + (-1)^n W^{-1} \left[ \left( \partial_{k} \partial_{l\mu}, D_{r} D_{i\mu} \right) g^{uv} D_{r} D_{u\mu} + \left( \partial_{k} \partial_{l\mu}, D_{v\mu} \right) \right. \\
\left. g^{uv} \overline{\partial}_{j} D_{r} D_{u\mu} \right].
$$

Using Propositions 4.4 and 4.9, and the first conclusion of this proposition,

$$
\overline{\partial}_{j} P_{ikr} = \partial_{r} \left( W''_{rskj} g^{uv} D_{u\mu} \right) - (-1)^n W^{-1} \left[ \left( D_{r} D_{k} D_{i\mu}, \mu \right) D_{j\mu} \right. \\
\left. + \left( \partial_{k} \partial_{l\mu}, \overline{\partial}_{j} D_{v\mu} \right) g^{uv} D_{r} D_{u\mu} + \left( \partial_{k} \partial_{l\mu}, D_{v\mu} \right) g^{uv} W''_{u\sigma r\sigma} g^{\sigma r} D_{i\mu} \right].
$$

Judicious cancellation reduces this to the second result:

$$
\overline{\partial}_{j} P_{ikr} = (-1)^{-1} W^{-1} \left( P_{ikr}, D_{s} D_{j\mu} \right) g^{uv} D_{u\mu} - (-1)^n \left( D_{r} D_{k} D_{i\mu}, \mu \right) W^{-1} D_{j\mu}.
$$

The final result follows quickly from second result, the proof of Lemma 4.1, and Proposition 4.9.

$$
\overline{\partial}_{j} U_{ikr} = (-1)^{-1} W^{-1} \left( U_{ikr}, D_{s} D_{j\mu} \right) g^{uv} D_{u\mu} - (-1)^n \left( D_{r} D_{k} D_{i\mu}, \mu \right) W^{-1} D_{j\mu} \\
- R_{ij\sigma r} g^{at} D_{k} D_{a\mu} - R_{kjl\sigma r} g^{at} D_{i} D_{a\mu} - R_{i\sigma jkl\sigma} g^{at} D_{r} D_{a\mu} \\
+ g_{\sigma r} D_{k} D_{i\mu} + g_{\sigma j} D_{r} D_{i\mu} + g_{\sigma r} D_{r} D_{k\mu}
$$

Since $(-1)^n \left( D_{r} D_{k} D_{i\mu}, \mu \right) = (-1)^{-1} \left( D_{k} D_{i\mu}, D_{r}\mu \right)$, the proposition holds.

The pieces are all now in place to compute the curvature of $h_{ij}$.

**Theorem 4.2.** Let $\kappa' = \kappa - m - 1$. The Riemann curvature tensor of the partial Hodge
Proposition 4.4 gives

\[ \tilde{R}_{ijkl} = \kappa' R_{ijkl} - W''_{ijkl} + W''_{ik} g^{il} g^{kl} W''_{ijkl} + W''_{ir} g^{il} g^{kl} W''_{ijkl} \]

Proof. By the definition of \( \kappa' \),

\[ \partial_i h_{ij} = \kappa' \partial_i g_{ij} + \left( \partial_i g^{kl} \right) W''_{ijkl} + g^{kl} \left( \partial_i W''_{ijkl} \right) \]

\[ = \kappa' \partial_i g_{ij} - \Gamma_{ri}^k g^{il} W''_{ijkl} - g^{kl} (\partial_i W) W^{-1} W''_{ijkl} \]

\[ + (-1)^{n-1} g^{kl} W^{-1} \left( \partial_i D_k D_i \mu, D_i D_j \mu \right), \]

since \( D_i D_j \mu \in H^{n+1,\alpha} \) by Proposition 4.9. Substituting in for \( \partial_i D_k D_i \mu \) and applying Proposition 4.4 gives

\[ \partial_i h_{ij} = \kappa' \Gamma_{ri}^u g_{uj} - \Gamma_{ri}^k g^{il} W''_{ijkl} - g^{kl} (\partial_i W) W^{-1} W''_{ijkl} + (-1)^{n-1} g^{kl} W^{-1} \left( P_{ikr}, D_i D_j \mu \right) \]

\[ - (-1)^{n-1} W^{-1} \left( \partial_i D_i \mu, D_i D_j \mu \right) g^{il} g^{kl} W''_{ijkl} \]

\[ = \kappa' \Gamma_{ri}^u g_{uj} - \Gamma_{ri}^k g^{il} W''_{ijkl} - g^{kl} (\partial_i W) W^{-1} W''_{ijkl} + (-1)^{n-1} g^{kl} W^{-1} \left( P_{ikr}, D_i D_j \mu \right) \]

\[ - \Gamma_{kl}^i g^{kl} W''_{ijkl} - (\partial_i W) W^{-1} g^{kl} W''_{ijkl} - (\partial_k W) W^{-1} g^{kl} W''_{ijkl} \]

Re-indexing several of the terms, using the symmetries of \( W''_{ijkl} \) and \( \Gamma_{kr}^i \), and substituting \( h_{ij} \) and \( U_{ikr} \) reduces this to

\[ \partial_i h_{ij} = \kappa' h_{ij} + (-1)^{n-1} g^{kl} W^{-1} \left( U_{ikr}, D_i D_j \mu \right) . \]
Then Proposition 4.9, Proposition 4.12, and Theorem 4.1 give

\[
\partial_s \partial_r h_{ij} = \partial_s \Gamma^k_{ri} h_{kj} + \Gamma^k_{ri} \partial_s h_{kj} + (-1)^{n-1} \left( \partial_s g^{kl} \right) W^{-1} (U_{ikr}, D_l D_j \mu) - (-1)^{n-1} g^{kl} \frac{\partial_s W}{W} (U_{ikr}, D_l D_j \mu) + (-1)^{n-1} g^{kl} W^{-1} (\partial_s U_{ikr}, D_l D_j \mu) + (-1)^{n-1} g^{kl} W^{-1} \left( \partial_s D_l D_j \mu \right) 
\]

\[
= \kappa R_{ijrs} - W'_{ijrs} + \Gamma^k_{ri} \Gamma^l_{sj} h_{kl} + (-1)^{n-1} \Gamma^k_{ri} g^{\nu\tau} W^{-1} (D_u D_k \mu, U_{j\nu\tau}) + (-1)^{n-1} g^{kl} W^{-1} (U_{ikr}, D_l D_j \mu) + \Gamma^k_{ri} \Gamma^l_{sj} h_{kl} + (-1)^{n-1} g^{kl} W^{-1} (U_{ikr}, D_l D_j \mu),
\]

using the fact that

\[
\frac{(-1)^n (D_r D_k D_l \mu, D_s D_l D_j \mu)}{W} = (D_r D_k D_l \mu, \mu) \frac{D_s D_l D_j \mu}{(\mu, \mu)},
\]

the result immediately follows. \(\square\)

### 4.2.2 Sectional and Bisectional Curvature

For particular values of \(\kappa\), the partial Hodge metric can be guaranteed to have nice curvature properties. The primary result of this section is the following theorem.

**Theorem 4.3.** The partial Hodge metric \(\omega_{m+3} = (m + 3) \omega_M + \text{Ric}(\omega_M)\) has the following properties.

1. The holomorphic bisectional curvature of \(\omega_{m+3}\) is non-positive.

2. The holomorphic sectional curvature of \(\omega_{m+3}\) is bounded above by \(-\frac{2}{m+8} (h_{\bar{\nu}})^2 \leq -\frac{8}{m+9}\).

The choice of \(\omega_{m+3}\) as the partial Hodge metric under consideration is not coincidental. In
[26], Lu proved that this choice of $\kappa$ gives a multiple of the Hodge metric for a Calabi-Yau threefold.

Before computing the holomorphic sectional and bisectional curvatures, it is advantageous to perform some simplifications. First fix a point $x_0$ in the moduli space. For the rest of this subsection, any terms not explicitly dependent on local coordinates of the moduli space are assumed to be computed at $x_0$; in slight abuse of notation, the dependence on $x_0$ will be suppressed to reduce visual complexity.

Considering the moduli space as a Kähler manifold with respect to the modulus metric $\omega_M$, Theorem 2.1 guarantees the existence of local coordinates $t$ such that at $x_0$, $g_{ij} = \delta_{ij}$, $\partial_{k} g_{ij} = 0$, and $\overline{\partial_{k} g_{ij}} = 0$. Once in such coordinates, there are a few desirable simplifications possible via choice of $\mu$:

1. $W = 1$ (simplifying denominators),

2. $\partial_{i} W = 0$ (i.e. $D_{i} \mu = \partial_{i} \mu$, canceling several terms), and

3. $\partial_{k} \partial_{i} W = 0$,

where the second condition is equivalent to finding a choice of $f(t)$ for $\mu'$ such that

\[
(-1)^{n-1} \left( (\partial_{i} f(t)) \mu' + f(t) (\partial_{i} \mu') \overline{f(t) \mu'} \right) = 0,
\]

and the third condition is equivalent to

\[
(-1)^{n-1} \left( (\partial_{k} \partial_{i} f(t)) \mu' + (\partial_{k} f(t)) (\partial_{i} \mu') + (\partial_{i} f(t)) (\partial_{k} \mu') + f(t) (\partial_{k} \partial_{i} \mu') \overline{f(t) \mu'} \right) = 0.
\]

That is, the goal is to find an $f(t)$ such that

\[
f(0) = \left( (-1)^{n-1} (\mu', \overline{\mu'}) \right)^{-\frac{1}{2}},
\]
\[ \partial_t f(t)|_{t=0} = -f(0) \frac{\partial_i \mu', \mu'}{\partial'_i \mu'}, \]

and

\[ \partial_k \partial_i f(t)|_{t=0} = -f(0) \frac{\partial_k \partial_i \mu', \mu'}{\partial'_k \mu', \partial'_i \mu'} - \partial_k f(t)|_{t=0} \frac{\partial_i \mu', \mu'}{\partial'_i \mu'} - \partial_i f(t)|_{t=0} \frac{\partial_k \mu', \mu'}{\partial'_k \mu'} . \]

The resulting solution need only be quadratic and is a refinement of the choices in [46] and [27] for Calabi-Yau manifolds. Set

\[ \mu = f(t) \mu', \]

where

\[ f(t) = \frac{1}{\left[(-1)^{n-1}(\mu', \mu')\right]^\frac{1}{2}} \left[ 1 - \sum_{i=1}^{m} \frac{\partial_i \mu', \mu'}{\partial'_i \mu'} t^i \right]. \]

and \( \mu' \in H^{n+2,n-1} \) is arbitrary (though non-zero). For this choice of \( \mu \), \( W = 1 \), \( \partial_i W = 0 \), and \( \partial_k \partial_i W = 0 \) at \( x_0 \). Under these favorable conditions at \( x_0 \), several terms reduce to simpler forms at the fixed point. Aside from the deliberately chosen \( g_{i\bar{j}} = 0 \), \( \partial_k g_{i\bar{j}} = 0 \), \( \overline{\partial_k g_{i\bar{j}}} = 0 \), \( W = 1 \), \( \partial_i W = 0 \), and \( \partial_k \partial_i W = 0 \), there are the following simplifications.

1. \( \Gamma^i_{kr} = 0 \),
2. \( D_k D_i \mu = \partial_k \partial_i \mu \),
3. \( W''_{i\bar{j}kl} = (-1)^{n-1} (D_k D_i \mu, \overline{D_l D_j \mu}) \),
4. \( R_{i\bar{j}kl} = \delta_{i\bar{j}} \delta_{kl} + \delta_{i\bar{j}} \delta_{kl} - W''_{i\bar{j}kl} \),
5. \( h_{i\bar{j}} = k' \delta_{i\bar{j}} + W''_{i\bar{j}kl} \).
6. \( U_{ikr} = P_{ikr} \),

for all indices running from 1 to \( m \). Most importantly, though

\[
\tilde{R}_{ijr} = \kappa' R_{ijr} - W''_{ijk} W''_{jlk} + W''_{ijklk} + (-1)^{n-1} (P_{ikr}, P_{jks}) \\
- (-1)^n (D_j D_k D_i \mu, \overline{D_j D_k D_i \mu}) h^{\mu\tau} (P_{ikr}, \overline{D_v D_k \mu})
\]
in such coordinates.

With such a convenient choice of \( \mu \) and local coordinates now made, it is time to begin the proof proper. As almost all computations are valid for any positive \( \kappa' \), the condition \( \kappa' = 2 \) will only be invoked at the last possible moment.

**Proof of Theorem 4.3.** Consider first the tensor

\[
\tilde{R}_{ijj} = \kappa' R_{ijj} - W''_{ijj} + W''_{ijklk} + W''_{ijjlk} + (-1)^{n-1} (P_{ikj}, P_{ikj}) \\
- (-1)^n (D_j D_k D_i \mu, \overline{D_j D_k D_i \mu}) - (D_u D_l \mu, \overline{P_{ij}}) h^{\mu\tau} (P_{ikr}, \overline{D_v D_k \mu})
\]

Showing this tensor is positive, as required, is a matter of showing various combinations of terms are either positive or non-negative. Let \( \| \cdot \| \) denote the norm induced by the inner product induced by \((\cdot, \cdot)\). This is of course actually two norms: on \( H^{n+2,n-1} \oplus H^{n,n+1} \) the norm is induced by the inner product \((-1)^{n-1}(\cdot, \cdot)\), while on \( H^{n+1,n} \oplus H^{n-1,n+2} \), the norm is induced by the inner product \((-1)^n(\cdot, \cdot)\). Which norm is being referenced should be clear from context. In what follows, it is the norm on \( H^{n,n+1} \).

To aid in this computation, define

\[
A_{ikr} := (-1)^{n-1} (D_k D_i \mu, \overline{D_r \mu})
\]
recalling that \( \{ D_r \mu^r | r = 1, \ldots, m \} \) is a harmonic basis of \( H^{n, n+1} \). Simple computation shows that \( A_{ikr} \) is symmetric in all three indices and

\[
A_{ikr} = (-1)^n (D_r D_k D_i \mu, \mu)
\]

It follows immediately that at the point \( x_0 \),

\[
W''_{ijkl} = A_{ikr} \overline{A_{jlr}}
\]

and

\[
(-1)^n (D_r D_k D_i \mu, D_s D_j \mu) = A_{ikr} \overline{A_{jls}}.
\]

By exchanging the first two indices and using Proposition 4.6, \( W''_{iik} = W''_{iik} \), and thus \( W''_{ijkl} \) is a real number for fixed \( i, k, \) and \( l \). It immediately follows that \( W''_{ijkl} = W''_{iik} = W''_{iik} \), and by extension,

\[
A_{ikr} \overline{A_{itr}} = A_{itr} \overline{A_{ikr}},
\]

to be used as convenient.

This pieces are now in place to consider bounds on the curvature tensor.

1. \( \kappa' R_{iij} = W''_{iiij} + W''_{iik} W''_{ijik} + W''_{iikl} W''_{ijik} - (-1)^n (D_j D_k D_i \mu, D_j D_k D_i \mu) \).

Using the new notation, this reduces to

\[
\kappa' (1 + \delta_{ij}) - (\kappa' + 1) A_{ijr} \overline{A_{ijr}} + A_{ikr} \overline{A_{jls}} \overline{A_{iks}} + A_{jkr} \overline{A_{jls}} \overline{A_{iks}} - A_{jki} \overline{A_{jki}},
\]

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which by clever reindexing becomes

\[ \kappa' (1 + \delta_{ij}) - (\kappa' + 2) \sum_{r=1}^{m} |A_{ijr}|^2 + 2A_{jkr} \overline{A_{jlr}} \overline{A_{iks}}. \]

Since \( \kappa = m + 3 \) implies \( \kappa' = 2 \) and

\[ A_{jkr} \overline{A_{jlr}} \overline{A_{ils}} \overline{A_{iks}} = \sum_{r,s=1}^{m} \sum_{k=1}^{m} W''_{i\bar{j}r} \overline{W''_{i\bar{j}s}} = \left( \sum_{r,s=1}^{m} \left| W''_{i\bar{j}r} \right| \right)^2 \]

the terms in question simplify to

\[ 2 \left( 1 - W''_{i\bar{j}j} \right)^2 + 2 \delta_{ij} + 2 \sum_{(r,s) \neq (j,i)} \left| W''_{i\bar{j}r} \right|^2. \]

When \( i \neq j \), this means

\[ \kappa'R_{i\bar{j}j} - W''_{i\bar{j}j} + W''_{i\bar{j}ki} W''_{j\bar{k}i} + W''_{i\bar{j}kl} W''_{j\bar{k}l} - (-1)^n \left( D_j D_k D_{i\mu}, D_j D_k D_{i\mu} \right) \]

\[ \geq 2 \left( 1 - W''_{i\bar{j}j} \right)^2 + 2 \sum_{(r,s) \neq (j,i)} \left| W''_{i\bar{j}r} \right|^2 \geq 0. \]

2. \((-1)^{n-1} \left( P_{ikj}, \overline{P_{ikj}} \right) - \left( D_u D_{i\mu}, D_{i\mu} \right) h^{\mu\nu} \left( P_{ikj}, D_{v} D_{k\mu} \right). \]

The final two terms are bounded below by a norm squared. To see this, consider

\[ 0 \leq \left\| P_{ikj} - (-1)^{n-1} \left( P_{ilj}, \overline{D_u D_{l\mu}} \right) h^{\mu\nu} D_u D_{k\mu} \right\|^2 \]

\[ = (-1)^{n-1} \left( P_{ikj}, \overline{P_{ikj}} \right) + (-1)^{n-1} \left( P_{ilj}, \overline{D_u D_{l\mu}} \right) h^{\mu\nu} \left( D_r D_{l\mu}, \overline{P_{isj}} \right) h^{\sigma\tau} W''_{k\bar{k}u\bar{s}} \]

\[ - \left( P_{ilj}, \overline{D_v D_{l\mu}} \right) h^{\mu\nu} \left( D_u D_{k\mu}, \overline{P_{ikj}} \right) - \left( D_r D_{l\mu}, \overline{P_{isj}} \right) h^{\sigma\tau} \left( P_{ikj}, \overline{D_u D_{k\mu}} \right). \]

Reindexing and substituting \( W''_{k\bar{k}u\bar{s}} = h_{\mu\nu} - \kappa' \delta_{\mu\nu} \) gives

\[ \kappa' \left\| \left( P_{ilj}, \overline{D_v D_{l\mu}} \right) h^{\mu\nu} \frac{g_{\mu\nu}}{g_u} \right\|^2 \leq (-1)^{n-1} \left( P_{ikj}, \overline{P_{ikj}} \right) - \left( P_{ilj}, \overline{D_v D_{l\mu}} \right) h^{\mu\nu} \left( D_u D_{k\mu}, \overline{P_{ikj}} \right). \]
Hence

\[-1]^{n-1} (P_{ikj}, \overline{P}_{ikj}) - (P_{ilj}, D_{v}D_{l}\mu) \ h^{uv} (D_{u}D_{k}\mu, \overline{P}_{ikj}) \]

has a non-negative lower bound. Notice that the existence of this bound does not depend on the value of \( \kappa' > 0 \), unlike with the previous term.

Combining these two facts, \( \tilde{R}_{\alpha j} \leq 0 \), so the holomorphic bisectional curvature of the partial Hodge metric is non-positive at \( x_0 \). Since this holds true for any \( x_0 \) that defines a manifold of Calabi-Yau type, it holds true on the moduli space.

For holomorphic sectional curvature, i.e. the case where \( i = j \), it’s possible to do one better by utilizing \( h_{ij} = 2 + W_{iikl}'' \) at \( x_0 \). The curvature tensor is now

\[
\tilde{R}_{\alpha i} = \kappa' R_{\alpha i} - W_{iis}'' + 2 \left| W_{iikl}'' \right|^2 + (-1)^{n-1} (P_{iki}, \overline{P}_{ikj}) \]

\[-1) (D_{i}D_{k}D_{l}\mu, \overline{D}_{i}D_{k}D_{l}\mu) - (D_{u}D_{l}\mu, \overline{P}_{ilj}) \ h^{uv} (P_{iki}, \overline{D}_{v}D_{k}\mu) . \]

The two bounds before both hold, but the first one can be refined. Applying the bounds,

\[
\tilde{R}_{\alpha i} \geq 2 \left( 1 - W_{iis}'' \right)^2 + 2 + 2 \sum_{(r,s) \neq (i,i)} \left| W_{iirs}'' \right|^2 ,
\]

which produces an obvious lower bound of 2. Another bound, depending on \( h_{ij} \), can also be found. Define

\[
b_k := \begin{cases} 
W_{iikl}'' & 1 \leq k \leq m \\
\frac{1}{3} & m + 1 \leq k \leq m + 9 
\end{cases}
\]
Each of these terms is non-negative. Then by the Cauchy-Schwarz inequality,
\[
\left( \sum_{k=1}^{m+9} b_k \cdot 1 \right)^2 \leq \left( \sum_{k=1}^{m+9} b_k^2 \right) \left( \sum_{k=1}^{m+9} 1^2 \right).
\]

However,
\[
\left( \sum_{k=1}^{m+9} b_k \cdot 1 \right)^2 = \left( 2 + \sum_{k=1}^{m} W''_{iikk} \right)^2 = (h_{ii})^2,
\]
\[
\sum_{k=1}^{m+9} 1^2 = m + 9,
\]
and
\[
\sum_{k=1}^{m+9} b_k^2 = \left( 1 - W''_{iiii} \right)^2 + 1 + \sum_{k \neq i} |W''_{iikk}|^2 \leq \frac{1}{2} \tilde{R}_{iiii},
\]
so
\[
\tilde{R}_{iiii} \geq \frac{2}{m + 9} (h_{ii})^2.
\]

Thus the holomorphic sectional curvature is bounded above by \(- \frac{2}{m + 9} (h_{ii})^2\). 

These boundedness properties provide some further results about the moduli space \( \mathcal{M} \). First, manifolds with holomorphic sectional curvature bounded above by negative constant are Kobayashi hyperbolic [21]. Thus, the moduli space is Kobayashi hyperbolic with respect to this partial Hodge metric, a special case of the result of To and Yeung in [44] for sufficient conditions for a manifold of have such a property.

Additionally, two of Royden’s results in [38] provide versions of the Schwarz Lemma for maps into the moduli space.

**Lemma 4.2** (Schwarz Lemma [38]). Let \( \mathcal{M} \) be the \( m \)-dimensional moduli space of manifolds
of Calabi-Yau type equipped with partial Hodge metric $\omega_{m+3}$ (tensor form $h_{ij}$). Let $Y$ be a complete Kähler manifold with either

1. Ricci curvature bounded below by a non-positive constant $k$; or

2. holomorphic sectional curvature bounded below by a non-positive constant $k$, and holomorphic bisectional curvature bounded from below.

Then for any holomorphic map $f : Y \to \mathcal{M}$, there exists a constant $C \geq 1$ such that

$$\|df\|^2 \leq C \frac{k}{K}.$$ 

$C$ depends only on which bounds were used (not even what they are) and the maximal possible rank of $df$.

### 4.3 BCOV Torsion

Before determining the BCOV torsion (Definition 2.8) on manifolds of Calabi-Yau type, recall the notations of Chapter 2. Let $X$ be a polarized manifold of Calabi-Yau type of dimension $2n+1$. Let $\mathcal{X}'$ be the local deformation space parameterized by $\mathcal{M}'$, with projection $\pi' : \mathcal{X}' \to \mathcal{M}'$ such that for some interior point $O \in \mathcal{M}'$, $\pi'^{-1}(O) \cong X$. Let $X_t$ denote $\pi'^{-1}(t)$. In particular, $X_O \cong X$. Assume without loss of generality that

- $\mathcal{M}'$ is an open neighborhood of the origin in $\mathbb{C}^m$, and
- If $\omega_t$ is the polarized Kähler form on $X_t$, then $\int_{X_t} \omega_t^n = 1$.

On the moduli space there exist the relative Hodge bundles $R^q\pi'_*\Omega^p_{\mathcal{X}'/\mathcal{M}'} \to \mathcal{M}'$ in place of the cohomology groups. Let $\omega_{\mathcal{H}'}$ denote the generalized Hodge metrics of the moduli space.
Define the following holomorphic coefficient vector bundle.

\[ E = \bigoplus_{p=1}^{n} (-1)^{p} \Omega_{X'/M}^{p}. \]

In this framework, we require a new map. Define \( S : R^{2n+1} \pi_{*} \Omega_{X}^{2n+1} \mathcal{M}' \rightarrow R^{0} \pi_{*} \Omega_{X}^{0} \mathcal{M}' \) by

\[ S(a \, dz^{N_{2n+1}} \wedge d\bar{z}^{N_{2n+1}}) = a. \]

Notice that for any \( 2n + 1 \)-form \( \psi \) on \( X_{t} \), \( S(\psi \wedge \bar{\psi}) \) is a non-negative real number. Finally, unlike in earlier sections, let \( \partial \) and \( \bar{\partial} \) denote those operators on \( \mathcal{X}' \), not the fibers \( X_{t} \). When needed, \( \partial_{X} \) and \( \bar{\partial}_{X} \) will be the operators in the direction of the fibers and \( \partial_{M} \) and \( \bar{\partial}_{M} \) will be the operators in the direction of the moduli space.

**Theorem 4.4.** Let \( \chi(X) \) denote the Euler characteristic of the compact manifold \( X \) of Calabi-Yau type of dimension \( 2n + 1 \). Let \( \omega_{M} \) denote the Kähler form of the modulus metric. Let \( \mu \) denote the harmonic \((n+2, n-1)\)-form defining the modulus metric. Let \( T_{X/M} \) be the holomorphic relative tangent bundle. Then

\[
\sum_{i=1}^{2n+1} (-1)^{i} \omega_{H}^{i} - \frac{\sqrt{-1}}{2\pi} \partial_{M} \bar{\partial}_{M} \log T = \frac{\chi(X)}{12} \omega_{M} + \frac{\sqrt{-1}}{24\pi} \partial_{M} \bar{\partial}_{M} \int_{X} c_{2n+1}(X_{t}) \log S(\mu \wedge \bar{\mu}),
\]

where the Chern class \( c_{2n+1} \) is taken with respect to the Levi-Civita connection compatible with \( \omega_{t} \).

While the individual terms \( c_{2n+1}(X) \) and \( \log S(\mu \wedge \bar{\mu}) \) both depend on the Kähler metric \( \omega \) on \( X \), their integrals over \( X \) do not due to the rescaling assumption \( \int_{X} \omega_{t}^{n} = 1 \).

It is with a formula similar to this one that Fang, Lu, and Yoshikawa were able to explicitly compute the BCOV torsion of some Calabi-Yau threefolds in [10]. It is possible this formula will lead to a similar result.
4.3.1 Proof of Theorem 4.4

Before proving some new facts about manifolds of Calabi-Yau type, we recall some general theorems. The first is from Bismut, Gillet, and Soulé [3].

**Theorem 4.5** (Riemann-Roch-Grothendieck Theorem for the Quillen metric). Let $\Theta$ denote the curvature of

$$\lambda = \bigwedge_{0 \leq p,q \leq n} \det(H^{p,q}(X, E, \overline{\partial}))(-1)^{p+q}$$

compatible with the unique holomorphic Hermitian connection. Then

$$\Theta = 2\pi \sqrt{-1} \left[ \int_X Td(\mathcal{T}_{X'/\mathcal{M}'}) Ch(E) \right]^{(2)}$$

where $Td$ denotes the Todd class, $Ch$ denotes the Chern character, and $[\cdot]^{(2)}$ denotes the 2-form component.

**Lemma 4.3.** [9] There exists a Kähler metric $g^{\mathcal{X}'}$ on $\mathcal{X}'$ such that for each $t \in \mathcal{M}'$, the restriction $g^{(t)} = g^{\mathcal{X}'}|_{\mathcal{X}_t}$ is the polarized metric on $\mathcal{X}_t$.

**Lemma 4.4.** [2] For $E$ associated with the induced metric from $g^\mathcal{X}'$, as forms on $\mathcal{X}'$

$$Td(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) Ch(E) = -c_{2n}(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) + \frac{2n+1}{2}c_{2n+1}(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) - \frac{1}{12}c_1(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'})c_{2n+1}(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) + \ldots,$$

where $\ldots$ refers to higher order terms.

The higher-order terms in this lemma will not be relevant later. In fact, every term other than $-\frac{1}{12}c_1(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'})c_{2n+1}(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'})$ will drop out after use of Theorem 4.5 later. This lemma is best understood as the Chern classes $c_k(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'})$ being the elementary symmetric polynomials of the eigenvalues of the curvature tensor of the relative tangent bundle $\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}$. 
With these facts established, we are now ready to move on to proving facts specific to manifolds of Calabi-Yau type.

**Lemma 4.5.** Let $\mu$ be a harmonic, holomorphic generator of the relative Hodge bundle $R^{n-1} \pi_* \Omega^{n+2}_{X'/M'}$. Then

$$c_1(T_{X'/M'}) = -\pi^*(\omega_M) - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log S(\mu \wedge \bar{\mu}).$$

**Proof.** Let $\mu = \mu_I d\bar{z}^I \wedge d\bar{z}'^I$ in local coordinates. Then there exists a smooth function $f(t)$ on $M'$ such that

$$\mu \wedge \bar{\mu} = \pi^*(f(t)) \omega^n_t.$$

Then

$$\int_{X_t} \mu \wedge \bar{\mu} = \int_{X_t} \pi^*(f(t)) \omega^n_t = f(t).$$

Hence $f(t) = \int_{X_t} \mu \wedge \bar{\mu}$. Recall from Section 4.1 that $-\sqrt{-1} f(t)$ is real-valued and positive, so then

$$-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log S(\mu \wedge \bar{\mu}) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \pi^* \left[ -\int_{X_t} \mu \wedge \bar{\mu} \right] - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left[ \det \left( g_{ij}^{(t)} \right) \right].$$

Without loss of generality, assume that $\left( \frac{\partial}{\partial x^i} \right)$ is an orthonormal frame on $M'$ with respect to $g^X$ and that $M'$ is orthogonal to $X_t$. Then $\det \left( g_{ij}^{x} \right) = \det \left( g_{ij}^{(t)} \right) \cdot 1^m$, so

$$-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left[ \det \left( g_{ij}^{(t)} \right) \right] = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left[ \det \left( g_{ij}^{x} \right) \right] = c_1(T_{X'/M'}).$$
Additionally, since $\pi^* \left[-\sqrt{-1} \int_{X_t} \mu \wedge \ovl{\mu} \right]$ depends only on the coordinates $t$ of $\mathcal{M}'$, then

$$ -\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log \pi^* \left[-\sqrt{-1} \int_{X_t} \mu \wedge \ovl{\mu} \right] = -\frac{\sqrt{-1}}{2\pi} \partial_{\mathcal{M}} \overline{\partial}_{\mathcal{M}} \log \pi^* \left[-\sqrt{-1} \int_{X_t} \mu \wedge \ovl{\mu} \right] = \pi^* (\omega_M) $$

Therefore

$$ -\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log S(\mu \wedge \ovl{\mu}) = \pi^* (\omega_M) + c_1(T_{X'/\mathcal{M}}). $$

For any nowhere 0, holomorphic $g(t)$,

$$ \partial \overline{\partial} \log S(g(t)\mu \wedge \ovl{g(t)\mu}) = \partial \overline{\partial} \log \left(g(t)g(t)S(\mu \wedge \ovl{\mu})\right) $$

$$ = \partial \overline{\partial} \log \left(g(t)g(t)\right) + \partial \overline{\partial} \log S(\mu \wedge \ovl{\mu}) $$

$$ = \partial \overline{\partial} \log S(\mu \wedge \ovl{\mu}). $$

Hence the equation is well-defined. \hfill \Box

When $X$ is a Calabi-Yau threefold (when $n = 1$), then $\mu = \mu_0 dz^1 \wedge dz^2 \wedge dz^3$ where $\mu_0$ is a never zero section such that $\frac{\partial \mu_0}{\partial x_j} = 0 = \frac{\partial \mu_0}{\partial x_l}$ for $1 \leq j \leq 2n + 1$ and $1 \leq l \leq m$. Thus

$$ -\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log S(\mu \wedge \ovl{\mu}) = -\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log (-\sqrt{-1}\mu_0\ovl{\mu_0}) = 0, $$

and $c_1(T_{X'/\mathcal{M}'})$ only has components parallel to $\mathcal{M}'$. Additionally, in this case $\omega_M$ corresponds to the alternate presentation of the Weil-Petersson metric first demonstrated by Tian in [43]. Under these conditions, the lemma simplifies to $c_1(T_{X'/\mathcal{M}'}) = -\pi^*(\omega_{WP})$ with proof identical to that of Fang-Lu in [9] of an equivalent lemma used by Bershadsky-CEccotti-Ooguri-Vafa in [2].
Proposition 4.13. The first Chern class of $\lambda$ with respect to the $L^2$ metric is

$$c_1(\lambda, \|\cdot\|_{L^2}) = \sum_{i=1}^{2n+1} (-1)^i \omega_{H^i}.$$ 

Proof. Since $\det(R^q_{\pi_*} \Omega^p_{\mathcal{X}'/\mathcal{M}'})$ is a line bundle,

$$c_1(\lambda, \|\cdot\|_{L^2}) = c_1 \left( \bigwedge_{0 \leq p, q \leq 2n+1} \det(H^{p,q}(X, E, \mathcal{O}))^{(-1)^{p+q}p}, \|\cdot\|_{L^2} \right)$$

$$= \sum_{0 \leq p, q \leq 2n+1} (-1)^{p+q} p \ c_1 \left( \det(R^q_{\pi_*} \Omega^p_{\mathcal{X}'/\mathcal{M}'}) \right) = \sum_{0 \leq p, q \leq 2n+1} (-1)^{p+q} p \ c_1 \left( R^q_{\pi_*} \Omega^p_{\mathcal{X}'/\mathcal{M}'} \right)$$

By Proposition 2.7, this reduces to

$$c_1(\lambda, \|\cdot\|_{L^2}) = \sum_{i=1}^{2n+1} (-1)^i \omega_{H^i},$$

as desired. 

The pieces are now in place to prove Theorem 4.4.

Proof of Theorem 4.4. By Theorem 4.5,

$$c_1(\lambda, \|\cdot\|_{\mathcal{Q}}^2) = \left[ \int_Z Td(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) Ch(E) \right]^{(1,1)}$$

(4.1)

where $[\cdot]^{(1,1)}$ denotes the $(1, 1)$-component – the only 2-form component present. Thus by Lemma 4.4,

$$c_1(\lambda, \|\cdot\|_{\mathcal{Q}}^2) = \left[ \int_X Td(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) Ch(E) \right]^{(1,1)}$$

$$= \left[ \int_X -c_{n-1}(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) + \frac{2n+1}{2} c_{2n+1}(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) - \frac{1}{12} c_1(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) c_{2n+1}(\mathcal{T}_{\mathcal{X}'/\mathcal{M}'}) + \ldots \right]^{(1,1)}$$

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After integrating, the only original terms that produce \((1, 1)\)-forms are the \((2n + 2, 2n + 2)\)-forms. Thus, the only term that matters is \(-\frac{1}{12} c_1(T_{X'/M'}) c_{2n+1}(T_{X'/M'})\), since all others are the wrong form type. Thus

\[
c_1(\lambda, \| \cdot \|^2_Q) = -\int_X \frac{1}{12} c_1(T_{X'/M'}) c_{2n+1}(T_{X'/M'}).
\]

On the other hand, since \(\| \cdot \|^2_Q = \| \cdot \|^2_{L^2 T}\), then by the definition of the first Chern class

\[
c_1(\lambda, \| \cdot \|^2_Q) = c_1(\lambda, \| \cdot \|^2_{L^2 T}) = c_1(\lambda, \| \cdot \|^2_{L^2}) - \frac{\sqrt{-1}}{2\pi} \partial_M \bar{\partial}_M \log T.
\]

Therefore, by Proposition 4.13, Lemma 4.5, and the Gauss-Bonnet-Chern Theorem

\[
\sum_{i=1}^{2n+1} (-1)^i \omega_M - \frac{\sqrt{-1}}{2\pi} \partial_M \bar{\partial}_M \log T
\]

\[
= -\frac{1}{12} \int_X c_1(T_{X'/M'}) c_{2n+1}(T_{X'/M'})
\]

\[
= \frac{1}{12} \int_X \pi^*(\omega_M) c_{2n+1}(T_{X'/M'}) + \frac{1}{12} \int_X \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log S(\mu \wedge \bar{\mu}) c_{2n+1}(T_{X'/M'})
\]

\[
= \frac{\chi(X)}{12} \omega_M + \frac{\sqrt{-1}}{24\pi} \int_X c_{2n+1}(T_{X'/M'}) \partial \bar{\partial} \log S(\mu \wedge \bar{\mu}).
\]

The term \(\int_X c_{2n+1}(T_{X'/M'}) \partial \bar{\partial} \log S(\mu \wedge \bar{\mu})\) can be further simplified. For ease of saving space, we abuse notation by letting \(c_{2n+1} = c_{2n+1}(T_{X'/M'})\) and \(S = S(\mu \wedge \bar{\mu})\). Then via integration by parts on \(X_t\),

\[
\int_X c_{2n+1}(\partial_M + \partial_X)(\bar{\partial}_M + \bar{\partial}_X) \log S
\]

\[
= \int_X c_{2n+1} \partial_M(\bar{\partial}_M + \bar{\partial}_X) \log S - \int_X (\partial_X c_{2n+1})(\bar{\partial}_M + \bar{\partial}_X) \log S
\]

\[
= \int_X c_{2n+1} \partial_M \bar{\partial}_M \log S - \int_X (\partial_X c_{2n+1}) \bar{\partial}_M \log S
\]

\[
+ \int_X (\bar{\partial}_X c_{2n+1}) \partial_M \log S - \int_X (\bar{\partial}_X \partial_X c_{2n+1}) \log S.
\]
Since both $dc_{2n+1}(T_{X'}/M') = 0$ and $\bar{\partial}c_{2n+1}(T_{X'}/M') = 0$ ([29]), then $\partial c_{2n+1}(T_{X'}/M') = 0$ as well, and therefore we can continue to

\[
\begin{align*}
= \int_X c_{2n+1} \partial_M \bar{\partial}_M \log S + \int_X (\partial_M c_{2n+1}) \bar{\partial}_M \log S \\
- \int_X (\bar{\partial}_M c_{2n+1}) \partial_M \log S + \int_X (\partial_M \bar{\partial}_M c_{2n+1}) \log S \\
= \int_X \partial_M \bar{\partial}_M \left( \log S \cdot c_{2n+1} \right).
\end{align*}
\]

Since $\log S$ is simply a function and the differential operators only add terms parallel to $M'$, the only terms that matter in $c_{2n+1}$ are the ones parallel to $X$. Then $c_{2n+1} = c_{2n+1}(X_t)$ and the expression becomes

\[
\partial_M \bar{\partial}_M \int_X c_{2n+1}(X_t) \log S.
\]

This completes the proof. \qed


