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Phonon-mediated tuning of instabilities in the Hubbard model at half-filling

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We obtain the phase diagram of the half-filled two-dimensional Hubbard model on a square lattice in the presence of Einstein phonons. We find that the interplay between the instantaneous electron-electron repulsion and electron-phonon interaction leads to new phases. In particular, a $d_{x^2-y^2}$-wave superconducting phase emerges when both anisotropic phonons and repulsive Hubbard interaction are present. For large electron-phonon couplings, charge-density-wave and $s$-wave superconducting regions also appear in the phase diagram, and the widths of these regions are strongly dependent on the phonon frequency, indicating that retardation effects play an important role. Since at half filling the Fermi surface is nested, a spin-density wave is recovered when the repulsive interaction dominates. We employ a functional multiscale renormalization-group method [Tsai et al., Phys. Rev. B 72, 054531 (2005)] that includes both electron-electron and electron-phonon interactions, and take retardation effects fully into account.

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I. INTRODUCTION

The renormalization-group (RG) approach to interacting fermions has recently been extended to include the study of interacting fermions coupled to bosonic modes, such as phonons. Experimental evidence indicates that in many strongly correlated systems, such as organic conductors and superconductors, cigars, cobaltates, and conducting polymers, both electron-electron interactions and phonons may play an important role. Also, recent advances in the field of cold atoms have made possible the creation of fermion-boson mixtures on artificial lattices. In these mixtures the fermions interact through instantaneous on-site repulsion, and when the bosonic atoms condense, there is an additional retarded attractive interaction mediated by the fluctuations of the bosonic condensate; a scenario similar to that of electron and phonon interaction in solid state systems. The physics of the interplay between the repulsive electron-electron ($e-e$) and attractive electron-phonon ($e-ph$) interaction is not well understood and fundamental questions arise that include a full understanding of retardation effects and whether competition or cooperation between these interactions can lead to new phases.

One of the most extensively studied model is the two-dimensional repulsive Hubbard model, which in the absence of phonons and at half filling becomes an antiferromagnet due to nesting in the Fermi surface (FS) which drives an $s$-wave spin-density wave (sSDW) instability. In the presence of isotropic phonons and in the strongly coupled regime, it has been shown that the $s$-wave charge-density wave instability dominates over antiferromagnetism and coexists with $s$-wave superconductivity (sSC). When anisotropic phonons are present and full retardation is taken into account the phase diagram in parameter space associated with this model is summarized in Fig. 1, which reproduces the usual sSDW and sCDW instabilities but additionally a $d_{x^2-y^2}$-wave superconducting phase (dSC). Therefore, in this work we show that this dSC phase that appears at half filling is a result of cooperation between the two interactions. We stress that the role of the phonons goes beyond simply creating an effective attractive force between the electrons. Retardation plays a crucial role in generating the new phases, and the size of each region depends on the phonon frequency. This model can also be used to study various other systems where the $e-ph$ interaction is present, such as the quasi-two-dimensional (quasi-2D) $\kappa$-(BEDT-TTF)$_2$X materials, which are stoichiometric with a fixed density of one electron per BEDT-TTF dimer, or as we mentioned in the beginning, for fermion-boson optical lattice mixtures (where there is experimental control on the number of fermions per lattice point) deep in the bosonic condensate phase. This study is by no means complete in its results since it is limited to the weak-coupling regime but is intended to chart the vast phase space of these type of systems with some preliminary results that will help arrange the important underlying physics and contribute to a better understanding of the competition and cooperation between the interactions involved.

This paper is organized as follows. In Sec. II we introduce the theoretical model and the RG method of analysis we
employ. In Sec. III we provide our results and draw the phase diagrams associated with the different orders in the system. In Sec. IV we discuss the summary and the basic physics our work has highlighted and finally in the appendix we provide more details for the RG flow of the couplings and susceptibilities for the reader that is interested on the theoretical details.

II. THEORETICAL MODEL

The approach we employ in this study is based on a general RG analysis of a system of electrons recently expanded to involve the coupling of the electrons with phonons as well.1 We use a generic model of electrons on a Hubbard lattice at half filling interacting through the repulsive Coulomb interaction and being isotropically and anisotropically coupled to dispersionless bosonic excitations (Einstein phonons). The Hamiltonian associated with this type of system is

\[ H = \sum_{k,\sigma} \xi_k c_{k,\sigma}^\dagger c_{k,\sigma} + \sum_{k_1, k_2, k_3} U_0 c_{k_1,\sigma}^\dagger c_{k_2,\sigma}^\dagger c_{k_3,\sigma} c_{k_4,\sigma} + \omega_F \sum_q b_q^\dagger b_q + \sum_{q, \sigma} g(q) c_{k,\sigma}^\dagger c_{k,\sigma} (b_q + b_{-q}) \]  

(1)

where \( c_{k,\sigma}^\dagger \) (\( c_{k,\sigma} \)) is the creation (annihilation) operator of an electron with momentum \( k \) and spin \( \sigma \), and \( b_q^\dagger \) (\( b_q \)) is the corresponding creation (annihilation) operator of a phonon with momentum \( q \). Also, \( \xi_k = -2t(\cos k_x + \cos k_y) \) is the non-interacting electron energy at half filling, \( \omega_F \) is the Einstein frequency, \( U_0 \) is the e-e on-site repulsion, while the e-ph coupling \( g(q) \) is taken to be momentum dependent. Momentum conservation (up to reciprocal lattice vectors) implies \( k_1 = k_1 + k_2 - k_3 \). Going to the path-integral formulation, the bosonic fields can be integrated out exactly1 to give an effective e-e interaction

\[ U(k_1,k_2,k_3) = U_0 - \lambda (k_1 - k_2) \frac{\omega_F^2}{\omega_F^2 + (\omega_1 - \omega_3)^2}. \]  

(2)

where \( k_i = (i\omega_1,k_i) \) and \( \lambda = 2g(q)^2/\omega_F \). The bare interaction therefore contains an instantaneous on-site Coulomb repulsion term and a retarded attractive part. This is the interaction used in the functional RG analysis in this work.

A. RG flow of the couplings

The general idea of RG theory is to integrate out self-consistently the degrees of freedom in the electronic system in successive steps (called RG time), so that the electronic interaction becomes renormalized with each RG time step until it diverges, which is a signature of an instability. The approximate character of this method lies in the self-consistent evaluation of the renormalization of the electronic interactions, where bare values are assumed to be smaller in strength than the bandwidth of the available electronic energies. The same approximation applies for the electron-phonon coupling strength as well (weak coupling) so that the parameters of Eq. (1) obey \( u_0, \omega_F, g(q) \ll \Lambda_0 = 4t \). We also remain within one-loop accuracy2 which is adequate to capture the essential physics, but we expand the analysis to include dynamics (frequency dependence) in the electronic couplings. In previous work16-18 the couplings involved in the RG study were frequency independent, which led to self-energy corrections at one loop that were neglected in order to keep the total number of electrons fixed. In this work, the implicit frequency dependence generates an imaginary part in the self-energy \( \Sigma_\ell(k) \) which we calculate in order to use the full cutoff-dependent electron propagator, given by

\[ C_\ell(p) = \frac{\theta(\xi_{\rho} - \Lambda_\ell)}{i\omega - \xi_{\rho} - \Sigma_\ell(p)}, \]  

(3)

where \( \theta(x) \) is the Heaviside step function, \( \Lambda_\ell \) is the value of the RG cutoff at the RG time \( \ell = \ln(\Lambda_0/\Lambda_1) \), and \( \ell_\rho = \ln(\Lambda_0/|\xi_{\rho}|) \), where \( \Lambda_0 \) is the initial cutoff corresponding to half the bandwidth (4\( t \)) as we defined above. As we have already mentioned the RG flow equations are evaluated at one-loop2 accuracy using the general Polchinski equation15 applied for this specific lattice model.16 The zero-temperature RG flow equations for the couplings and the self-energy can be written as1,16,18

\[ \partial_\ell U_\ell(k_1,k_2,k_3) = \int_p \frac{d\ell}{d\ell} [C_\ell(p)C_\ell(q_{pp} - p)] U_{\ell pp}(k_1,k_2,p) U_{\ell pp}(k_3,k_4,p) + \int_p \frac{d\ell}{d\ell} [C_\ell(p)C_\ell(p + q)] U_{\ell ph}(k_1,p,k_2) U_{\ell ph}(p,k_3,k_4) + \int_p \frac{d\ell}{d\ell} [C_\ell(p)C_\ell(p + q_{ph})] U_{\ell pp}(k_1,p,k_3) U_{\ell ph}(k_4,p,k_2) + \int_p \frac{d\ell}{d\ell} [C_\ell(p)C_\ell(p + q_{ph})] U_{\ell pp}(k_1,p,k_3) U_{\ell ph}(k_4,p,k_2) \]  

(4)

\[ \partial_\ell \Sigma_\ell(k) = \int_p \frac{d\ell}{d\ell} [C_\ell(p)] [U_{\ell pp}(k_1,p,k_1) - 2U_{\ell pp}(k_1,p,k_1)], \]  

(5)

where we have defined \( q_{pp} = k_1 + k_2 \), \( q_{ph} = k_1 - k_3 \), \( q_{ph} = k_1 - k_4 \), \( q_{pp} = \min(\ell_p,\ell_{pp-p}) \), \( q_{ph} = \min(\ell_p,\ell_{pp+q_{ph}}) \), and \( q_{ph} = \min(\ell_p,\ell_{pp+q_{ph}}) \). We have also used the shorthand notation

\[ \int_p = \int_{-\pi/2}^{\pi/2} \frac{d\rho}{2\pi^2}. \]  

As we see, the above RG flow equations are in general nonlocal in RG parameter time \( \ell \).

For the square lattice at half filling the FS contains van Hove points where the density of states is logarithmically singular. It is a well-justified approximation9 to divide the FS into regions around the singular points \( \mathbf{Q}_1 = (\pi,0) + \mathbf{k} \) and...
Q_2 = (0, \pi) + k \text{ with } |k| \ll \pi (\text{called the two-patch approximation}), where the majority of electronic states are expected to reside. This results in four types of e-e scattering processes defined as \( g_1 = U(Q_1, Q_2), \) \( g_2 = U(Q_1, Q_1), \) \( g_3 = U(Q_1, Q_2), \) and \( g_4 = U(Q_1, Q_2), \) which in the fully retarded case become frequency dependent \( g_i(\omega_1, \omega_2, \omega_3). \) Deformations of the FS have been shown to be stable with respect to corrections due to e-e interactions\(^{11,19}\) and need not be of concern in this approximation. During the RG flow the electronic density is kept fixed, which results in the chemical potential flow canceling the flow of the real part of the electronic density. Nonlocality in the RG equations is lifted because momentum transfers can only be zero or (\( \pi, \pi \)) and since \( \xi_k = -\xi_{k + (\pi, \pi)} \) all RG parameters map back to \( \ell. \) Phononic anisotropy is introduced by distinguishing e-ph scattering processes that involve electrons from the same patch \((g_2, g_4)\) and those that scatter an electron from one patch to another \((g_1, g_3)\), and by assigning different coupling strengths \((\lambda_0, \lambda_\pi)\) to them. The only place that phonons enter into the RG flow is through the following initial conditions:

\[
\begin{align*}
\frac{\delta_g}{\delta g} \left( \omega_1, \omega_2, \omega_3 \right) &= u_0 - \lambda_\pi \frac{\omega_1^2}{\omega_2^2} (\omega_1 - \omega_3)^2, \\
\frac{\delta_g}{\delta g} \left( \omega_1, \omega_2, \omega_3 \right) &= u_0 - \lambda_0 \frac{\omega_1^2}{\omega_2^2} (\omega_1 - \omega_3)^2.
\end{align*}
\]

The reader interested in the analytic details of the flow equations can consult Appendix A where for the sake of completeness we provide the exact RG flow equations for the couplings and the self-energy.

### B. RG flow of the susceptibilities

As we mentioned in the Introduction, the different instabilities associated with the Hubbard model at half filling are superconductivity (sSC), antiferromagnetism (sSDW, dSDW), and charge-density waves (sCDW, dCDW) with the corresponding couplings\(^9\) (we suppress the implicit frequency dependence for clarity)

\[
\begin{align*}
\sigma_{\text{sd,SC}} &= -2(g_2 \pm g_3), \\
\sigma_{\text{sd,SDW}} &= 2(g_4 \pm g_3), \\
\sigma_{\text{sd,CDW}} &= -2(2g_1 \pm g_3 - g_4),
\end{align*}
\]

where the signs are chosen so that strong fluctuations in a channel produce a positive value for the corresponding coupling, irrespective of its attractive (SC,CDW) or repulsive (SDW) nature. Since the couplings are functions of frequency and the most divergent ones are not necessarily at zero frequency,\(^{21}\) only a divergent susceptibility can determine which phase has dominant fluctuations. The corresponding order parameters for the different phases are

\[
\begin{align*}
\Delta_{\text{SC}}(\xi, \theta, \tau) &= \sum_{\sigma} \sigma \Psi_{\sigma, k}(\tau) \Psi_{-\sigma, -k}(\tau), \\
\Delta_{\text{SDW}}(\xi, \theta, \tau) &= \sum_{\sigma} \tilde{\Psi}_{\sigma, k}(\tau) \Psi_{-\sigma, k + (\pi, \pi)}(\tau),
\end{align*}
\]

and involve the creation of particle-particle (\(p-p\)) and particle-hole (\(p-h\)) pairs at given angle \(\theta\) and energy \(\xi\) set by \(k.\) The homogeneous frequency-dependent susceptibilities associated with these order parameters are calculated by extending the one-loop RG scheme of Zanchi and Schulz.\(^{16}\) Their definition involves scattering processes between pairs at different angles \(\theta\) with energies corresponding to fast modes and are given by

\[
\chi^{\xi}(\theta_1, \theta_2, |\tau_1 - \tau_2|) = \prod_{i=1}^{2} \int d\xi \Theta(\xi - \Lambda_i) J(\xi, \theta) \times \langle \Delta^{\xi}(\xi_1, \theta_1, \tau_1) \Delta^{\xi}(\xi_2, \theta_2, \tau_2) \rangle,
\]

where \(J(\xi, \theta)\) is the Jacobian for the \(k \rightarrow (\xi, \theta)\) transformation, and \(\delta = \text{SC,SDW,CDW}.\) In the two-patch approximation \(\theta\) can take only two values and each susceptibility becomes a \(2 \times 2\) matrix that we diagonalize to extract the symmetric (\(s\)-wave) and antisymmetric (\(d\)-wave) eigenvectors of each corresponding order. The interested reader is urged again to refer to Appendix B where we provide the full expressions for the susceptibility flow equations of all the different orders calculated to one-loop accuracy.

### III. RESULTS

In Fig. 2 we show the numerical solution of the RG flow equations for the couplings with \(\omega_1 = \omega_2 = \omega_3 = 0\) (inset) and the corresponding static (\(\omega = 0\)) homogeneous susceptibilities
for the case of $\lambda_0=0.6$, $\lambda_\pi=0.4$, $\omega_F=1.0$, and $u_0=0.5$. All parameters are expressed in energy units of $2t$ (a quarter of the bandwidth) and for the numerical implementation the frequencies are discretized into a total of $N=41$ divisions up to a maximum value of $\omega_{\text{max}}=6.0$. The particular choice of $\lambda_0>\lambda_\pi$ enhances the attractive BCS-type pairing processes associated with $g_2$, while suppressing the dominant repulsive nesting channels of $g_3$ [Eqs. (6) and (7)], and tilts the balance between the usually dominant sSDW and subdominant dSC phases. When reversed ($\lambda_\pi>\lambda_0$), the attractive channels of $g_3$ combined with $g_1$ lead to a CDW instability.

The general phase diagram for fixed values of $u_0=0.5$ and $\omega_F=1.0$ is shown in Fig. 1, where the e-ph coupling is parametrized by the mean value $\tilde{\lambda}=(\lambda_0+\lambda_\pi)/2$ and relative anisotropy $r=\lambda_0/(\lambda_0+\lambda_\pi)$. Previous work on the phononic effects in this type of system was limited along $r=0.5$ (isotropic phonons) and $\tilde{\lambda} \gg u_0$ and found the nesting-related dominant CDW competing with sSC, while dSC was suppressed. In our generic study, extending to all possible configurations of coupled e-ph systems, we find that close to $\tilde{\lambda}=u_0$ and along $r=0.5$ there are four competing phases including dSC. Deep in the repulsive region antiferromagnetism prevails as expected (the counterpart of sCDW), but for $r>0.5$ type of anisotropy ($\lambda_0>\lambda_\pi$), there is a large region of dSC dominance (the case of $r=2$ is a point in that region). Another candidate for this parameter space is the dCDW (charge flux phase) but we find that while this channel does get renormalized significantly (Fig. 2), the suppressed $g_1$ and $g_3$ couplings undermine its strength.

It should be pointed out explicitly that the generic characteristics of the phase diagram in Fig. 1 are independent of the actual value of the repulsive $u_0$. The same phase diagram is always expected near $\tilde{\lambda}=u_0$ and details such as adding a next-nearest-neighbor hopping term in the Hamiltonian or doping away from half filling will only enhance our findings associated with dSC since all nesting-related processes will then be additionally suppressed. In the absence of Hubbard on-site repulsion ($u_0=0$), not only does SDDW not occur, as expected, but the dSC phase also disappears completely as shown in Fig. 3 for the $\omega_F=1.0$ case. Therefore, the dSC phase is not being driven solely by anisotropic phonons, but by the combined effect of anisotropic phonons, on-site repulsion and nesting of the FS.

In order to demonstrate the importance of retardation, we show in Fig. 4 the corresponding phase diagram for slower phonons of $\omega_F=0.1$, keeping $u_0=0.5$ fixed. We see that as compared to Fig. 1, the SC regions are suppressed and pushed toward larger values of $\tilde{\lambda}$. This is a clear indication that the internal dynamics of the coupling functions extend beyond renormalizing the effective $e-e$ interaction to attractive values. There is a rich internal frequency structure among the couplings as we show in Fig. 5, where we plot the evolution of the pairing channel of $g_2(\omega_1,-\omega_1,\omega_3)$ for the case referring to the dSC instability of Fig. 2. At the beginning of the flow we see the attractive part of the coupling (limited around $\omega_1=\omega_3$), to be much weaker compared to the repulsive parts. As the flow progresses, and we go beyond the initial conditions of Eqs. (6) and (7), the repulsive parts

![Figure 3](image3.png)

**FIG. 3.** (Color online) Phase diagram ($u_0=0.0$ and $\omega_F=1.0$). Without on-site repulsion, the dSC phase disappears completely even for large phonon anisotropy. Color scheme is identical to Fig. 1.

![Figure 4](image4.png)

**FIG. 4.** (Color online) Phase diagram for the same on-site repulsion ($u_0=0.5$) as in Fig. 1, but for a smaller phonon frequency ($\omega_F=0.1$). Slower phonons suppress superconductivity over the nesting-associated CDW and SDW phases. Key is identical to Fig. 1.

**IV. CONCLUSIONS**

In conclusion, we have presented a one-loop functional renormalization group analysis of the Hubbard model at half
filling, self-consistently including the electron-phonon interaction. We find that different values of the e-ph coupling energy and anisotropy can tune the system into different instabilities, including sSDW, sCDW, sSC, and dSC. When only the instantaneous on-site repulsion is present, sSDW is well known to dominate. In the absence of on-site repulsion, phonon-mediated attraction drives the system into sCDW and sSC phases. In the presence of both interactions, depending on the competition between them, these phases appear in different parts of the phase diagram. In addition, these interactions also cooperate to generate a dSC phase. This phase therefore emerges only because of the interplay between the physics of Coulomb interactions and phons. Retardation effects play an important role in the onset of these phases and also determine the size of the different regions in the phase diagram.

Our work is not to be considered as a complete study of the two-dimensional Hubbard model in the presence of the e-ph interaction but as a preliminary but self-consistent study that hints toward the right physics of such a system. In other words, what the functional RG study can beautifully provide to us is an unbiased study of the instabilities associated with this type of system in a self-consistent manner to the given one-loop accuracy order, which allows us the flexibility of probing an immense parameter phase space in a rather inexpensive way (numerically and analytically). The assumptions of this approach are the weak-coupling regime in the electron-phonon interaction and that the largest energy scale of the system is the bandwidth. Within this approximate method we are able to draw the first charts of the rich phase space when the fermion-boson interaction is no longer neglected but self-consistently included as well.

**ACKNOWLEDGMENTS**


**APPENDIX A: COUPLING AND SELF-ENERGY FLOW EQUATIONS**

The renormalization-group flow equations for the four couplings and the self-energy are directly derived from Eqs. (4) and (5) when the two-patch approximation is used which maps all momenta transfers either to $(0,0)$ or $(\pi, \pi)$ converting the intrinsically nonlocal flow equations to local. This procedure is very well presented in the work of Zanchi and Schult,\(^{16}\) for the reader interested in the full exposition of details. Here we highlight only the basic points along with presenting the final formulas for the sake of completeness.

In the two-patch approximation the whole FS is divided into two patches (each patch has its redundant mirror image). The RG flow Eqs. (4) and (5) involves a $\delta$ function constraining the electronic energies $\xi_k$ to be “on shell” which can be above $(+\Lambda_\ell)$ or below $(-\Lambda_\ell)$ the FS. This constraint induces a one-to-one correspondence between $k$ and $(\theta, |\xi_k|=\Lambda_\ell)$ which consequently is employed to simplify the 2D $k$ integration into an azimuthal integration over $\theta$, with the proper Jacobian $J(\theta, |\xi_k|$ introduced. The integrand is $\theta$-independent and one can define a general operator $\hat{F}_\nu$ acting on any product of two coupling functions $g_i(\omega_1, \omega_2, \omega_3)g_j(\omega_2, \omega, \omega_3)$ according to

$$
\hat{F}_\nu[\Omega][g_i g_j](\omega_1, \omega_2, \omega_3) \propto \frac{\Lambda_\ell}{\pi^2} \int_0^{\pi^2} d\theta \frac{k(\theta-\Lambda_\ell)}{d\xi_{\Lambda_\ell}(\theta-\Lambda_\ell)} \int_{-\infty}^{+\infty} d\omega g_i(\omega_1, \omega_2, \omega_3)g_j(\omega_2, \omega, \omega_3)
$$

where $\Omega$ depends on the external frequencies and $\omega$ (which is integrated out), $\nu=\pm$ and effectively defines two types of $\hat{F}$ operators, while $\Sigma''(\omega)$ is the imaginary part of the self-energy. The Jacobian at half filling has the convenient property $\hat{J}(\theta, -\Lambda_\ell) = \hat{J}(\theta, +\Lambda_\ell)$. The complete RG flow equations for the four couplings and the imaginary part of the self-energy can then be written as

$$
\frac{d}{dt} \sigma_i(\omega) = \sum_{\nu=\pm} \frac{\nu}{2} \left( \frac{\sigma_i''(\omega)}{\sigma_i''(\omega) + \nu \Lambda_\ell^2} + \frac{\nu \Lambda_\ell^2}{\sigma_i''(\omega) + \nu \Lambda_\ell^2} \right)
$$

where $\omega$ is the Matsubara frequency.
\[
\frac{\partial g_1}{\partial \ell}(w_1, w_2, w_3) = F[\omega + \omega_1 - \omega_3][-2[g_{1g_1} + g_{1g_3}][\omega_1, \omega, \omega_2] + [g_{3g_3} + g_{1g_4}][\omega_1, \omega, \omega_3][\omega, \omega_4, \omega_2] + [g_{3g_3} + g_{4g_1}] 
\times (\omega, \omega_1, \omega_2, \omega_3) + F[\omega + \omega_2 - \omega_1][-2[g_{1g_3} + g_{1g_1}][\omega_3, \omega, \omega_1] + [g_{3g_3} + g_{1g_4}][\omega_3, \omega_1] 
\times (\omega, \omega_2, \omega_4) + [g_{3g_3} + g_{4g_1}][\omega, \omega_3, \omega_1]),
\]

\[
\frac{\partial g_2}{\partial \ell}(w_1, w_2, w_3) = F[\omega + \omega - \omega_1][[g_{3g_3} + g_{2g_2}][\omega, \omega_1, \omega_2, \omega] + [g_{3g_3} + g_{2g_2}][\omega, \omega_1, \omega_3][\omega, \omega_2, \omega_4]]
\]

The above equations are numerically solved for each RG step \( \ell \) until any coupling for any frequency channel diverges to values greater than 20, at which point we stop the algorithm and form all the couplings associated with the different instabilities given by Eqs. (8)–(10). For frequency-independent interactions the above equations reduce to the usual16

\[
\frac{\partial g_1}{\partial \ell} = -2g_1(g_1 - g_4),
\]

\[
\frac{\partial g_2}{\partial \ell} = \frac{\partial g_3}{\partial \ell} = -g_2 - g_3.
\]

\[
\frac{\partial g_4}{\partial \ell} = g_3^2 + g_4.
\]

**APPENDIX B: SUSCEPTIBILITY FLOW EQUATIONS**

Once the RG flow for the couplings is numerically solved and the divergence point toward strong coupling is reached we calculate the susceptibilities associated with the major orders in the system. As we mentioned in the text, a general susceptibility calculation involves Eq. (14) for all different orders. By including (to one-loop) all RG vertex corrections16 we obtain the general homogeneous susceptibility flow equation which in the two-patch approximation reduces to a 2 \times 2 matrix given by

\[
\frac{\partial \Sigma^\mu}{\partial \ell}(\omega_1) = \frac{\Lambda_\ell}{\pi^4} \int_0^{\pi/4} d\theta J[\theta, -\Lambda_\ell] \int_{-\infty}^{\infty} d\omega \omega \left[ g_1 + g_2 \omega_1, \omega_2 - 2[g_2 + g_3][\omega, \omega, \omega_1] + \Sigma^\mu(\omega) \right] \left[ \omega - \Sigma^\mu(\omega) \right]^2 + \Lambda_\ell^2.
\]

\[
\delta(\omega) = [\omega - \Sigma^\mu(\omega)] \left[ \omega + \omega_1 - \Sigma^\mu(\omega + \omega_1) \right]
\]

\[
\Sigma^\mu(\omega) = \Lambda_\ell^2 \left[ \omega + \omega_1 - \Sigma^\mu(\omega) \right] \left[ \omega + \omega_1 - \Sigma^\mu(\omega + \omega_1) \right] + \Lambda_\ell^2,
\]

\[
W^{AF}_\ell(\omega_1, \omega) = W^{CDW}_\ell(\omega_1, \omega)
\]

\[
W^{SC}_\ell(\omega_1, \omega)
\]
PHONON-MEDIATED TUNING OF INSTABILITIES IN

and


\[ z_{ijgk} = z_{ijgk}(\theta_1, \theta_2; \omega_1, \omega_2)g_{ik}(\omega_2, \omega_1 + \omega_2) \text{ (B5)} \]

where \( z_{ijgk} = z^\delta(\theta_1, \theta_2; \omega_1, \omega_2)g_{ik}(\omega_2, \omega_1 + \omega_2) \) and we used the additional definitions

\[ g_\delta(\omega_2, \omega_1 + \omega_2, \omega) = 2g_3(\omega_1 + \omega_2, \omega) - g_3(\omega_2, \omega_1 + \omega_2), \quad (B6) \]

\[ g_\delta(\omega_2, \omega_1 + \omega_2, \omega) = 2g_1(\omega_1 + \omega_2, \omega) - g_4(\omega_2, \omega_1 + \omega_2). \quad (B7) \]

For the SC-related vertex function we have

\[ \partial_c z^{\text{SC}} \left( \omega_1, \omega_2 \right) = \frac{\Lambda_c}{\pi} \int_{-\infty}^{\infty} d\omega W^{\text{SC}}_\ell \left( \omega_1, \omega \right) \]

and

\[ \partial_c z^{\text{CDW}} \left( \omega_1, \omega_2 \right) = \frac{\Lambda_c}{\pi} \int_{-\infty}^{\infty} d\omega W^{\text{CDW}}_\ell \left( \omega_1, \omega \right) \]

where \( z_{ijgk} = z(\theta_1, \theta_2; \omega_1, \omega_2)g_{ik}(\omega_2, \omega_1 + \omega_2) \). All susceptibilities are zero at the initial RG step \( \ell = 0 \), and the vertex functions obey

\[ z_{\ell,0}^\delta(\theta_1, \theta_2; \omega_1, \omega_2) = \frac{1}{4} \delta_{\ell,0}^\delta(\omega_1 - \omega_2). \quad (B9) \]

Also, due to the fact that we are at half filling the flow equations for the nested related phases are local.16

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